

Secure domination of mesh and pyramid networks

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Abstract

Given a dominating set $S \subseteq V(G)$ for a graph G , each vertex of the graph is either in S or is adjacent to at least one vertex in S . The set S is a secure dominating set of G if, for each vertex $u \in V(G) \setminus S$, there exists a vertex $w \in S$ adjacent to u such that $(S \setminus \{w\}) \cup \{u\}$ is also a dominating set of G . The minimum cardinality of a secure dominating set in G determines the secure domination number of G . A prior investigation has established an upper bound for the secure domination number of $P_p \times P_q$. Also, the exact values of the secure domination numbers of $P_2 \times P_n$ and $P_3 \times P_n$ have been reported in the literature. We present an improved upper bound for the secure domination number of $P_n \times P_n$, when $n \geq 4$ and $n \equiv s \pmod{3}$, where $s \in \{1, 2\}$. Using the improved upper bound, we provide an upper bound for the secure domination number of pyramid networks.

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1. Introduction

Domination in graphs has wide applications and has been studied extensively as a research topic. In a graph G , a vertex b is known to be a *neighbor* of another vertex c if they are adjacent. A vertex in the *open neighborhood* of b is a vertex that is adjacent to it. The set is denoted by $N(b)$. In addition, the *closed neighborhood* of b consists of the vertices of $N(b) \cup \{b\}$. Consider S to be a dominating set of a graph G ; then $\bigcup_{u \in S} N[u] = V(G)$.

Cockayne et al. [1] introduced the concept of secure domination to identify an optimized set of locations for allocating defense units. Consider $W \subseteq V(G)$ and $w \in W$. The set $N[w] \setminus N[W \setminus \{w\}]$ is the *private neighborhood of w with respect to W* and is denoted by $pn(w, W)$. The set $epn(w, W) = pn(w, W) \setminus \{w\}$ is the *external private neighborhood of w with respect to W* . The set S is a *secure dominating set* (SDS) of G if each vertex $u \in V(G) \setminus S$ has a neighbor $w \in S$ such that $(S \setminus \{w\}) \cup \{u\}$ is a dominating set of G [1]. Here, u is *S -defended* by w . It can also be stated that w *S -defends* u . If the set S is known from the context, we say that u is *defended* by w , or that w *defends* u . If S has the minimum cardinality, then it is a γ_s -set of G , and its cardinality is the secure domination number of G , denoted by $\gamma_s(G)$. The complexity results related to secure domination are discussed in [2], [3]. If $G \neq C_5$ is connected with $|V(G)| = n$ and $\delta(G) \geq 2$, then $\gamma_s(G) \leq \frac{n}{2}$ [4]. Graphs with secure domination numbers 1, 2, and 3 are characterized in [5]. The effect of edge removal on $\gamma_s(G)$ is discussed in [6]. Several significant results on secure domination and certain variants can be found in [2], [7]-[13]. A dynamic domination problem in graphs is studied in [14]. Some variants of domination in certain Cartesian product graphs are studied in [15], [16].

The progression in the development of supercomputers is coupled with numerous computational experiments and continuous improvements in algorithm design. The use of mesh, and other grid structures in the topology of interconnection networks employed in parallel computing systems provides an efficient communication framework for deploying many algorithms [17]. The pyramid networks are useful structures in image processing and parallel computing [17], [18]. They are a combination of mesh and tree structures. Algorithms that leverage both connection types are demonstrated in [18].

2. Preliminaries

Definition 1: [17] A mesh network $G(p, q)$ has its topological structure given by $P_p \times P_q$ where P_p and P_q are undirected paths.

When $p = q = t$, the vertex set is defined to be the set $\{(a, b): 1 \leq a, b \leq t\}$ and two vertices (a_1, b_1) and (a_2, b_2) are adjacent if and only if $|a_1 - a_2| + |b_1 - b_2| = 1$.

Definition 2: [17] A d -dimensional pyramid network $PN(d)$ has its vertex set defined as:

$$V(PN(d)) = \{(a, b, f): 1 \leq a, b \leq 2^f, 0 \leq f \leq d\}$$

$V_f = \{(a, b, f) : 1 \leq a, b \leq 2^f\}$ for each fixed f ($0 \leq f \leq d$) denotes the set of vertices on level f . An induced subgraph with V_f as the vertex set is $G(2^f, 2^f)$. Each $(a, b, f) \in V_f$, $f < d$ has four neighbors in V_{f+1} : $(2a - 1, 2b, f + 1)$, $(2a, 2b, f + 1)$, $(2a - 1, 2b - 1, f + 1)$, $(2a, 2b - 1, f + 1)$. The resulting graph for $d = 2$ is shown in Figure 1.

We define $J_{(a,b,f)} = \{(a, b, f), (2a - 1, 2b, f + 1), (2a, 2b, f + 1), (2a - 1, 2b - 1, f + 1), (2a, 2b - 1, f + 1)\}$.

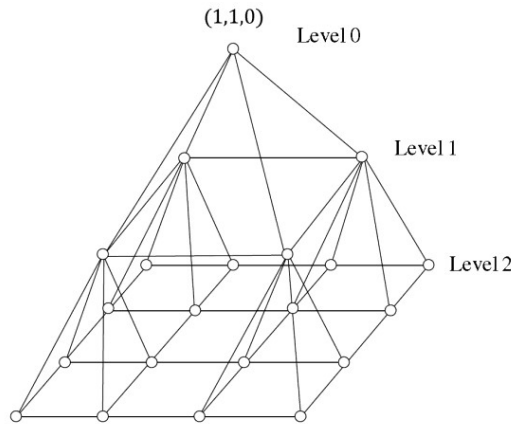


Figure 1
The pyramid network $PN(2)$

Theorem 3: [1] $\gamma_s(P_u \times P_v) \leq \left\lceil \frac{uv}{3} \right\rceil + 2$.

Theorem 4: [16] $\gamma_s(P_2 \times P_l) = \left\lceil \frac{3l+1}{4} \right\rceil$ for $l \geq 2$

Theorem 5: [16] For $b \geq 2$, $\gamma_s(P_3 \times P_b) = \begin{cases} b + 1, & b \leq 8 \text{ or } b = 10 \\ b + 2, & b = 9 \text{ or } b \geq 11 \end{cases}$

3. Main Results

An upper bound for $\gamma_s(P_t \times P_t)$ is established in the following theorem, for $t \geq 4$ and $t \equiv s \pmod{3}$, where $s \in \{1, 2\}$. The vertices of $P_t \times P_t$ are labeled as specified in Definition 1.

Theorem 6: If $t \geq 4$ and $t \equiv r \pmod{3}$, where $r \in \{1, 2\}$, then $\gamma_s(P_t \times P_t) \leq \left\lceil \frac{t^2}{3} \right\rceil + 1$.

Proof:

Case 1: $t \equiv 1 \pmod{3}$

Consider the set,

$$\begin{aligned}
Y_t = & \left\{ (2 + 3(c - 1), 1 + 3(r - 1)) : 1 \leq c \leq \left\lfloor \frac{t}{3} \right\rfloor, 1 \leq r \leq \left\lfloor \frac{t}{3} \right\rfloor \right\} \\
& \cup \left\{ (3 + 3(c - 1), 2 + 3(r - 1)) : 1 \leq c \leq \left\lfloor \frac{t}{3} \right\rfloor, 1 \leq r \leq \left\lfloor \frac{t}{3} \right\rfloor \right\} \\
& \cup \left\{ (1 + 3(c - 1), 3 + 3(r - 1)) : 1 \leq c \leq \left\lfloor \frac{t}{3} \right\rfloor, 1 \leq r \leq \left\lfloor \frac{t}{3} \right\rfloor \right\} \\
& \cup \{(t, 1), (3, t)\}
\end{aligned} \tag{1}$$

The following vertices of Y_t defend their external private neighbors:

For each $q \in \left\{1, 2, \dots, \left\lfloor \frac{t}{3} \right\rfloor\right\}$,

- $(1, 3 + 3(q - 1))$ defends $(1, 2 + 3(q - 1))$
- $(t, 3 + 3(q - 1))$ defends $(t, 4 + 3(q - 1))$
- $(2 + 3(q - 1), 1)$ defends $(1 + 3(q - 1), 1)$.

For $t > 4$ and each $q \in \left\{2, 3, \dots, \left\lfloor \frac{t}{3} \right\rfloor\right\}$, $(2 + 3(q - 1), t)$ defends $(3 + 3(q - 1), t)$.

The remaining vertices of Y_t do not have external private neighbors and they defend all their neighbors in $V(P_t \times P_t) \setminus Y_t$. Since Y_t is a secure dominating set, $\gamma_s(P_t \times P_t) \leq \left\lfloor \frac{t^2}{3} \right\rfloor + 1$ when $t \equiv 1 \pmod{3}$.

Case 2: $t \equiv 2 \pmod{3}$

Consider the set,

$$\begin{aligned}
Y_t = & \left\{ (2 + 3(z - 1), 1 + 3(s - 1)) : 1 \leq z \leq \left\lfloor \frac{t}{3} \right\rfloor, 1 \leq s \leq \left\lfloor \frac{t}{3} \right\rfloor \right\} \\
& \cup \left\{ (3 + 3(z - 1), 2 + 3(s - 1)) : 1 \leq z \leq \left\lfloor \frac{t}{3} \right\rfloor, 1 \leq s \leq \left\lfloor \frac{t}{3} \right\rfloor \right\} \\
& \cup \left\{ (1 + 3(z - 1), 3 + 3(s - 1)) : 1 \leq z \leq \left\lfloor \frac{t}{3} \right\rfloor, 1 \leq s \leq \left\lfloor \frac{t}{3} \right\rfloor \right\} \\
& \cup \{(t, 2), (1, t)\}
\end{aligned} \tag{2}$$

The following vertices of Y_t defend their external private neighbors:

For each $q \in \left\{1, 2, \dots, \left\lfloor \frac{t}{3} \right\rfloor\right\}$,

- $(1, 3 + 3(q - 1))$ defends $(1, 2 + 3(q - 1))$
- $(3 + 3(q - 1), t)$ defends $(4 + 3(q - 1), t)$.

For each $q \in \left\{1, 2, \dots, \left\lfloor \frac{t}{3} \right\rfloor\right\}$, $(2 + 3(q - 1), 1)$ defends $(1 + 3(q - 1), 1)$.

For every $q \in \left\{2, 3, \dots, \left\lfloor \frac{t}{3} \right\rfloor\right\}$, $(t, 1 + 3(q - 1))$ defends $(t, 2 + 3(q - 1))$.

The remaining vertices of Y_t do not have external private neighbors and they defend all their neighbors in $V(P_t \times P_t) \setminus Y_t$. Since Y_t is a secure dominating set, $\gamma_s(P_t \times P_t) \leq \left\lfloor \frac{t^2}{3} \right\rfloor + 1$ when $t \equiv 2 \pmod{3}$.

The set Y_{16} of vertices of $P_{16} \times P_{16}$ is shown in Figure 2 and the set Y_8 of vertices of $P_8 \times P_8$ is shown in Figure 3. These sets are defined in Theorem 6.

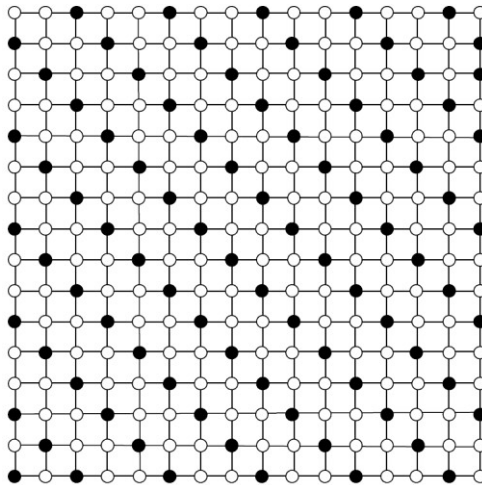


Figure 2

The shaded vertices represent the vertices of Y_{16} in $P_{16} \times P_{16}$

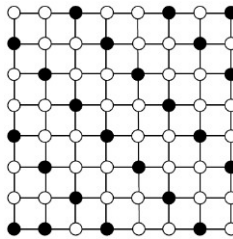


Figure 3

The shaded vertices represent the vertices of Y_8 in $P_8 \times P_8$

The set Y_{2^f} defined in Equations 1 and 2 of Theorem 6 forms a secure dominating set of $G(2^f, 2^f)$ when $f \geq 2$. Let $T_f \subset V_f$, for $2 \leq f \leq d$ be a parallel representation of the set Y_{2^f} with respect to $PN(d)$. Hence, if a vertex $(a, b) \in Y_{2^f}$, then $(a, b, f) \in T_f$ for $f \in \{2, 3, \dots, d\}$.

The following theorems provide the upper bound for the secure domination number of pyramid networks $PN(d')$ for different values of d' .

Theorem 7:

$$\gamma_s(PN(d')) \leq \begin{cases} 2, & \text{when } d' = 1 \\ 7, & \text{when } d' = 2 \\ 25, & \text{when } d' = 3 \end{cases}$$

Proof: The set $S_1 = \{(1, 1, 1), (2, 2, 1)\}$ forms a secure dominating set of $PN(1)$.

Consider $S_2 = \{(1, 3, 2), (2, 1, 2), (2, 4, 2), (3, 2, 2)\} \cup \{(2, 1, 1), (2, 2, 1)\} \cup \{(1, 1, 0)\}$ to be a dominating set of $PN(2)$. The vertices of $\{(1, 3, 2), (2, 1, 2),$

$(2, 4, 2), (3, 2, 2)\}$ defend the vertices of $(V_2 \setminus S_2) \setminus \{(4, 1, 2), (4, 3, 2), (4, 4, 2)\}$ and the vertices of $\{(1, 1, 1), (1, 2, 1)\}$. Also, the vertices of $\{(2, 1, 1), (2, 2, 1)\}$ defend the vertices of $\{(4, 1, 2), (4, 3, 2), (4, 4, 2)\}$. Hence, S_2 is an SDS of $PN(2)$.

Let $S_3 = T_3 \cup S_1$ be a dominating set of $PN(3)$. The vertices of the set T_3 defend the vertices of $(V_3 \cup V_2) \setminus S_3$ and the vertices of S_1 defend the vertices of $(V_1 \setminus S_1) \cup \{(1, 1, 0)\}$. Thus, $PN(3)$ has S_3 as a secure dominating set.

Theorem 8: For $d' \geq 4$,

$$\gamma_s(PN(d')) \leq \begin{cases} \left(\sum_{j=0}^{\lfloor \frac{d'}{4} \rfloor - 1} \left(\left\lfloor \frac{(2^{d'-4j})^2}{3} \right\rfloor + \left\lfloor \frac{(2^{d'-4j-2})^2}{3} \right\rfloor + 4 \left\lfloor \frac{(2^{d'-4j-2})}{3} \right\rfloor \right) \right) - \left\lfloor \frac{d'}{4} \right\rfloor + q_1, & \text{when } d' \text{ is odd} \\ \left(\sum_{j=0}^{\lfloor \frac{d'}{4} \rfloor - 1} \left(\left\lfloor \frac{(2^{d'-4j})^2}{3} \right\rfloor + \left\lfloor \frac{(2^{d'-4j-2})^2}{3} \right\rfloor + 4 \left\lfloor \frac{(2^{d'-4j-2})}{3} \right\rfloor \right) \right) - 2 \left\lfloor \frac{d'}{4} \right\rfloor + q_2, & \text{when } d' \text{ is even} \end{cases}$$

where

$$q_1 = \begin{cases} 25, & \text{when } d' \equiv 3 \pmod{4} \\ 2, & \text{when } d' \equiv 1 \pmod{4} \end{cases}$$

$$q_2 = \begin{cases} 2, & \text{when } d' \equiv 0 \pmod{4} \\ 7, & \text{when } d' \equiv 2 \pmod{4} \end{cases}$$

Proof:

Case 1: When d' is odd

Consider the set,

$$X = \left(\bigcup_{f=0}^{\lfloor \frac{d'}{4} \rfloor - 1} T_{d'-4f} \right) \cup \{(1 + 3q, 1 + 3r, d' - 2 - 4f), (2 + 3q, 2 + 3r, d' - 2 - 4f) : 0 \leq f \leq \left\lfloor \frac{d'}{4} \right\rfloor - 1, 0 \leq q \leq \left\lfloor \frac{2^{d'-2-4f}}{3} \right\rfloor - 1, 0 \leq r \leq \left\lfloor \frac{2^{d'-2-4f}}{3} \right\rfloor - 1\} \setminus \{(2, 2, d' - 2 - 4f), (2^{d'-2-4f} - 1, 2^{d'-2-4f} - 1, d' - 2 - 4f) : 0 \leq f \leq \left\lfloor \frac{d'}{4} \right\rfloor - 1\} \cup \{(3 + 3q, 3 + 3r, d' - 2 - 4f) : 0 \leq f \leq \left\lfloor \frac{d'}{4} \right\rfloor - 1, 0 \leq q \leq \left\lfloor \frac{2^{d'-2-4f}}{3} \right\rfloor - 1, 0 \leq r \leq \left\lfloor \frac{2^{d'-2-4f}}{3} \right\rfloor - 1\} \cup \{(3 + 3q, 1, d' - 2 - 4f),$$

$$\begin{aligned}
 & (4 + 3q, 2^{d'-2-4f}, d' - 2 - 4f), (1, 2 + 3q, d' - 2 - 4f), \\
 & (2^{d'-2-4f}, 3 + 3q, d' - 2 - 4f): 0 \leq f \leq \left\lfloor \frac{d'}{4} \right\rfloor - 1, \\
 & 0 \leq q \leq \left\lfloor \frac{2^{d'-2-4f}}{3} \right\rfloor - 1\}
 \end{aligned}$$

Let $S = X \cup \{(1,1,1), (2,2,1)\}$ when $d' \equiv 1 \pmod{4}$ and let $S = X \cup T_3 \cup \{(1,1,1), (2,2,1)\}$ when $d' \equiv 3 \pmod{4}$. Consider,

$$\begin{aligned}
 X_{c'} = & \{(1, 2, d' - 2 - 4c'), (3, 3, d' - 2 - 4c'), \\
 & (2^{d'-2-4c'} - 1, 2^{d'-2-4c'}, d' - 2 - 4c'), \\
 & (2^{d'-2-4c'} - 2, 2^{d'-2-4c'} - 2, d' - 2 - 4c')\}
 \end{aligned}$$

for $0 \leq c' \leq \left\lfloor \frac{d'}{4} \right\rfloor - 1$. Then, for each $h' \in \{0, 1, \dots, \left\lfloor \frac{d'}{4} \right\rfloor - 1\}$, the following vertices defend their external private neighbors:

- $(1, 2, d' - 2 - 4h')$ defends $(2, 2, d' - 2 - 4h')$
- $(3, 3, d' - 2 - 4h')$ defends $(2, 3, d' - 2 - 4h')$
- $(2^{d'-2-4h'} - 1, 2^{d'-2-4h'}, d' - 2 - 4h')$ defends $(2^{d'-2-4h'} - 1, 2^{d'-2-4h'} - 1, d' - 2 - 4h')$
- $(2^{d'-2-4h'} - 2, 2^{d'-2-4h'} - 2, d' - 2 - 4h')$ defends $(2^{d'-2-4h'} - 2, 2^{d'-2-4h'} - 1, d' - 2 - 4h')$.

Let $Z \subseteq \bigcup_{f=0}^{\left\lfloor \frac{d'}{4} \right\rfloor - 1} T_{d'-4f}$ denote the set of vertices that have external private neighbors in their same level. The set Z can be inferred from Theorem 6. The vertices of Z defend their corresponding external private neighbors.

Consider a vertex $(x, y, l) \in X \setminus \left(\left(\bigcup_{c'=0}^{\left\lfloor \frac{d'}{4} \right\rfloor - 1} X_{c'} \right) \cup Z \right)$ where

$l \in \bigcup_{s=0}^{\left\lfloor \frac{d'}{4} \right\rfloor - 1} \{d' - 4s, d' - 2 - 4s\}$. Suppose that $(x, y, l) \in J_{(u,v,l-1)}$. If the neighbor of (x, y, l) in V_{l-1} is its external private neighbor, then (x, y, l) defends its neighbors in $J_{(u,v,l-1)}$. If $|X \cap J_{(u,v,l-1)}| > 1$, then (x, y, l) defends all its neighbors. Any vertex of $X \cap V_l$ that has an external private neighbor in V_l defends its neighbor in V_{l-1} when that neighbor in V_{l-1} is adjacent to its external private neighbors in V_l .

If a vertex is in $\{(2, r, d' - 4s), (2^{d'-4s} - 1, r, d' - 4s), (r, 2, d' - 4s), (r, 2^{d'-4s} - 1, d' - 4s)\} \cap X$, where $0 \leq s \leq \left\lfloor \frac{d'}{4} \right\rfloor - 1$ and $2 \leq r \leq 2^{d'-4s} - 1$, then the vertex defends all its neighbors. Any vertex of S that does not have an external private neighbor defends all its neighbors in $V(PN(d')) \setminus S$.

Subcase 1: When $d' \equiv 1 \pmod{4}$

When $S = X \cup \{(1,1,1), (2,2,1)\}$, the vertices of $(V_{d'} \cup V_{d'-1} \cup \dots \cup V_2) \setminus X$ are defended by the vertices of X . The vertices of $(V_1 \cup \{(1,1,0)\}) \setminus S$ are defended by the vertices of $\{(1,1,1), (2,2,1)\}$. Thus, S is an SDS of the graph.

Subcase 2: When $d' \equiv 3 \pmod{4}$

Considering $S = X \cup T_3 \cup \{(1,1,1), (2,2,1)\}$, the vertices of $(V_{d'} \cup V_{d'-1} \cup \dots \cup V_4) \setminus X$ are defended by the vertices of X . The vertices of $(V_3 \cup V_2 \cup V_1 \cup \{(1,1,0)\}) \setminus S$ are defended by the vertices of $\{(1,1,1), (2,2,1)\} \cup T_3$. Hence, S is an SDS of the graph.

Case 2: When d' is even

Consider the set

$$\begin{aligned} X = & \left(\bigcup_{p=0}^{\left\lfloor \frac{d'}{4} \right\rfloor - 1} T_{d'-4p} \right) \cup \{(2+3q, 2+3r, d'-2-4p), \\ & (3+3q, 3+3r, d'-2-4p): 0 \leq p \leq \left\lfloor \frac{d'}{4} \right\rfloor - 1, \\ & 0 \leq q \leq \left\lfloor \frac{2^{d'-2-4p}}{3} \right\rfloor - 1, 0 \leq r \leq \left\lfloor \frac{2^{d'-2-4p}}{3} \right\rfloor - 1\} \setminus \\ & \{(2,2, d'-2-4p), (2^{d'-2-4p}-1, 2^{d'-2-4p}-1, d'-2-4p): \\ & 0 \leq p \leq \left\lfloor \frac{d'}{4} \right\rfloor - 1\} \cup \{(1+3q, 1+3r, d'-2-4p): \\ & 0 \leq p \leq \left\lfloor \frac{d'}{4} \right\rfloor - 1, 0 \leq q \leq \left\lfloor \frac{2^{d'-2-4p}}{3} \right\rfloor - 1, \\ & 0 \leq r \leq \left\lfloor \frac{2^{d'-2-4p}}{3} \right\rfloor - 1\} \setminus \{(2^{d'-2-4p}, 1, d'-2-4p): \\ & 0 \leq p \leq \left\lfloor \frac{d'}{4} \right\rfloor - 1\} \cup \{(3+3q, 1, d'-2-4p), \\ & (3+3q, 2^{d'-2-4p}, d'-2-4p), (1, 2+3q, d'-2-4p), \\ & (2^{d'-2-4p}, 2+3q, d'-2-4p): 0 \leq p \leq \left\lfloor \frac{d'}{4} \right\rfloor - 1, \\ & 0 \leq q \leq \left\lfloor \frac{2^{d'-2-4p}}{3} \right\rfloor - 1\} \end{aligned}$$

Let $S = X \cup \{(1,2,1), (1,1,0)\}$ when $d' \equiv 0 \pmod{4}$ and let $S = X \cup \{(1,3,2), (2,1,2), (2,4,2), (3,2,2)\} \cup \{(2,1,1), (2,2,1)\} \cup \{(1,1,0)\}$ when $d' \equiv 2 \pmod{4}$. Consider

$$\begin{aligned} X_{x'} = & \{(1,2, d'-2-4x'), (3,3, d'-2-4x'), \\ & (2^{d'-2-4x'}-1, 2^{d'-2-4x'}, d'-2-4x'), \\ & (2^{d'-2-4x'}-2, 2^{d'-2-4x'}-2, d'-2-4x')\} \end{aligned}$$

with $0 \leq x' \leq \left\lfloor \frac{d'}{4} \right\rfloor - 1$ when $d' \equiv 2 \pmod{4}$ and when $d'-2-4x' > 2$ for $d' \equiv 0 \pmod{4}$. When $d'-2-4x' = 2$ for $d' \equiv 0 \pmod{4}$, we have

corresponding $X_{x'} = \{(1, 2, 2), (3, 4, 2)\}$. Then, for each $y' \in \{0, 1, \dots, \lfloor \frac{d'}{4} \rfloor - 1\}$, the following vertices defend their external private neighbors:

- $(1, 2, d' - 2 - 4y')$ defends $(2, 2, d' - 2 - 4y')$
- $(3, 3, d' - 2 - 4y')$ defends $(2, 3, d' - 2 - 4y')$ if $(3, 3, d' - 2 - 4y') \in X_{y'}$
- $(2^{d'-2-4y'} - 1, 2^{d'-2-4y'}, d' - 2 - 4y')$ defends $(2^{d'-2-4y'} - 1, 2^{d'-2-4y'} - 1, d' - 2 - 4y')$
- $(2^{d'-2-4y'} - 2, 2^{d'-2-4y'} - 2, d' - 2 - 4y')$ defends $(2^{d'-2-4y'} - 2, 2^{d'-2-4y'} - 1, d' - 2 - 4y')$ if $(2^{d'-2-4y'} - 2, 2^{d'-2-4y'} - 2, d' - 2 - 4y') \in X_{y'}$.

Let $Z \subseteq \bigcup_{p=0}^{\lfloor \frac{d'}{4} \rfloor - 1} T_{d'-4p}$ denote the set of vertices that have external private neighbors in their same level. The set Z can be inferred from Theorem 6. The vertices of Z defend their corresponding external private neighbors.

Consider a vertex $(x, y, l) \in X \setminus \left(\left(\bigcup_{x'=0}^{\lfloor \frac{d'}{4} \rfloor - 1} X_{x'} \right) \cup Z \right)$ and suppose that

$(x, y, l) \in J_{(u,v,l-1)}$ where $l \in \bigcup_{b=0}^{\lfloor \frac{d'}{4} \rfloor - 1} \{d' - 4b, d' - 2 - 4b\}$. If the neighbor of (x, y, l) in V_{l-1} is its external private neighbor, then (x, y, l) defends its neighbors in $J_{(u,v,l-1)}$. If $|X \cap J_{(u,v,l-1)}| > 1$, then (x, y, l) defends all its neighbors. Any vertex of $X \cap V_l$ that has an external private neighbor in V_l defends its neighbor in V_{l-1} when that neighbor in V_{l-1} is adjacent to its external private neighbors in V_l .

If a vertex is in $\{(2, r, d' - 4b), (2^{d'-4b} - 1, r, d' - 4b), (r, 2, d' - 4b), (r, 2^{d'-4b} - 1, d' - 4b)\} \cap X$, where $0 \leq b \leq \lfloor \frac{d'}{4} \rfloor - 1$ and $2 \leq r \leq 2^{d'-4b} - 1$, then the vertex defends all its neighbors. Any vertex of S that does not have an external private neighbor defends all its neighbors in $V(PN(d')) \setminus S$.

Subcase 1: When $d' \equiv 0 \pmod{4}$

When $S = X \cup \{(1, 2, 1), (1, 1, 0)\}$, the vertices of $(V_{d'} \cup V_{d'-1} \cup \dots \cup V_1) \setminus (X \cup \{(2, 3, 2)\})$ are defended by the vertices of X . The vertex $(2, 3, 2)$ is defended by the vertex $(1, 2, 1)$. Thus, S is an SDS of the graph.

Subcase 2: When $d' \equiv 2 \pmod{4}$

When $S = X \cup \{(1, 3, 2), (2, 1, 2), (2, 4, 2), (3, 2, 2)\} \cup \{(2, 1, 1), (2, 2, 1)\} \cup \{(1, 1, 0)\}$, the vertices of $(V_{d'} \cup V_{d'-1} \cup \dots \cup V_3) \setminus X$ are defended by the vertices of X . The vertices of $(V_2 \cup V_1 \cup \{(1, 1, 0)\}) \setminus S$ are defended by the vertices of $\{(1, 3, 2), (2, 1, 2), (2, 4, 2), (3, 2, 2)\} \cup \{(2, 1, 1), (2, 2, 1)\} \cup \{(1, 1, 0)\}$. Therefore, S is an SDS of the graph.

4. Conclusion

The existing upper bound for the secure domination number of $P_u \times P_v$ has been improved in the case of $P_n \times P_n$, for $n \geq 4$ with $n \equiv s \pmod{3}$, where $s \in \{1, 2\}$. Consequently, the upper bound for $\gamma_s(PN(d))$ has been presented. A further objective is to determine the exact secure domination numbers of these graphs. The grid-based structures have been extensively utilized in the topology of interconnection networks. Thus, we are motivated to extend the investigation to other grid-based interconnection networks.

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