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A combinatorial approach towards Ramanujan's partition congruences modulo 5, 7 and 11 in terms of its corresponding cranks

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Abstract

This paper presents a combinatorial interpretation of Ramanujan's celebrated partition congruences: $p(5n+4) \equiv 0(mod\ 5)$, $p(7n+5) \equiv 0(mod\ 7)$ and, $p(11n+6) \equiv 0(mod\ 11)$, where $p(n)$ denotes the number of partitions of n. We interpret these congruences using the notions of rank and crank, and vector partitions—concepts developed by Dyson, Atkin, Swinnerton-Dyer, Andrews, and Garvan via elegant combinatorial arguments and q-series identities.

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1. Theory of Partitions : A Brief Survey

1.1 *Introduction*

Partition theory originated with Leibniz and matured through Euler, Sylvester, and Ramanujan. The following excerpt can be found in the book *Combinatorics: Ancient & Modern* [1, Chapt. 9],

> "*While Leibniz appears to have been the earliest to consider the partitioning of integers into sums, Euler was the frst person to make truly deep discoveries. J. J. Sylvester was the next researcher to amke major contributions, followed by Fabian Franklin. The* 20^{th} *Century saw an explosion of research with contributions from L. J. Rogers, G. H. Hardy, Percy MacMahon, Srinivasa Ramanujan and Hans Rademacher.*"

- George E. Andrews.

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A partition of $n \in \mathbb{N}$ is a non-increasing sequence of natural numbers whose sum is n, denoted λn. The number of such partitions is $p(n)$.

Example 1.1.1: For example, $2+13$, $5+3+210$ denotes partitions of 3 and 10 respectively.

A more pertinent question may pop up in a curious mind,

Are there infinitely many integers n *such that,* $p(n)$ *is prime* ?

Unsurprisingly, the problem is not as trivial as it seems and is still open. The exact behavior of $p(n)$, its prime values, and divisibility by primes are key themes in number theory, often approached via the notion of *generating functions*.

Suppose, $D(m,n)$ denotes the number of partitions of n into m distinct parts. Furthermore, we choose q (called the *base*), such that, $|q|<1$. Then, we can deduce the following identity,

$$\sum_{m,n\geq 0} D(m,n)z^m q^n = \prod_{j=1}^{\infty} (1+zq^j) \tag{1.1}$$

For every such j, we index the distinct parts as $(i_1, i_2, \ldots\ldots, i_j)$. Consequently,

$$\sum_{m,n\geq 0} D(m,n)z^m q^n = (1+zq)\sum_{m,n\geq 0} D(m,n)z^m q^{n+m} \tag{1.2}$$

Comparing co-efficients of $z^m q^n$ on both sides,

$$D(m,n) = D(m,n-m) + D(m-1,n-m) \tag{1.3}$$

1.2 *A Note on q-Series and Theta Functions*

Definition 1.2.1: (q-Series) We define,

$$(a)_n := (a;q) := \prod_{k=0}^{n-1} (1-aq^k), n \geq 1.$$
$$(a)_\infty := (a;q) := \prod_{k=0}^{\infty} (1-aq^k), |q|<1.$$

Definition 1.2.2: (Ramanujan's Theta Function) Ramanujan's General Theta Function $f(a,b)$ is defined by,

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}, \quad |ab|<1. \tag{1.4}$$

Ramanujan introduced notations for three special cases applying definition (1.2.2).

$$\phi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2 (q^2;q^2)_\infty \tag{1.5}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} \tag{1.6}$$

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty \tag{1.7}$$

The expressions in R.H.S. of (1.5), (1.6), (1.7) follows from the *Jacobi-Triple Product Identity*.

Theorem 1.2.1: (Jacobi-Triple Product Identity) *For* $z \neq 0$ *and* $|q|<1$,

$$\sum_{n=-\infty}^{\infty} z^n q^{n^2} = (-zq;q^2)_\infty \left(-\frac{q}{z};q^2\right)_\infty (q^2;q^2)_\infty$$

Combining *Theorem* (1.2.1) and (1.7), we obtain the **Euler's Pentagonal Theorem**,

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2} = (q;q)_\infty \quad (1.8)$$

1.3. *An Estimate for the Partition Function* $p(n)$

A priori from (1.8), we can in fact conclude about the following recurrence relation which eventually leads us to approximate the value of $p(n)$.

Corollary 1.3.1: *For brevity, for every integer* j*, we set,* $w_j = \frac{j(3j-1)}{2}$*. Then,*

$$p(n) = \sum_{0<w_j\le n} (-1)^{j+1} p(n-w_j) \quad (1.9)$$

A priori given *Ramanujan*'s generalizations of the results involving the partition congruences [2],

$$p(5n+4) \equiv 0 \ (\mathrm{mod}\ 5) \quad (1.10)$$

$$p(7n+5) \equiv 0 \ (\mathrm{mod}\ 7) \quad (1.11)$$

$$p(11n+6) \equiv 0 \ (\mathrm{mod}\ 11) \quad (1.12)$$

The following conjectures have already been established for $j \ge 1$, $\delta_{l,j}$ denoting the l^j th reciprocal *modulo* 24.

$$p(5^j n + \delta_{5,j}) \equiv (\mathrm{mod}\ 5^j) \quad (1.13)$$

$$p\left(7^j n + \delta_{7,j}\right) \equiv 0 \ (\mathrm{mod} 7^{\left\lfloor \frac{j}{2} \right\rfloor + 1}) \quad (1.14)$$

$$p\left(11^j n + \delta_{11,j}\right) \equiv 0 \ (\mathrm{mod} 11^j) \quad (1.15)$$

Watson provided the proofs of (1.13) and (1.14) [3], whereas, *Hirschhorn* and *Hunt* derived (1.13) [interested readers can look up [4]]. Using similar techniques, *Garvan* [5] proved (1.14). More notably, the congruence (1.15) was established by *Atkin* [6].

1.4 *Ramanujan's Congruence,* $p(5n+4) \equiv 0(mod\ 5)$

In this section, we intend to provide a detailed version of the proof of the *Ramanujan's Congruence*. It can also be observed as an adaptation of the proof proposed by *Ramanujan* in his papers [2] and [7, pp. 201–213], as mentioned in *Berndt*'s book [8, pp. 31].

We shall be needing the following result in order to understand the proof.

Theorem 1.4.1: *(Jacobi's Identity)*

$$\sum_{n=0}^{\infty} (-1)^n (2n+1) q^{n(n+1)/2} = (q;q)_\infty^3 \quad (1.16)$$

A priori from definition of *q-Series*, we obtain,

$$q(q;q)_\infty^4 . \frac{(q^5;q^5)_\infty}{(q;q)_\infty^5} = q . \frac{(q^5;q^5)_\infty}{(q;q)_\infty} = (q^5;q^5)_\infty \sum_{m=0}^{\infty} p(m) q^{m+1} \quad (1.17)$$

Hence, a simple application of *Binomial Theorem* yields,

$$\frac{(q;q)_\infty^5}{(q^5;q^5)_\infty} \equiv 1 (\mathrm{mod}\ 5) \quad (1.18)$$

From (1.17) and (1.18), we can conclude,

$$(q;q)_\infty^4 \equiv (q^5;q^5)_\infty \sum_{m=0}^{\infty} p(m)q^{m+1} (\text{mod } 5) \tag{1.19}$$

Thus it only suffices to show that, the co-efficients of q^{5n+5} on the L.H.S. of (1.19) are multiples of 5.

A priori from the statements of *Euler Pentagonal Theorem* (1.8) and *Jacobi's Identity* (cf. Th. (1.4.1)), (1.19) can also be interpreted as,

$$q(q;q)_\infty^4 = \sum_{j=-\infty}^{\infty} \sum_{k=0}^{\infty} (-1)^{j+k}(2k+1)q^{1+j\left(\frac{3j+1}{2}\right)+k\left(\frac{k+1}{2}\right)} \tag{1.20}$$

Our objetcive is to determine when the exponents on the R.H.S. are multiples of 5. Observe that,

$$2(j+1)^2 + (2k+1)^2 = 8\{1 + \tfrac{1}{2}j(3j+1) + \tfrac{1}{2}k(k+1)\} - 10j^2 - 5$$

Therefore, $\{1 + j\left(\frac{3j+1}{2}\right) + k\left(\frac{k+1}{2}\right)\}$ is a multiple of 5 *iff*,

$$2(j+1)^2 + (2k+1)^2 \equiv 0 (\text{mod } 5) \tag{1.21}$$

Equivalently,

$$2(j+1)^2 \equiv 0(\text{mod } 5) \text{ and}, (2k+1)^2 \equiv 0(\text{mod } 5)$$

In particular, we shall have, $(2k+1) \equiv 0(\text{mod } 5)$, which, by (1.20), further implies that, the co-efficient of q^{5n+5} (where, $n > 0$) in the term $q(q;q)_\infty^4$ is a multiple of 5. The co-efficient of q^{5n+5} on the R.H.S. of (1.19) is therefore, also a multiple of 5, i.e., $p(5n+4)$ is a multiple of 5, and our result is proved.

Remark 1.4.2: Andrews provided an alternative proof of the above result. [cf. [8, pp. 33]]. As for other versions of the problem, we can further study Ramanujan's work [2], as cited in [8, pp. 34] and also the work of Andrews in [8, pp. 102] applying Eisenstein Series and Generating Function for $p(n)$.

2. Combinatorial Interpretation for Ramanujan Congruences

2.1 *Rank of a Partition*

Now that, we have already discussed about the *partition congruences*, a more instinctive question which should arise is,

Is there any method to classify the partitions of integers of the form $(an+b)$ *in a number of groups ?*

Dyson [9] first gave an answer to this question, when he established some significant empirical combinatorial results in this context.

Definition 2.1.1: (Rank of a Partition) Given any $n \in \mathbb{N}$, and λn, suppose, we introduce the following notations :

1. $l(\lambda)$:= The largest part of λ.
2. $\#(\lambda)$:= Number of parts of λ.

Then, the Rank of λ, denoted by $r(\lambda)$, is defined as, $r(\lambda) := l(\lambda) - \#(\lambda)$.

The number of partitions of n ($\in \mathbb{N}$) with rank m is denoted by $N(m,n)$, and that with rank congruent to m modulo j is denoted by $N(m,j,n)$.

Example 2.1: Consider the following example for classifying partitions of 9 according to their respective ranks modulo 5, one can indeed derive the following table (1).

Table 1

Ranks of the Partitions of 9

Partition	Rank	Rank (mod 5)	Partition	Rank	Rank (mod 5)
9	$9-1=8$	3	$4+2+2+1$	$4-4=0$	0
$8+1$	$8-2=6$	1	$4+2+1_3$	-1	4
$7+2$	$7-2=5$	0	$4+1_5$	-2	3
$7+1+1$	$7-3=4$	4	3_3	0	0
$6+3$	$6-2=4$	4	$3+3+2+1$	-1	4
$6+2+1$	$6-3=3$	3	$3+3+1_3$	-2	3
$6+1_3$	$6-4=2$	2	$3+2_3$	-1	4
$5+4$	$5-2=3$	3	$3+2+2+1+1$	-2	3
$5+3+1$	$5-3=2$	2	$3+2+1_4$	-3	2
$5+2+2$	$5-3=2$	2	$3+1_6$	-4	1
$5+2+1+1$	$5-4=1$	1	2_4+1	-3	2
$4+4+1$	$4-3=1$	1	$2+2+1_5$	-5	0
$4+3+2$	$4-3=1$	1	$2+1_7$	-6	4
$4+3+1+1$	$4-4=0$	0	1_9	-8	2

Consequently, we obtain the following classification of partitions of 9 corresponding to their respective ranks provided in the table 2 below.

Table 2

Partitions of 9 with Rank $\equiv a \pmod 5$

$a=0$	$a=1$	$a=2$	$a=3$	$a=4$
$4+2+2+1$	$8+1$	$6+1_3$	9	$4+2+1_3$
$7+2$	$3+1_6$	$5+3+1$	$4+1_5$	$7+1+1$
3_3	$5+2+2+1$	$3+2+1_4$	$6+2+1$	$6+3$
$5+1_4$	2_3+1_3	$5+2+2$	$3+3+1_3$	$3+2+2+1$
$2+2+1_5$	$4+4+1$	2_4+1	$5+4$	$3+2+2+2$
$4+3+1+1$	$4+3+2$	1_9	$3+2+2+1+1$	$2+1_7$

Where, we followed the notation, $a_b := \underbrace{a+a+\cdots a}_{\text{b}-times}$.

Furthermore, *Dyson* (1944) conjectured the following.

Theorem 2.1.2: *The following holds true.*

$$N(m,5,5n+4)=\frac{1}{5}p(5n+4), 0\leq m\leq 4 \tag{2.1}$$

$$N(m,7,7n+5)=\frac{1}{7}p(7n+5), 0\leq m\leq 6 \tag{2.2}$$

Remark 2.1.3: One can in fact verify the result (2.1) from the table (2) in the example (2.1.1).

Important to note that, these conjectures were later derived by *Atkin* and *Swinnerton-Dyer* [12]. Although their proof is analytic, and relies significantly on the properties of *modular functions*, we must admit that, no such combinatorial proof is known as such. All we can comment in combinatorial context about the rank are the following.

Theorem 2.1.4: *The following holds for every chosen positive integer* n .

1. $N(m,n) = N(-m,n)$
2. $N(m,j,n) = N(j-m,j,n)$
3. $N(m,j,n) = \sum_{r=-\infty}^{\infty} N(m+rq,n)$

Proof: (1) follows from the fact that, the operation of conjugation reverses the sign of the rank. A trivial consequence of the above argument helps us conclude (2). A priori, combining (1) and (2), we obtain (3).

Similar concepts were incorporated by *Atkin* and *Hussain* [13] to study the case when $a = 11$, and, *O'Brien* 1965 [14] too opted for the same technique to evaluate the case, when $a = 13$.

2.2 *Generalization of Dyson's Rank*

Atkin [15] generalized the definition (2.1.1) of rank provided by *Dyson*. Geometrically, given any $n \in \mathbb{N}$, we can represent any partition of n as a set of *nested right angles* of nodes.

For example, the partition, $5+4+3+3+116$ has the following representation by 3 number of right angles. (These specific diagrams are also known as *Ferrers Graph* (cf. Fig. (1))).

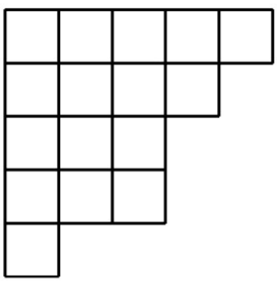

Figure 1
Ferrers Graph

2.3 *Successive Ranks of a Partition*

Given a partition λ, suppose $d_i(\lambda)$ denotes the number of *nodes* in the horizontal section of the i^{th} right angle in the graph of λ subtracted by the number of nodes in the vertical section of the same.

Thus, *Dyson's rank* is equal to $d_1(\lambda)$ and, $d_i(\lambda) = 0$, if λ doesn't have an i^{th} right angle. These $d_i(\lambda)$'s are termed as the **Successive Ranks** of λ. *Atkin* [15] provided alternate combinatorial interpretations analogous to (2.1) and (2.2) respectively.

We define $N^*(m,n)$ to be the number of partitions λ of any chosen n ($\in \mathbb{N}$) such that, $d_1(\lambda) - 2d_2(\lambda) = m$. Similarly, denote λn such that, $d_1(\lambda) - 2d_2(\lambda) \equiv m(\bmod j)$ and define it by $N^*(m,j,n)$. *Atkin* asserted that,

Theorem 2.3.1:

$$N^*(m,5,5n+4) = \frac{1}{5}p(5n+4), 0 \leq m \leq 4 \tag{2.3}$$

$$N^*(m,7,7n+5) = \frac{1}{7}p(7n+5), 0 \leq m \leq 6 \tag{2.4}$$

Moreover, given an i-conjugacy operation (denoted as C_i) acting on partitions satisfying, $d_1(C_i\lambda) = d_1(\lambda) - 2d_i(\lambda)$, we can therefore conclude that, $N^*(m,n) = N(m,n)$. Later on, *Andrews* extensively studied *Atkin's Successive Ranks*. [16, § 9.3]

Remark 2.3.2: A priori from the above identity, one can observe that, the statements (2.3) and (2.4) described in the Theorem(2.3.1) can be rendered as a trivial consequence of (2.1) and (2.2) respectively.

2.4 *Cranks of a Partition*

While rank explained Ramanujan's congruences modulo 5 and 7, it failed to do the same for modulo 11. Dyson conjectured another statistic, later termed the **crank**. Given any partition λn be any partition of some positive integer n, let us denote, $l(\lambda), \omega(\lambda)$ and $\mu(\lambda)$ to be the largest part of λ, number of ones in λ and number of parts of λ (strictly) larger than $\omega(\lambda)$ respectively. The **crank** of λ (denoted by $c(\lambda)$) has the following definition :

$$c(\lambda) := \begin{cases} l(\lambda) & \text{if, } \omega(\lambda) = 0 \\ \mu(\lambda) - \omega(\lambda) & \text{if, } \omega(\lambda) > 0. \end{cases} \tag{2.5}$$

If we denote the number of partitions of any positive integer n with *crank* congruent to m mod$j$ as $M(m,j,n)$. Then, $M(m,j,n) = M(j-m,j,n)$. *Dyson* (1944) [9] further conjectured the following.

$$M(m,11,11n+6) = \frac{1}{11}p(11n+6), \forall\ 0 \leq m \leq 10 \tag{2.6}$$

2.5 *Construction of Cranks relative to Vector Partitions*

Andrews and *Garvan* [10] [11] derived an explicit method in order to compute cranks relative to *vector partitions*.

For any partition λn, define, $\#(\lambda)$ to be the number of parts of λ and $\sigma(\lambda)$ to be the sum of parts of λ (or, the number 'λ' is partitioning). As per convention, us assume that, $\#(\phi) = \sigma(\phi) = 0$ for empty partition ϕ of 0. Furthermore, we denote,

$V := \{(\lambda_1, \lambda_2, \lambda_3) | \lambda_1$ is a partition into distinct parts, λ_2, λ_3 are unrestricted partitions$\}$

We define elements of V to be the *vector partitions*. For every $\lambda \in V$, suppose, $\mathfrak{S}_V, \mathfrak{W}_V$ and $\mathfrak{R}$ denotes the sum of parts, weight and crank respectively, hence,

$$\mathfrak{S}(\overleftarrow{\lambda}) := \sigma(\lambda_1) + \sigma(\lambda_2) + \sigma(\lambda_3). \quad (2.7)$$

$$\mathfrak{W}(\overleftarrow{\lambda}) := (-1)^{\#(\lambda_1)}. \quad (2.8)$$

$$\mathfrak{R}(\overleftarrow{\lambda}) := \#(\lambda_2) - \#(\lambda_3). \quad (2.9)$$

Here, $\overleftarrow{\lambda}$ is defined to be a *vector partition* of a positive integer n provided, $\mathfrak{S}(\overleftarrow{\lambda}) = n$.

Example 2.5.1: Consider, $\overleftarrow{\lambda} = (5+3+2,2+2+1,2+1+1)$. Then, $\mathfrak{S}(\overleftarrow{\lambda}) = 19, \mathfrak{W}(\overleftarrow{\lambda}) = -1$, $\mathfrak{R}(\overleftarrow{\lambda}) = 0$. Consequently, $\overleftarrow{\lambda}$ is a vector partition of 19.

Suppose, $N_V(m,n)$ denotes the number of vector partitions of n (counted in accordance to the weight $\mathfrak{W}$) having *crank* $= m$. Moreover, assuming $N_V(m,j,n)$ yields the number of vector partitions of n (counting in terms of the weight $\mathfrak{W}$) with *crank* congruent to m j, the following holds true.

$$N_V(m,j,n) = \sum_{k=-\infty}^{\infty} N_V(kj+m,n) = \sum_{\substack{\overleftarrow{\lambda} \\ \mathfrak{S}(\overleftarrow{\lambda})=n \\ \mathfrak{R}(\overleftarrow{\lambda})=m (\mathrm{mod}\ j)}} \mathfrak{W}(\overleftarrow{\lambda})$$

An application of transformation formula (interchanging the roles of λ_2 and λ_3) helps us assert that, $N_V(m,n) = N_V(-m,n)$. Subsequently, $N_V(j-m,j,n) = N_V(m,j,n)$.

Lemma 2.1: *The following identity does indeed illustrate a generating function for* $N_V(m,n)$.

$$\sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} N_V(m,n) z^m q^n = \prod_{n=1}^{\infty} \frac{(1-q^n)}{(1-zq^n)(1-z^{-1}q^n)} = \frac{(q;q)_\infty}{(zq;q)_\infty \left(\frac{q}{z};q\right)_\infty} \quad (2.10)$$

Whenever $z \neq 0$, $|q| < 1$ and, $|z| < \frac{1}{|q|}$.

Remark 2.5.2: Putting $z = 1$ in (2.10) gives,

$$\sum_{m=-\infty}^{\infty} N_V(m,n) = p(n). \quad (2.11)$$

Apparently, we can thus derive an important relation regarding cranks of vector partitions and the partition function corresponding to integers of the form $5n+4$, $7n+5$ and $11n+6$.

$$N_V(m,5,5n+4) = \frac{1}{5} p(5n+4) \quad (2.12)$$

$$N_V(m,7,7n+5)=\frac{1}{7}p(7n+5) \tag{2.13}$$

$$N_V(m,11,11n+6)=\frac{1}{11}p(11n+6) \tag{2.14}$$

One may subsequently pose this question :

Does there exists a crank for every ordinary partition ?

Not only the answer is indeed **"yes"**, but also, for every partition λn, given some $n \in \mathbb{N}$ and with $c(\lambda)=m$, $N_V(m,n)$ is in fact equal to the number of partitions λn with crank $=m$, denoted by $M(m,n)$. In other words, $N_V(m,n)=M(m,n)$.

Remark 2.5.3: Ardent readers can read [11, pp. 52-53] as reference for further explicit illustrations regarding Crank of Vector partitions.

2.6 *An Important Property of* $M(m,j,n)$

Given $n \in \mathbb{N}$, we fix $j \in \mathbb{N}$. Then, we intend to establish the following relation involving the number of partitions of any positive integer n with *crank* congruent to m mod$j$, usually denoted as $M(m,j,n)$.

$$M(m,j,n)=M(-m,j,n) \text{ for every } m=0,1,2,\cdots,j-1. \tag{2.15}$$

We provide a short proof of (2.15) for better understanding of our readers. A priori from the previous literature, we can deduce that, $N_V(m,n)=N_V(-m,n)$ and, $N_V(m,j,n)=N_V(-m,j,n)$ for every $n \in \mathbb{N}$. Observe that, the case : $n=1$ and $j=1$ holds trivially, as,

$$M(m,j,n)=M(0,1,1)=1=M(-0,1,1)=M(-m,j,n).$$

Consider the case when, $n>1$ and $j>1$.

We can deduce,

$$M(m,j,n)=\sum_{k\in\mathbb{Z}} N_V(jk+m,n)=\sum_{k\in\mathbb{Z}} N_V(jk-m,n)=M(-m,j,n).$$

3. Interpretation of the Partition Congruences mod 5, 7 and 11 in terms of their corresponding Cranks

3.1 *The Main Result*

We aim to establish the following relations involving partition congruences mod 5, 7 and 11 in terms of partitions of any positive integer n with crank congruent to m mod j, denoted by $M(m,j,n)$.

Theorem 3.1.1:

$$M(j,5,5n+4)=\frac{1}{5}p(5n+4),\ 0\le j\le 4 \tag{3.1}$$

$$M(j,7,7n+5)=\frac{1}{7}p(7n+5), 0\le j\le 6 \tag{3.2}$$

$$M(j,11,11n+6)=\frac{1}{11}p(11n+6), 0\le j\le 10 \tag{3.3}$$

Given a prime number s and r_s be the reciprocal of 24 modulo s, *Garvan* [11] made a significant observation by representing the identities in a more compact form.

$$N_V(j,s,sn+r_s)=\frac{1}{s}p(sn+r_s),0\leq j\leq s-1, \text{ iff}, a_{sn+r_s}=0. \tag{3.4}$$

where, $\sum_{n\geq 0} a_n q^n=\frac{(q;q)_\infty}{(\zeta_s q;q)_\infty(\zeta_s^{-1}q;q)_\infty}$, ζ_s being the primitive s^{th} root of unity.

Proposition 3.1.2: Suppose, s be a prime, proving the statement (3.4) is equivalent to estimating the co-efficient of q^{sn+r_s} in the product,

$$\prod_{n=1}^{\infty}\frac{(1-q^n)}{(1-\zeta_s q^n)(1-\zeta_s^{-1}q^n)} \tag{3.15}$$

to be equal to 0, *where,* $\zeta_s=exp(2\pi i/s)$ *is the primitive* s^{th} *root of unity.*

Proof: We rewrite (3.5) in terms of $N_V(k,s,n)$. Replacing $z=\zeta_s$ into L.H.S. of (2.10),

$$\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty} N_V(m,n)\zeta_s^m q^n=\sum_{k=0}^{s-1}\zeta_s^k\sum_{n=0}^{\infty}\left(\sum_{\substack{m\equiv k\\ (\text{mod } s)}} N_V(m,n)\right) q^n=\sum_{k=0}^{s-1}\zeta_s^k\sum_{n=0}^{\infty}N_V(k,s,n)q^n$$

Therefore,

$$\frac{(q)_\infty}{(\zeta_s q)_\infty(\zeta_s^{-1}q)_\infty}=\sum_{k=0}^{s-1}\zeta_s^k\sum_{n=0}^{\infty}N_V(k,s,n)q^n \tag{3.6}$$

Important to note that, $\sum_{k=0}^{s-1}N_V(k,s,sn+r_s)\zeta_s^k$ equals to the co-efficient of q^{sn+r_s} in the expression on the L.H.S. of (3.6), i.e., $\sum_{k=0}^{s-1}N_V(k,s,sn+r_s)\zeta_s^k=N_V(0,s,sn+r_s)\sum_{k=0}^{s-1}\zeta_s^k=0$, as expected.

Conversely, assuming that, the co-efficient of q^{sn+r_s} in L.H.S. of (3.6) is 0, it implies,

$$\sum_{k=0}^{s-1}N_V(k,s,sn+r_s)\zeta_s^k=0 \tag{3.7}$$

Note that, L.H.S. of (3.7) is indeed a polynomial in ζ_s over $\mathbb{Z}$. Subsequently,

$$N_V(j,s,ns+r_s)=N_V(j+1,s,ns+r_s)\ \forall\, j=0,1,2,\cdots,s-2. \tag{3.8}$$

A priori s being a prime and the minimal polynomial for ζ_s over $\mathbb{Q}$ being the *cyclotomic polynomial* $\Phi_s(x)$, thus combining (2.11) and (3.8), we obtain, $p(sn+r_s)=\sum_{k=0}^{s-1}N_V(k,s,sn+r_s)=sN_V(0,s,sn+r_s)$, and the proof is complete.

In the subsequent subsections, we shall establish each of the identities stated in (3.1), (3.2) and (3.3) in order to justify our claim.

3.2 *Special Case when* $s=5$

It only suffices to show that, $a_{5n+4}=0$. Assume ζ_5 be the primitive 5^{th} root of unity. By the *Euler Pentagonal Theorem*, it can be deduced that,

$$\sum_{n\geq 0} a_n q^n = \frac{(q;q)_\infty}{(\zeta_5 q;q)_\infty(\zeta^{-1}q;q)_\infty} = \frac{\sum_{n\in\mathbb{Z}}\sum_{m\geq 0}(-1)^{n+m} q^{\{\frac{n(3n-1)}{2}+\frac{m(m+1)}{2}\}}\zeta_5^{-2m}\left(1-\zeta_5^{2(2m+1)}\right)}{(q^5;q^5)_\infty(1-\zeta_5^2)}$$

Since, $\frac{n(3n-1)}{2} \equiv 0,1,2 \pmod 5$ and, $\frac{m(m+1)}{2} \equiv 0,1,3 \pmod 5$, the power of q is *congruent* to 4 mod 5 only when, $m \equiv 2$ (mod 5), and so, $2m+1 \equiv 0$ (mod 5). Therefore, $((\zeta_5^2)^{2m+1}-1)=0$ implying, $a_{5n+4}=0$. On the other hand, the relation, $N_V(m,j,n) = M(m,j,n)$ yields,

$$N_V(i,5,5n+4) = \frac{1}{5}p(5n+4) = M(i,5,5n+4), \ \ 0 \leq i \leq 4$$

as desired.

3.3 *Special Case when* $s = 7$

It only suffices to show that, $a_{7n+5}=0$. Suppose, ζ_7 be the primitive 7^{th} root of unity. A priori implementing similar arguments as in section 3.1,

$$\sum_{n\geq 0} a_n q^n = \frac{\sum_{n,m\geq 0}(-1)^{n+m} q^{\{\frac{n(n+1)}{2}+\frac{m(m+1)}{2}\}}\zeta_7^{-2n-3m}\left(1-\zeta_7^{2(2n+1)}\right)\left(1-\zeta_7^{3(2m+1)}\right)}{(q^7;q^7)_\infty(1-\zeta_7^5)(1-\zeta_7^3)} \quad (3.9)$$

Since, $\frac{n(n+1)}{2} \equiv 0,1,3,6 \pmod 7$, implies that, the power of q in R.H.S. of (3.9) is in fact congruent to 5 modulo 7 provided, $n \equiv m \equiv 3 \pmod 7 2m + 1 \equiv 0 \pmod 7$, and, $2n+1 \equiv 0 \pmod 7$. As a consequence, $((\zeta_7^2)^{2n+1}-1) = 0$, and, $((\zeta_7^3)^{2m+1}-1)=0$. Thus, $a_{7n+5}=0$. In conclusion,

$$N_V(i,7,7n+5) = \frac{1}{7}p(7n+5) = M(i,7,7n+5), 0 \leq i \leq 6$$

as required.

3.4 *Special Case when* $s = 11$

We are requiredto establish, $a_{11n+6}=0$. Assuming ζ_{11} be the primitive 11^{th} root of unity, the proof shall follow the *Winquist*'s Identity [7].

Theorem 3.4.1: *(Winquist's Identity)*

$$(q;q)_\infty^2(y;q)_\infty\left(\frac{q}{y};q\right)_\infty(z;q)_\infty\left(\frac{q}{z};q\right)_\infty\left(\frac{y}{z};q\right)_\infty\left(\frac{zq}{y};q\right)_\infty(yz;q)_\infty\left(\frac{q}{yz};q\right)_\infty =$$
$$\sum_{i\geq 0}\sum_{j\in\mathbb{Z}}(-1)^{i+j}\left((y^{-3i}-y^{3i+3})(z^{-3j}-z^{3j+1})+(y^{-3j+1}-y^{3j+2})(z^{3i+2}-z^{-3i-1})\right)q^{\left\{\frac{3i(i+1)}{2}+\frac{j(3j+1)}{2}\right\}}$$

Replacing y by ζ_{11}^9 and z by ζ_{11}^5 in (3.4.1) yields,

$$\sum_{n\geq 0} a_n q^n = \frac{\sum_{i\geq 0}\sum_{j\in\mathbb{Z}}(-1)^{i+j}\left((\zeta_{11}^{6i}-\zeta_{11}^{5i+5})(\zeta_{11}^{7j}-\zeta_{11}^{4j+5})+(\zeta_{11}^{6j+9}-\zeta_{11}^{5j+7})(\zeta_{11}^{4i+10}-\zeta_{11}^{7i+6})\right)q^{\left\{\frac{3i(i+1)}{2}+\frac{j(3j+1)}{2}\right\}}}{(1-\zeta_{11}^3)(1-\zeta_{11}^4)(1-\zeta_{11}^5)(1-\zeta_{11}^9)(q^{11};q^{11})_\infty}$$

Observe that, $\frac{3i(i+1)}{2} \equiv 0,1,3,7,8,9 \pmod{11}$ and, $\frac{j(3j+1)}{2} \equiv 0,1,2,4,5,7$ (mod 11), implying that the power of q is indeed congruent to 6 modulo 11 provided, $i \equiv 5 \pmod{11}$ and, $j \equiv 9 \pmod{11}$.

Therefore, $\left(\zeta_{11}^{6i}-\zeta_{11}^{5i+5}\right)=0$, and, $\left(\zeta_{11}^{6j+9}-\zeta_{11}^{5j+7}\right)=0$. Subsequently, $a_{11n+6}=0$. Accordingly,

$$N_V(i,11,11n+6)=\frac{1}{11}p(11n+6)=M(i,11,11n+6), 0\le i\le 10.$$

Remark 3.4.2: We can in fact, infer from the results proven in the above three special cases in sections 3.2, 3.3 and, 3.4 that, the Crank does indeed provides an explicit and thorough explanation to the Partition Congruences modulo 5, 7 and 11.

4. Conclusion : Future Scope for Research

In this article, we have provided an explicit combinatorial interpretation of Ramanujan's Partition Congruences, specifically for the cases $p(5n+4)\equiv 0(\mathrm{mod}\ 5)$, $p(7n+5)\equiv 0(\mathrm{mod}\ 7)$, and $p(11n+6)\equiv 0(\mathrm{mod}\ 11)$. By employing the concepts and techniques developed by notable mathematicians such as *Dyson* [9], *Atkin* [6], [12], [20], *Swinnerton-Dyer* [12], *Andrews* [16], [18], [19], [21], [22], [23], and *Garvan* [5], [10], [11], [21], we have illuminated the relationship between these congruences and their corresponding cranks.

Our study of Partitions, q-Series, and Ramanujanâ€™s Theta Function lays a strong foundation for understanding the structure and behavior of partition functions. Using the *Jacobi Triple Product Identity* and the *Euler Pentagonal Theorem*, we provided an approximate estimate for $p(n)$, with particular emphasis on the case $n\equiv 4(\mathrm{mod}\ 5)$.

We explored the roles of *Rank* and *Crank*, highlighting *Dyson*’s pivotal contributions. The main result—centered around the partition function $M(m,j,n)$, which counts partitions with crank congruent to $m \mathrm{mod} j$ offers a powerful framework for analyzing partition congruences.

While our focus has been on congruences modulo 5, 7, and 11, the methods employed can be extended to other moduli, potentially revealing deeper combinatorial patterns. Future directions include generalizing rank and crank to *overpartitions* and *multipartitions*, and applying more refined analytical and computational techniques to improve estimates of $p(n)$.

Finally, connecting partition theory with other domains like *algebraic geometry*, *modular forms*, and *number theory* may lead to fruitful interdisciplinary developments. In sum, this work contributes to the combinatorial understanding of Ramanujanâ€™s Congruences and sets the stage for further exploration in this rich mathematical domain.

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