

Some remarks on the index of a broom and equitable partitions

Carlos M. Da Fonseca [§]

Kuwait College of Science and Technology

Doha District

Safat 13133

Kuwait

and

Faculty of Applied Mathematics and Informatics

Technical University of Sofia

Kliment Ohridski Blvd. 8

1000 Sofia

Bulgaria

Paulo Saraiva [†]

Faculty of Economics

University of Coimbra

Av. Dias da Silva, 165

3004-512 Coimbra

Portugal

and

Department of Mathematics

Centre for Mathematics

University of Coimbra

3000-143 Coimbra

Portugal

and

[§] ORCID-ID: 0000-0001-7742-4416

[†] ORCID-ID: 0000-0002-2856-5033

[§] E-mail: c.dafonseca@kcst.edu.kw

[†] E-mail: psaraiva@fe.uc.pt

Centre for Business and Economics Research
 Faculty of Economics
 University of Coimbra
 Av. Dias da Silva, 165
 3004-512 Coimbra
 Portugal

Fatih Yılmaz *
 Department of Mathematics
 Ankara Hacı Bayram Veli University
 Ankara
 Turkey

Abstract

In this paper, we present an alternative and short proof for the expression of the index of a broom, based on equitable partitions and perturbed Toeplitz tridiagonal matrices. A similar general result for a double-broom is established.

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1. Introduction

The $(0,1)$ -adjacency matrix $A = (a_{ij})$ of order n of a simple graph G having vertex-set $V(G) = \{1, 2, \dots, n\}$ is defined as $a_{ij} = 1$ if and only if (i, j) is an edge in G ([8]). The eigenvalues of the adjacency matrix are real and are called the eigenvalues of the graph. The *nullity* of G , denoted by $\eta(G)$, is the multiplicity of the eigenvalue zero in the spectrum of its adjacency matrix. The *index* of G refers to its largest eigenvalue.

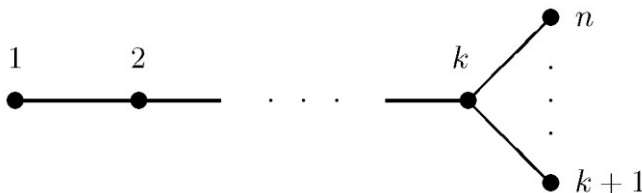


Figure 1
 $B_{n,k}$

* ORCID-ID: 0000-0001-7873-1979

* E-mail: fatih.yilmaz@hbu.edu.tr (Corresponding author)

A *broom* is a tree formed by attaching one or more pendant vertices to a pendant vertex of a path. This is a widely studied graph and we can find instances or general cases of these trees in [3, 17, 22], just to cite a few examples. Our motivation starts with the recent article [13] in which, according to the authors, a new formula for the index of the broom with n vertices $B_{n,k}$ is derived (see Figure 1).

Namely, the authors prove the following result.

Theorem 1: ([13]).1 *The index of $B_{n,k}$ is*

$$\lambda_n(k) = \sqrt{x_n(k)} + \frac{1}{\sqrt{x_n(k)}},$$

where $x_n(k)$ is a solution to the equation

$$x^{k+1} = \frac{(n-k-1)x-1}{(n-k-1)-x}$$

in the interval $(1, n - k - 1)$.

Originally, Theorem 1 also considered the monotonicity, uniqueness, and location of the index. However, as the authors mentioned, these are known in the literature (see, for example, [9, 22]). Here we will only focus on the formula for the eigenvalues.

In the next section, we will present the three main basic techniques that will be used in our approach. As a result, the formula for the index follows. Subsequently, we derive a general formula for the characteristic polynomial of a double-broom graph using our proposed approach. In the last section we construct two trees with the same index.

2. Main tools

In this section, we describe the main tools for our approach.

2.1 Equitable partitions

Let G be a simple graph. A partition $V(G) = V_1 \sqcup V_2 \sqcup \dots \sqcup V_k$ is called *equitable* if, for every $i, j \in \{1, 2, \dots, k\}$, each vertex in V_i has the same number of neighbours in V_j . Furthermore, we consider the weighted digraph with vertex set $\{V_1, V_2, \dots, V_k\}$, where the weight of the arc (i, j) is the number b_{ij} of neighbours each vertex of V_i has in V_j . This digraph is called a *divisor* of G and its $k \times k$ (weighted) adjacency matrix is called a *divisor matrix*.

For example, an equitable partition for $B_{n,k}$ is

$$\Pi : V_1 = \{1\}, \dots, V_k = \{k\}, V_{k+1} = \{k+1, \dots, n\},$$

with the associated divisor $(k+1) \times (k+1)$ matrix

$$D_\Pi = \begin{pmatrix} 0 & 1 & & & \\ 1 & \ddots & \ddots & & \\ & \ddots & \ddots & 1 & \\ & & 1 & 0 & n-k \\ & & & 1 & 0 \end{pmatrix}. \quad (1)$$

The concept of the divisor or quotient matrix (referred to as *Teiler*) was introduced in the late 1960s in two foundational papers by German mathematician Horst Sachs, who was awarded the Euler Medal in 2000 [18, 19] (see also [5]). Among its many notable properties, the characteristic polynomial of a divisor matrix is known to divide the characteristic polynomial of the graph. [7, Theorem 3.9.5]. This means, for example, that an eigenvalue of a divisor matrix is also an eigenvalue of the original graph [7, Theorem 3.9.5]. Of course, the multiplicity of any eigenvalue of the divisor is less than or equal to its multiplicity as eigenvalue of the graph. Furthermore, the index (i.e., the largest eigenvalue) of a graph is an eigenvalue of any of its divisors (cf. [7, Corollary 3.9.11]).

2.2 Nullity of a tree

A matching of a graph G is a subset of edges without common vertices in G . The matching number $\nu = \nu(G)$ of a graph G is the size of a largest matching, that is, a matching containing the maximum possible number of edges.

Lemma 1: ([6, 12, 20]) *If T is an n -vertex tree with a matching number ν , then its nullity is equal to*

$$\eta = n - 2\nu.$$

2.3 Perturbed tridiagonal Toeplitz matrices

The spectral analysis of tridiagonal Toeplitz matrices is a subject of study in distinct areas with numerous applications in pure and applied sciences. In particular, those matrices with perturbations on the most extreme entries are relevant in many instances. For a brief introduction to this topic, the reader is referred to [4, 10, 11, 14, 21, 23] for results on general perturbations. Among all the references just mentioned, perhaps the most comprehensive one that encompasses all possible cases is [23]. As applications to arrays of linear dynamical systems with nearest-neighbour coupling and to discretizations of second-order linear partial differential equations, the authors considered the $(n + 1) \times (n + 1)$ matrix

$$A_{n+1} = \begin{pmatrix} -b_0 & 1 - b_1 & & & \\ 1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 0 & 1 \\ & & & 1 - c_{-1} & -c_0 \end{pmatrix}.$$

The order of the matrix and the entries are set for convenience as we will see next. The function h defined by

$$\begin{aligned} h(x) = & x^{2n}(x^2 + b_0x + b_1)(x^2 + c_0x + c_{-1}) \\ & - (b_1x^2 + b_0x + 1)(c_{-1}x^2 + c_0x + 1) \end{aligned} \quad (2)$$

is a polynomial of degree $2n + 4$. It is not difficult to see that the set of roots of h remains unchanged by the reciprocal transformation $x \rightarrow x^{-1}$. Having said that, we have an elegant form for the spectrum of A_{n+1} .

Theorem 2 ([23]). *Let S be the set of roots of the rational function defined by*

$$\frac{h(x)}{x^2-1}.$$

Then, the eigenvalues of A_{n+1} are

$$x + \frac{1}{x}, \text{ for } x \text{ in } S. \quad (3)$$

We can clearly feel in (3) a taste of the Joukowski transform used in applied mathematics, namely in aerodynamics.

3. The spectrum of a broom

The spectrum of the broom $B_{n,k}$ and, in particular, its index follow now from the previous section. In fact, taking the divisor matrix D_Π defined in (1), from (2) we get

$$h(x) = x^{2k+2}(x^2 + 1 + k - n) - ((1 + k - n)x^2 + 1)$$

and from Theorem 2, we obtain all nonzero eigenvalues of $B_{n,k}$. All the others are zero by Lemma 1. Therefore, in particular, we have Theorem 1. The case where $k = n - 2$ was recently considered in [3] in a particular example.

Moreover, since the matching number of $B_{n,k}$ is $\nu = \lfloor n/2 \rfloor$, the nullity is $n - k$, if k is even and $n - k - 1$, otherwise. The eigenvalues of D_Π defined in (1) are all simple since it is a tridiagonal matrix with nonnegative entries. We conclude that they are all simple eigenvalues of $B_{n,k}$, possibly plus zero, if k is odd. This conclusion is in accordance with [22, (5)].

4. Double brooms

Among the different ways to generalize a broom, we have the acyclic double brooms. In [17], the authors analysed the index of such trees. The main technique for determining the spectra is based on the powerful algorithm by Jacobs and Trevisan [15]. Here we discuss their spectra. The reader is also referred to [16] for some co-spectral discussions. Our approach relies once again on equitable partitions.

A double broom $D(p, r, q)$ is a tree formed by attaching p leaves to one end-vertex of a path P_r and q leaves to the other end-vertex (see Figure 2).

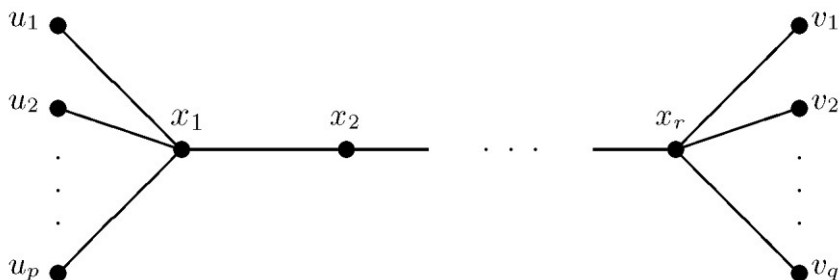


Figure 2
 $D(p, r, q)$

In this case, we consider the equitable partition

$$V_1 = \{u_1, \dots, u_p\}, V_2 = \{x_1\}, \dots, V_{r+1} = \{x_r\}, V_{r+2} = \{v_1, \dots, v_q\}.$$

In this case, the divisor matrix is

$$A_{r+2} = \begin{pmatrix} 0 & 1 & & & & & \\ p & 0 & 1 & & & & \\ & 1 & \ddots & \ddots & & & \\ & & \ddots & \ddots & 1 & & \\ & & & 1 & 0 & q & \\ & & & & 1 & 0 & \end{pmatrix}.$$

Therefore, the divisor matrix provides all non-zero eigenvalues of the double broom and they are simple. They can all be obtained from Theorem 2. If r is odd, the nullity is $p + q + 1$ and 0 is also an eigenvalue of A_{r+2} . Otherwise, the nullity is $p + q$ and 0 of course is not an eigenvalue of A_{r+2} . For a different approach, where the characteristic polynomial is explicitly calculated, the reader is referred to [22, Theorem 2].

Another particular case is obtained when $r = 2$. In this case, the nullity is $n - 4$ and the corresponding divisor matrix is

$$A_4 = \begin{pmatrix} 0 & 1 & & \\ p & 0 & 1 & \\ & 1 & 0 & q \\ & & 1 & 0 \end{pmatrix}.$$

This means that the nonzero eigenvalues are the roots of the polynomial

$$p(x) = x^4 - (p + q + 1)x^2 + pq,$$

which are

$$\pm \sqrt{\frac{1}{2}(p + q + 1 \pm \sqrt{(p + q + 1)^2 - 4pq})}.$$

In this case, if we take both $\pm$ positive, we get the index claimed recently in [2]. There, we can also find the tree designated by $S(p, q, k, 1)$ (see Figure 3).

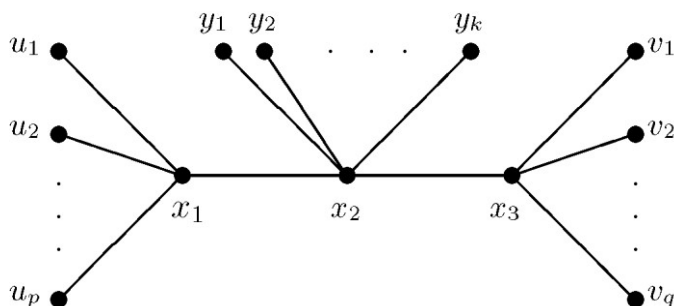


Figure 3
 $S(p, q, k, 1)$

This tree has nullity $n - 6$ and the nonzero eigenvalues from those of the divisor matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ p & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & k \\ 0 & 0 & 1 & 0 & q & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

corresponding to the equitable partition

$$V_1 = \{u_1, \dots, u_p\}, V_2 = \{x_1\}, V_3 = \{x_2\}, V_4 = \{x_3\},$$

$$V_5 = \{v_1, \dots, v_q\}, V_6 = \{y_1, \dots, y_k\}.$$

Basically, it corresponds to the characteristic polynomial in [2 (5)].

5. Families of trees with same index

In the note [2] the authors established necessary and sufficient conditions for the trees $D(p, r, q)$ and $S(p, q, k, 1)$ defined in the previous section have the same index. Using the techniques described before, we establish an analogous result for a distinct family of trees.

Let us consider a star S_{r+1} and to each pendant vertex we add the same number k of vertices. We denote such tree by $S_{r+1,s}$ and its order is $rs + 1$. From Theorem 1 the nullity for such tree is $s + 1$. Let c be the central vertex of the star S_{r+1} and $u_1, \dots, u_r$ be its pendant vertices. Now, let $v_1, \dots, v_s$ be a set of vertices attached to the same pendant vertex of S_{r+1} . Then

$$V_1 = \{c\}, V_2 = \{u_1, \dots, u_r\}, V_3 = \{v_1, \dots, v_s\}$$

is an equitable partition of $S_{r+1,s}$. Therefore, a divisor matrix of $S_{r+1,s}$ is

$$\begin{pmatrix} 0 & r & 0 \\ 1 & 0 & s \\ 0 & 1 & 0 \end{pmatrix}$$

with eigenvalues

$$-\sqrt{r+s}, 0, \sqrt{r+s}.$$

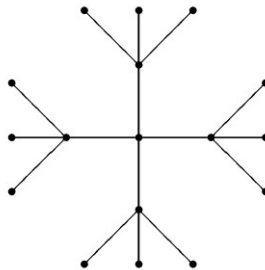


Figure 4

$S_{5,3}$

Therefore the index of $S_{r+1,s}$ is $\sqrt{r+s}$. Consequently, we may conclude the following:

Theorem 3: Let k, r, s be positive integers with $k \leq \min\{r+1, s\}$. Then $S_{r+1-k, s+k}$, $S_{r+1+k, s-k}$, and $S_{r+1, s}$ share the same index $\sqrt{r+s}$.

As an example, let us consider the tree $S_{5,3}$ as depicted in Figure 4.

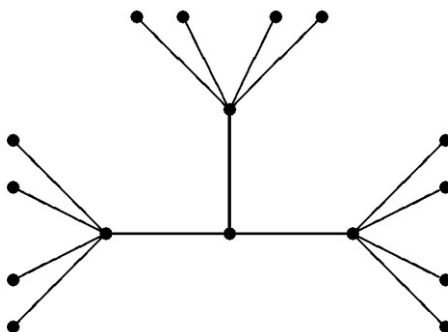


Figure 5
 $S_{4,4}$

Its index is $\sqrt{7}$ the same as, for instance, $S_{4,4}$ (see Figure 5) or even the star S_8 .

In fact, these trees can be found in distinct settings throughout the literature. They are highly symmetric and for more details and general results the reader is referred to the survey [1].

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