

Maximum subgraph and wirelength analysis of extended Sierpiński graph $S^+(n, K_3)$

P. Leo Joshua [†]

*Department of Mathematics
Hindustan Institute of Technology & Science
Chennai 603103
Tamil Nadu
India*

R. Sundara Rajan ^{*}

*Department of Mathematics
Rajalakshmi Engineering College
Chennai 602105
Tamil Nadu
India*

D. Ahima Emilet [§]

*Department of Mathematics
Hindustan Institute of Technology & Science
Chennai 603103
Tamil Nadu
India*

S. Arulanand [‡]

*Department of Mathematics
Rajalakshmi Institute of Technology
Chennai 600124
Tamil Nadu
India*

[†] ORCID-ID: 0000-0002-0365-2309

^{*} ORCID-ID: 0000-0002-1851-6334

[§] ORCID-ID: 0000-0002-3424-2126

[‡] ORCID-ID: 0000-0001-9862-0213

[†] E-mail: joseleowa18@gmail.com

^{*} E-mail: vprsundar@gmail.com (Corresponding Author)

[§] E-mail: ahi.180492@gmail.com

[‡] E-mail: arulanand7875@gmail.com

Abstract

The Sierpiński graph $S(n, K_m)$, known for its recursive and hierarchical construction, serves as a significant model for multiprocessor interconnection architectures. Its extended variant, denoted $S^*(n, K_m)$, further strengthens these properties and has practical relevance in areas such as parallel computation and VLSI circuit design. This work investigates the unresolved Maximum Subgraph Problem (MSP) for the extended Sierpiński graph $S^*(n, K_3)$, with $n \geq 2$, a problem originally proposed by Joshwa et al. [1], and still open for general values of m . We introduce a computational framework, implemented in SageMath, for determining the maximal edge count among all subgraphs induced by r vertices, where $1 \leq r \leq 3^n + 1$. In addition, we examine minimum wirelength embeddings of $S^*(n, K_3)$ into various hierarchical graph topologies, a key factor in optimizing communication efficiency and layout complexity in networked systems.

Subject Classification: 05C10, 05C60, 05C62, 05C75.

Keywords: Embedding, Wirelength, Extended Sierpiński graph $S^*(n, K_3)$, Maximum subgraph, Hierarchical structures.

1. Introduction and Preliminaries

Graph embeddings have broad applications across fields such as communication networks, data management, and computational modeling. By embedding one graph into another, researchers can simulate task distribution, assess communication paths, and improve network efficiency. This is typically achieved by minimizing key embedding parameters such as dilation, edge congestion, and wirelength which directly influence system performance and resource optimization.

Let $A = (V_A, E_A)$ and $B = (V_B, E_B)$ be two finite graphs. An embedding of A into B [1], denoted by $\varphi: G \rightarrow H$, consists of:

1. Each vertex of A is mapped to a unique vertex in B .
2. Each edge $(uv) \in E(A)$ is mapped to a path in B connecting $\varphi(u)$ and $\varphi(v)$.

Edge Congestion: For an embedding φ , the *edge congestion* [2] on an edge $e \in E_B$ is the number of guest edges $e' \in E_A$ whose paths $P_\varphi(e')$ include e . Formally,

$$C_\varphi(A, B(e)) = |\{e' \in E_A : e \in P_\varphi(e')\}|.$$

Wirelength: The total communication cost of an embedding is measured by its *wirelength* [3], defined as the sum of the congestions over all host edges:

$$WL_\varphi(A, B) = \sum_{e \in E_B} C_\varphi(A, B(e)), \text{ Then } WL(A, B) = \min_{\varphi} WL_\varphi(A, B).$$

Maximum Subgraph Problem (MSP): Another classical problem closely related to embeddings is the *Maximum Subgraph Problem* [4]. Given a graph

$A = (V, E)$ and integer $1 \leq m \leq |V|$, MSP seeks a subset $W \subseteq V$, $|W| = m$, such that the induced subgraph $A[W]$ has the maximum number of edges:

$$I_A(m) = \max_{W \subseteq V, |W|=m} |E(A[W])|.$$

Lemma 1.1: [6] *Let $\varphi: A \rightarrow B$ be an embedding. Suppose an edge cut $S \subseteq E(B)$ separates B into two components B^1 and B^2 . Let $A^j = A[\varphi^{-1}(V(B^j))]$ for $j = 1, 2$, and assume:*

- *Each edge between vertices in A^j has no edge in S .*
- *Each edge between vertices in A^1 and A^2 use exactly one edge from S .*
- *$V(A^1)$ and $V(A^2)$ are optimal subsets.*

Then,

$$C_\varphi(S) = \sum_{u \in V(A^1)} \deg_A(u) - 2|E(A^1)| = \sum_{u \in V(A^2)} \deg_A(u) - 2|E(A^2)|.$$

Lemma 1.2: [3] *Let $\varphi: A \rightarrow B$ be an embedding, and let $[tE(B)]$ denote the multiset where each edge appears r times. $\{S_y: 1 \leq y \leq k\}$ be a partition of $[tE(B)]$, with each S_t satisfying the Modified Congestion Lemma. Then, $WL_\varphi(A, B) = \frac{1}{t} \sum_{y=1}^k C_\varphi(S_y)$.*

Extended Sierpiński graph $S^+(n, K_m)$: The Sierpiński graph, denoted by $S(n, K_m)$, was introduced by Klavžar and Milutinović [5]. It is inspired by the Sierpiński triangle, a well-known fractal, and exhibits a hierarchical structure that makes it particularly useful for recursive algorithms and network modeling. In Klavžar and Mohar [14] extended this concept by introducing the extended Sierpiński graph, denoted by $S^+(n, K_3)$. This variant builds on the original Sierpiński graph by incorporating additional connections to improve its structural properties, such as fault tolerance and connectivity. The extended network finds applications in parallel computing and distributed systems, where efficient communication and robustness are critical.

Definition 3.1: For integers $n, m \geq 1$, the *generalized Sierpiński graph* $S(n, K_m)$ consists of vertices that are all sequences of length n over the set $\{0, 1, \dots, m-1\}$:

$$V(S(n, K_m)) = \{(a_1, a_2, \dots, a_n) | a_i \in \{0, 1, \dots, m-1\}\}.$$

Two vertices $(a_1, \dots, a_n)$ and $(b_1, \dots, b_n)$ are linked by an edge if there is an index $t \in \{1, \dots, n\}$ such that:

- the coordinates after t are identical: $a_i = b_i$ for all $i > t$,
- the t -th coordinates differ: $a_t \neq b_t$,
- the coordinates before t are swapped: $a_i = b_t$ and $b_i = a_t$ for all $i < t$.

Definition 3.2: Given a generalized Sierpiński graph $S(n, K_m)$, the *extended Sierpiński graph* $S^+(n, K_m)$ is formed by introducing a new vertex s and

connecting it to every vertex considered *extreme* in the original graph:
 $V(S^+(n, K_m)) = V(S(n, K_m)) \cup \{s\},$

$$E(S^+(n, K_m)) = E(S(n, K_m)) \cup \{(s, v) | v \text{ is extreme in } S(n, K_m)\}.$$

In particular, for $m = 3$, the resulting graph is denoted $S^+(n, K_3)$, which is the primary focus in this study.

2. Maximum Subgraph Problem of $S^+(n, K_3)$

The *Maximum Subgraph Problem* seeks to identify, for a given integer l , the l -vertex induced subgraph of a graph A that contains the maximum edges. This problem is a special case of the edge-isoperimetric problem and is known to be NP-complete. Previous results by Harper [7] and Rajan, Greeni, and Joshwa [6] have solved the MSP for classical Sierpiński graphs $S(n, K_m)$. Recently, Joshwa, Rajan, and Rajalaxmi [9] extended these findings to the further-extended Sierpiński graph $S^{++}(n, K_m)$ for $n \geq 2$, $m \geq 3$. However, the MSP for the extended Sierpiński graph $S^+(n, K_m)$, particularly for $m = 3$, remains unresolved.

Lexicographic Optimality

Vertex Labeling Scheme : Let the vertices of $S^+(n, K_3)$ be labeled as $v_0, v_1, \dots, v_{3^n-1}$ using a lexicographic order consistent with the construction of $S(n, K_3)$. The additional vertex v_s , which connects to the extreme vertices, is assigned the label $v_{(3^n+5)/2}$. Remaining vertices follow sequential lexicographic order, shifted accordingly.

Theorem 2.1: [7] *In $S(n, K_m)$, selecting the initial l vertices according to lexicographic ordering results in a subgraph with the maximum number of induced edges.*

Theorem 2.2: *Let A denote the extended Sierpiński graph $S^+(n, K_m)$, and let l be an integer satisfying $1 \leq l \leq 3^n + 1$. Among all subsets $Z \subseteq \{0, 1, \dots, 3^n\}$ with cardinality $|Z| = l$, the induced subgraph $A[Z]$ reaches its optimum edges when $Z = \{0, 1, \dots, l - 1\}$.*

Proof: According to Theorem 2.1, in the Sierpiński graph $S(n, K_m)$, selecting the first l vertices in lexicographic order yields a subset that maximizes the number of internal edges. Since the extended structure $S^+(n, K_3)$ retains the fundamental adjacency characteristics of $S(n, K_m)$, this optimality result continues to hold.

Therefore, for any subset $Z \subseteq \{0, 1, \dots, 3^n\}$ with $|Z| = l$, the configuration $Z = \{0, 1, \dots, l - 1\}$ produces the highest edge count in the induced subgraph $A[Z]$.

Theorem 2.3: *Let an integer l be expressed as $l = \sum_{i=0}^t a_i 3^i$ with digits $a_i \in \{0, 1, 2\}$. For each $i = 1, 2, \dots, t$, define*

$$\beta_i = \min \left\{ a_i, \left\lceil \frac{3-i}{3^{i-1}} \sum_{j=0}^{i-1} a_j 3^j \right\rceil \right\}, \text{ and}$$

$$\mu_l = \frac{1}{2} \sum_{i=0}^t a_i (3^{i+1} - 3 + a_i - 1) + \sum_{i=1}^t \beta_i.$$

Then, the induced edge count $I_A(l)$ in $S^+(n, K_3)$ is given by:

$$I_A(l) = \begin{cases} \mu_l & \text{if } 1 \leq l \leq \frac{3^{n+3}}{2}, \\ \mu_{l-1} + 2 & \text{if } \frac{3^{n+5}}{2} \leq l \leq 3^n, \\ \mu_{l-1} + 3 & \text{if } l = 3^n + 1. \end{cases}$$

Proof: We consider the vertex labeling in $S^+(n, K_3)$ as described in Section 2, where each label corresponds to a unique vertex.

Case 1: $1 \leq l \leq \frac{3^{n+3}}{2}$. We prove the result by induction on t .

Base case ($t = 0$): Here $l = a_0 < 3$, so the vertices form a complete graph K_{a_0} . The induced edges are

$$I_A(l) = \binom{a_0}{2} = \frac{1}{2} a_0 (3^{0+1} - 3 + a_0 - 1),$$

matching μ_l for $t = 0$.

Inductive step: Assume the formula holds for $t - 1$. For $t \geq 1$, the first l vertices (in lexicographic order) consist of:

- a_t full copies of $S^+(t, K_3)$, contributing $\binom{a_t}{2}$ edges between these copies.
- A partial copy on the remaining $l_t = \sum_{i=0}^{t-1} a_i 3^i$ vertices, plus β_t edges linking it to the a_t full copies.

By the induction hypothesis, the partial copy contributes $I_A(l_t) = \mu_{l_t}$ edges. Therefore, the total induced edges are:

$$\begin{aligned} I_A(l) &= a_t \cdot E(S^+(t, K_3)) + \binom{a_t}{2} + I_A(l_t) + \beta_t \\ &= a_t \cdot \frac{3^t - 3}{2} + \frac{a_t(a_t - 1)}{2} + \frac{1}{2} \sum_{i=0}^{t-1} a_i (3^{i+1} - 3 + a_i - 1) + \sum_{i=1}^{t-1} \beta_i + \beta_t \\ &= \frac{1}{2} \sum_{i=0}^t a_i (3^{i+1} - 3 + a_i - 1) + \sum_{i=1}^t \beta_i = \mu_l. \end{aligned}$$

Case 2: $\frac{3^{n+5}}{2} \leq l \leq 3^n$. Let v_l be the vertex labeled l . For l above the midpoint, each new vertex contributes exactly two edges connecting it to neighboring vertices, following the structure of $S^+(n, K_3)$. Hence,

$$I_A(l) = \mu_{l-1} + 2.$$

Case 3: $l = 3^n + 1$. The final vertex connects to three existing vertices, adding three edges, so

$$I_A(l) = \mu_{l-1} + 3.$$

Algorithm 1: Computation of $I_A(l)$ for Extended Sierpiński Graphs $S^+(n, K_3)$

Input: An extended Sierpiński graph $S^+(n, K_3)$ and an integer l

Output: Value of $I_A(l)$.

```

1 /* Step 1: Convert  $l$  to base- $m$  representation */ ;
2 if  $l = 0$  then
3   return 0;
4 Initialize  $a \leftarrow$  Base- $m$  digits of  $l$ ;
5 Set  $t \leftarrow$  length of  $a - 1$ ;
6 /* Step 2: Compute correction factors  $\alpha_i$  */ ;
7 Initialize list  $\alpha \leftarrow []$ ;
8 for  $i \leftarrow 1$  to  $t$  do
9   Compute  $\alpha_i = \min \left\{ a_i, \left\lceil \frac{m-1}{m^i-1} \sum_{j=0}^{i-1} a_j m^j \right\rceil \right\}$ ;
10  Append  $\alpha_i$  to list  $\alpha$ ;
11 /* Step 3: Evaluate  $I_A(l)$  using case-based rules */ ;
12 if  $1 \leq l \leq \frac{3^n+3}{2}$  then
13   Compute  $\mu_r = \frac{1}{2} \sum_{i=0}^t a_i (a_i - 1 + 3^{i+1} - 3) + \sum_{i=1}^t \beta_i$ ;
14   return  $\mu_r$ ;
15 else if  $\frac{3^n+5}{2} \leq l \leq 3^n$  then
16   return  $\mu_{t-1} + 2$ ;
17 else if  $l = 3^n + 1$  then
18   return  $\mu_{t-1} + 3$ ;

```

3. Main Results

In this section, we present exact computations of the wirelength required to embed extended Sierpiński graphs into several host structures, including paths, caterpillars, 1-hierarchical caterpillars, and banana trees. Accurately determining these wirelengths is essential for reducing communication overhead in distributed architectures and plays a key role in improving the efficiency of network-based algorithms.

3.1 Minimum Linear Arrangement

The *Minimum Linear Arrangement* (MinLA) problem involves assigning the vertices of a graph A to positions along a linear sequence (or path) so that the sum of the distances between connected vertices is minimized. This problem was first studied by Harper [15] in the context of designing efficient error-correcting codes, where minimizing the total separation between related elements is important. Finding an optimal linear placement is computationally challenging, and in fact, determining the minimum arrangement has been proven to be NP-complete [1].

MinLA Embedding Algorithm

Input: Extended Sierpiński graph $S^+(n, K_3)$ and a path graph $P_{3^{n+1}}$, where $n \geq 2$.

Algorithm:

- Assign the vertex labeling scheme to $S^+(n, K_3)$ as described in Section 2.
- Label the vertices of the path $P_{3^{n+1}}$ sequentially from 1 to $3^n + 1$, starting from the leftmost node.
- Define a vertex mapping $\varphi: V(A) \rightarrow V(B)$ such that each vertex $x \in V(A)$ is mapped to $\varphi(x) = x$.

Output: The mapping φ yields a minimum linear arrangement of the extended Sierpiński graph.

Proof of correctness : For each edge $(y, y + 1): 1 \leq y \leq 3n$, in the graph $P_{3^{n+1}}$, it forms an edge cut S^y . It disconnects $P_{3^{n+1}}$ into two components, denoted as B^y and $\overline{B^y}$, where $V(B^y) = \{1, 2, \dots, y\}$ and $V(\overline{B^y}) = \{y + 1, \dots, 3^n + 1\}$. Using Theorem 2.3, G^y is the optimal on $S^+(n, K_3)$. Thus, each S^y fulfils the condition stated in Lemma 1.1. Then, for $1 \leq y \leq 3n$, $C_\varphi(S^y)$ is minimum. The linear arrangement is considered to be a minimum by Lemma 1.2.

Theorem 3.1: The MinLA of $S^+(n, K_3)$ is given by

$$\text{MinLA}(S^+(n, K_3)) = \frac{1}{2} 3^{n+1} (3^n + 1) - 2 \sum_{y=1}^{3^n} I_A(y).$$

Proof: Apply the MinLA algorithm to assign labels to the vertices of $S^+(n, K_3)$ and to the path $P_{3^{n+1}}$. The congestion on each $S^y, 1 \leq y \leq 3^n + 1$ is minimum by MinLA Algorithm and is given by $C_\varphi(S^y) = 3y - 2I_A(y), 1 \leq y \leq 3^n$. By Lemma 1.2, we get

$$\text{MinLA}(S^+(n, K_3)) = \sum_{y=1}^{3^n} C_\varphi(S^y) = \frac{1}{2} 3^{n+1} (3^n + 1) - 2 \sum_{y=1}^{3^n} I_A(y).$$

3.2 Caterpillar

A tree is referred to as a *caterpillar* if removing all its leaf nodes (vertices of degree one) results in a simple path, known as the *spine*. For $n \geq 2$, the caterpillar structure, denoted by CAT, is formed by starting with a path $v_1, v_2, \dots, v_k$. Each internal vertex along this path is then extended by attaching two leaf nodes, except for the last vertex v_k , which has exactly three leaf neighbors [9, 11].

Embedding Algorithm A

Input: The $S^+(n, K_3)$ and a CAT tree with $3^n + 1$ vertices.

Algorithm:

- Assign the vertex labeling scheme to $S^+(n, K_3)$ as described in Section 2.
- For the caterpillar, assign labels based on a preorder traversal of its structure.
- Define a vertex mapping $\varphi: V(A) \rightarrow V(B)$ such that each vertex $x \in V(A)$ is mapped to $\varphi(x) = x$.

Output: The algorithm produces the exact wirelength of the embedding.

Proof of Correctness: Let S^y , with $1 \leq y \leq 3^{n-1} - 1$, denote a cut edge along the backbone of the caterpillar, which divides it into two components, B^y and $\overline{B^y}$, such that the vertices of B^y are $\{1, 2, \dots, 3y\}$. Let A^y and $\overline{A^y}$ be the preimages of B^y and $\overline{B^y}$ under the mapping φ . By Theorem 2.3, A^y constitutes a maximum subgraph in $S^+(n, K_3)$. Consequently, each cut edge S^y (for $1 \leq y \leq 3^{n-1} - 1$) meets the conditions of Lemma 1.1, and the corresponding cost $C_\varphi(S^y)$ is minimized.

Next, let S_y^1 , with $1 \leq y \leq 2 \cdot 3^{n-1} + 1$, represent a cut edge among the pendant vertices of the caterpillar, which separates it into components B_y^1 and $\overline{B_y^1}$, where B_y^1 consists solely of individual vertices. Denote their preimages under φ as A_y^1 and $\overline{A_y^1}$. Again, by Theorem 2.3, A_y^1 is an largest set in $S^+(n, K_3)$. Therefore, each cut edge S_y^1 (for $1 \leq y \leq 2 \cdot 3^{n-1} + 1$) meets the criteria of Lemma 1.1, and $C_\varphi(S_y^1)$ achieves its minimum.

Theorem 3.2: The wirelength of $S^+(n, K_3)$, $n \geq 2$ into a CAT is given by

$$WL(S^{++}(p, q), CAT) = \frac{1}{2}(3^n + 9^n + 6) - 2 \sum_{y=1}^{3^{n-1}-1} I_A(3y).$$

Proof: By Lemma 1.1, we have $C_\varphi(S^y) = 9y - 2I_A(3y)$ for $1 \leq y \leq 3^{n-1} - 1$ and $C_\varphi(S_y^1) = 3$ for $1 \leq y \leq 2 \cdot 3^{n-1} + 1$. Then by Lemma 1.2, we can get the minimum wirelength as:

$$\begin{aligned} WL(S^+(n, K_3), CAT) &= \sum_{y=1}^{3^{n-1}-1} C_\varphi(S^y) + \sum_{y=1}^{2 \times 3^{n-1}+1} C_\varphi(S_y^1) \\ &= \frac{1}{2}(3^n + 9^n + 6) - 2 \sum_{y=1}^{3^{n-1}-1} I_A(3y). \end{aligned}$$

3.3 1-Hierarchical Caterpillar

This caterpillar is a specialized tree structure built by integrating multiple star graphs. Let S_k represent a star with k leaves. To construct this structure, a single leaf from each S_k is connected to a distinct vertex on a central caterpillar C . The resulting graph, denoted by $C_{k,t}$, maintains a caterpillar structure when the star components are removed, i.e., $C \setminus S_k$ forms a caterpillar [9, 12].

Embedding Algorithm B

Input : The $S^+(n, K_3)$ and a $C_{\frac{3^{n+1}}{2}, 2}$, $p \geq 3, q \geq 3$.

Algorithm:

- Assign the vertex labeling scheme to $S^+(n, K_3)$ as described in Section 2.
- labeling of vertices in $C_{\frac{3^{n+1}}{2}, 2}$ is done through preorder traversal.
- Define a vertex mapping $\varphi: V(A) \rightarrow V(B)$ such that each vertex $x \in V(A)$ is mapped to $\varphi(x) = x$.

Output: The algorithm produces the exact wirelength of the embedding.

Proof of Correctness: Let S^1 be a cut edge of $C_{\frac{3^{n+1}}{2}, 2}$ such that removing S^1 from $C_{\frac{3^{n+1}}{2}, 2}$ produces two components, B^1 and $\overline{B^1}$, with $V(B^1) = \{1, 2, \dots, \frac{3^{n+1}}{2}\}$. By Theorem 2.3, A^1 forms an optimal set in $S^+(n, K_3)$.

Next, let S'_y , for $1 \leq y \leq 2$, be a cut edge of $C_{\frac{3^{n+1}}{2}, 2}$ such that removing S'_y yields components B'_y and $\overline{B'_y}$ where the vertices of B'_y are $(y-1) \cdot \frac{3^{n+1}}{2} + 2$ to $y \cdot \frac{3^{n+1}}{2}$. Let $A'_y = A[\varphi^{-1}(V(B'_y))]$ and $\overline{A'_y} = A[\varphi^{-1}(V(\overline{B'_y}))]$. Again, by Theorem 2.3, each A'_y is an largest set in $S^+(n, K_3)$. Similarly, for each pendant cut edge S_y^n , with $1 \leq n \leq 2$ and $1 \leq y \leq 3^n - 3$, the corresponding preimage sets are optimal in $S^+(n, K_3)$. Since all these cut edges satisfy the criteria given in Lemma 1.1, it follows that the Wiener index under φ ,

$$WL_\varphi(S^+(n, K_3), C_{\frac{3^{n+1}}{2}, 2}), \text{ is minimized.}$$

Theorem 3.3: *The wirelength of embedding $S^+(n, K_3)$, $n \geq 2$ into -hierarchical caterpillar $C_{\frac{3^{k+1}}{2}, 2}$ is*

$$WL\left(S^+(n, K_3), C_{\frac{3^{k+1}}{2}, 2}\right) = 3n + 3^{n+1} - 2.$$

Proof: By Lemma 1.1 and Lemma 1.2,

$$\begin{aligned} WL\left(S^+(n, K_3), C_{\frac{3^{k+1}}{2}, 2}\right) &= C_\varphi(S_1) + \sum_{y=1}^2 C_\varphi(S'_y) + \sum_{n=1}^2 \sum_{y=1}^{3^n-3} C_\varphi(S_y^n) \\ &= (n+3) + \sum_{y=1}^2 (n+2) + \sum_{n=1}^2 \sum_{y=1}^{3^n-3} 3 \\ &= 3n + 3^{n+1} - 2. \end{aligned}$$

3.4 Banana tree $B_{3, 3^{n-1}}$

The banana tree graph $B(k, t)$ was defined by Nandini et al. [10]. This graph is formed by connecting the leaves of k copies of a t -star graph to a single root vertex that is separate from all the stars. A t -star graph is characterized by t vertices, where one vertex has a degree of $t-1$, and the rest of the vertices each have a degree of 1 [8, 13].

Embedding Algorithm C

Input: The extended Sierpinski graph $S^+(n, K_3)$ and a banana tree $B_{3, 3^{n-1}}$, $n \geq 3$.

1. Assign the vertex labeling scheme to $S^+(n, K_3)$ as described in Section 2.
2. labeling of vertices in $B_{3, 3^{n-1}}$ is done through preorder traversal.

3. Define a vertex mapping $\varphi: V(A) \rightarrow V(B)$ such that each vertex $x \in V(A)$ is mapped to $\varphi(x) = x$.

Output: The algorithm produces the exact wirelength of the embedding.

Proof of Correctness: Let S^y , for $y = 1, 2, 3$, be cut edges of $B_{3,3^{n-1}}$. Each S^y separates $B_{3,3^{n-1}}$ into two components, B^y and $\overline{B^y}$, where $|V(B^y)| = 3^{n-1}$. By Theorem 2.3, each A^y forms an optimal set in $S^+(n, K_3)$. Similarly, let S'_y , for $y = 1, 2, 3$, be another set of cut edges of $B_{3,3^{n-1}}$. Removing S'_y produces components B'_y and $\overline{B'_y}$ where $|V(B'_y)| = 3^{n-1} - 1$. Let $A'_y = A[\phi^{-1}(V(B'_y))]$ and $\overline{A'_y} = A[\phi^{-1}(V(\overline{B'_y}))]$. By Theorem 2.3, each A'_y is an optimal set in $S^+(n, K_3)$. For the pendant edges, denote the cut edges as S_y^q , with $y = 1, 2, 3$ and $1 \leq q \leq 3^{n-1} - 2$. Removing each S_y^q produces two components of $B_{3,3^{n-1}}$, whose inverse images in $S^+(n, K_3)$ are also optimal sets. By Lemma 1.1, the congestion of each such edge cut is minimized. Therefore, the Wirelength under ϕ , $WL_\phi(S^+(n, K_3), B_{3,3^{n-1}})$, is minimized.

Theorem 3.4: *The wirelength of embedding $S^+(n, K_3)$, $n \geq 3$ into $B_{3,3^{n-1}}$ is given by*

$$WL(S^+(n, K_3), B_{3,3^{n-1}}) = 3^{n+1} + 3.$$

Proof: By Lemma 1.1 and Lemma 1.2, the resulting wirelength is the minimum.

$$\begin{aligned} WL(S^+(n, K_3), B_{3,3^{n-1}}) &= \sum_{y=1}^3 \sum_{q=1}^{3^{n-1}-2} C_\phi(S_y^q) + \sum_{y=1}^3 C_\phi(S^y l) + \sum_{y=1}^3 \\ C_\phi(S'_y) &= \sum_{y=1}^{3^{n-1}-6} 3 + \sum_{y=1}^3 4 + \sum_{y=1}^3 3 = 3^{n+1} + 3. \end{aligned}$$

4. Conclusion

This work addressed the Maximum Subgraph Problem (MSP) for the extended Sierpiński graph $S^+(n, K_3)$ with $n \geq 2$, offering an exact solution in this specific case. However, the problem remains open for general values of m . In addition, we analyzed the minimum wirelength required for embedding $S^+(n, K_3)$ into various structured trees, including paths, caterpillars, 1-hierarchical caterpillars, and banana trees. These findings contribute to a deeper understanding of embedding efficiency in hierarchical networks.

Acknowledgment

The work is supported by the I-DAPT HUB Foundation, IIT (BHU) Varanasi, for funding under Technology Innovation Hub (TIH), Department of Science & Technology (DST), Government of India, with No. I-DAPT / IIT (BHU) / 2025-26 / Project Sanction / 30.

References

- [1] P. L. Joshwa, R. S. Rajan, T. M. Rajalaxmi, and I. N. Cangul, "Embedding of extended Sierpiński network $S^{++}(k,m)$ into certain trees," *RAIRO Operations Research*, vol. 59, no. 4, pp. 2279-2301 (2025).
- [2] S. L. Bezrukov, J. D. Chavez, L. H. Harper, M. Rotteger, and U. P. Schroeder, "Embedding of hypercube into grids," in *Mathematical Foundations of Computer Science*, vol. 1450, pp. 693-701 (1998).
- [3] P. Manuel, I. Rajasingh, B. Rajan, and H. Mercy, "Exact wirelength of hypercubes on a grid," *Discrete Applied Mathematics*, vol. 157, no. 7, pp. 1486-1495 (2009).
- [4] S. L. Bezrukov, "Edge isoperimetric problems on graphs," *Graph Theory and Combinatorial Biology*, vol. 7, pp. 157-197 (1999).
- [5] S. Klavžar and U. Milutinović, "Graphs $S(n,k)$ and a variant of the Tower of Hanoi problem," *Czechoslovak Mathematical Journal*, vol. 47, no. 1, pp. 95-104 (1997).
- [6] R. S. Rajan, A. B. Greeni, and P. L. Joshwa, "Maximum subgraph problem and minimum linear arrangement of generalized Sierpiński graphs," *Journal of Graph Algorithms and Applications*, vol. 27, no. 9, pp. 767-782 (2023).
- [7] L. H. Harper, "The edge-isoperimetric problem on Sierpiński graph: Final resolution," arXiv preprint arXiv:1802.08355 (2018).
- [8] P. L. Joshwa, R. S. Rajan, and T. M. Rajalaxmi, "Maximum subgraph and wirelength optimization in cyclic bipartite networks $G_{k,3}$," *Int. J. Parallel, Emergent Distrib. Syst.*, pp. 1-21 (Jan. 2026), doi: 10.1080/17445760.2025.2610814.
- [9] P. L. Joshwa, R. S. Rajan, and T. M. Rajalaxmi, "Maximum subgraph and wirelength analysis of extended Sierpiński networks in parallel computing," *Discrete Applied Mathematics*, vol. 383, pp. 367-382 (2026).
- [10] G. K. Nandini, S. Klavžar, T. M. Rajalaxmi, and R. S. Rajan, "A note on eccentricity based topological indices of honeycomb, oxide and 2-power interconnection networks," *Journal of Discrete Mathematical Sciences and Cryptography*, vol. 26, no. 1, pp. 231-253 (2023).
- [11] S. Arulanand, R. S. Rajan, S. Prabhu, and S. Stephen, "Certain domination numbers for Cartesian product of graphs," *Journal of Discrete Mathematical Sciences and Cryptography*, vol. 27, no. 3, pp. 1045-1058 (2024).

- [12] A. Khobragade, R. Mahajan, H. Langi, R. Mundhe, and S. Ghumbre, "Effective negative triplet sampling for knowledge graph embedding," *Journal of Information and Optimization Sciences*, vol. 43, no. 8, pp. 2075-2087 (2022).
- [13] G. M. Jose, K. N. Geetha, and K. Somasundaram, "Double power domination in graphs," *Journal of Discrete Mathematical Sciences and Cryptography*, vol. 28, no. 6, pp. 2261-2278 (2025).
- [14] S. Klavzar and B. Mohar, "Crossing numbers of Sierpiński-like graphs," *Journal of Graph Theory*, Vol. 50, no. 3, pp. 186–198 (2005).
- [15] L. H. Harper, "Optimal assignments of numbers to vertices," *Journal of the Society for Industrial and Applied Mathematics*, Vol. 12, no. 1, pp. 131–135 (1964).

Received May, 2025