

Exploring Even-Even graceful labeling for product of star graphs

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Abstract

An Even-Even graceful labeling involves assigning even labels to both vertices and edges in a graph. The goal is to create a labeling where vertex and edge labels are even integers, and the differences between vertex and edge labels form a set of consecutive positive even integers. This concept explores the relationship between numbers and graph structures and has potential applications in computer science, optimization, and network analysis. This paper demonstrates some new results on Even-Even graceful labeling of m star graphs for $m = 4, 5, 6$. Additionally, it is shown that double, triple, and quadruple extended star graphs are Even-Even graceful under certain conditions, as discussed in this paper.

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Keywords: Graph labeling, Graceful labeling, Even-Even graceful labeling.

1. Introduction

An Even-Even graceful labeling is a mathematical concept related to graph theory and combinatorics. Given an undirected graph, a labeling of the vertices is considered “graceful” if edge labels, obtained by assigning non negative integers to vertices such that each edge is labeled with the absolute difference between its endpoint vertices’ labels, form a set of consecutive integers from 1 to q (number of edges). This concept explores the relationship between numbers and graph structures and has potential applications in computer science, optimization, and network analysis.

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Rosa [8] introduced a focus on graceful graphs and methods for graceful labeling. Earlier, he named graceful labeling as β -valuation. By referring to the latest survey of Gallian [6] on graph labeling problems one perceives the various graph labeling problems. Ahmed et al. [1] introduce and explore a new concept in graph theory that combines graceful and antimagic labeling methods. This research defines and investigates the properties of these combined labeling techniques within graph theory. Basher [2] proves that the splitting graphs of several standard graphs are odd-even graceful graphs. Basher [3] proves that specific classes of standard geometric graphs namely, planar grid graphs and prism graphs which admit an odd-even graceful labeling. Behera, Mishra, and Nayak [4] prove that several families of graphs adhere to Even-Even graceful and Odd-Even graceful labelings. Boonklurb, Ruamkaew, and Singhun [5] focus on a specific type of composite graph and prove conditions under which it can be assigned a directed edge-graceful labeling. Lau et al. [7] establish that some graphs possess an extremal maximum vertex degree. Kumar, Anbarasan, and Pushparaj [9] proved edge odd graceful labeling for some graphs. Senthilkumar and Venkatesan [10] proves several classes of graphs to prove they are odd vertex even edge root square mean graphs.

There are different types of graph labeling available so far like graceful labeling, Odd graceful labeling, Even graceful labeling, Odd-Even graceful labeling, Even-Even graceful labeling, prime labeling, super edge graceful labeling, etc. Our work is justified with Even-Even graceful labeling for product of m star graphs for $m = 4, 5, 6$ together with double, triple and quadruple extended star graphs for odd natural number.

Definition 1.1: Suppose G is a graph with p vertices and q edges, graceful labeling of graph is an one-one mapping from vertex set of the graph to a set whose elements are $0, 1, 2, \dots, q$ such that absolute value of difference between two consecutive vertex labeling is distinct integer being an element of the set $\{1, 2, 3, \dots, q\}$. A graceful graph is one that permits graceful labeling.

Definition 1.2: Let G is a graph with p vertices and q edges, Even-Even graceful labeling of graph is one-one mapping from set of vertices to set with even integers from set $\{0, 2, 4, \dots, 2k - 2\}$, where $k = \max\{p, q\}$ and absolute value of difference between two consecutive vertex labeling is a distinct even integer being an element of the set $\{2, 4, 6, \dots, 2q\}$, an Even-Even graceful graph is one that permits Even-Even graceful labeling.

Definition 1.3: A star graph is a complete bipartite graph with one central vertex joined with other remaining vertices of degree one. There are m central vertices in m -star graphs.

Definition 1.4: The central path of a graph is a path that passes through the centers of the graph.

2. Main Results

Theorem 2.1: The 4-star $S_n \oplus S_{n+2} \oplus S_{n+2} \oplus S_n$ exhibits Even-Even graceful labeling for $n \geq 2$.

Proof: Suppose $S_n \oplus S_{n+2} \oplus S_{n+2} \oplus S_n$ be a 4-star graph with central path P_4 such that both ends u_1, u_4 of the central path are joined to n number of vertices, while each internal vertices u_2, u_3 are joined to $n+2$ vertices. Total number of vertices and edges are $4n+4$ and $4n+3(=q)$ respectively. Suppose x_i, y_i, z_i and $v_i, i=1,2,3,\dots,n$ are joined with u_1, u_2, u_3 and u_4 respectively. The vertex labeling is given by

$$\begin{aligned} f(x_i) &= 2q - 2i + 2, f(y_i) = q - 2n - 2i - 1, f(z_i) = 2q - 2n - 2i, \\ f(v_i) &= q - 2i + 1, i = 1, 2, \dots, n, \\ f(u_1) &= 0, f(u_2) = 6n + 6, f(u_3) = 2n + 2, f(u_4) = 4n + 4 \end{aligned}$$

And the edge labeling is given by

$$\begin{aligned} f(u_1 x_i) &= 2q - 2i + 2, f(u_2 y_i) = 8n - q + 2i + 7, f(u_3 z_i) = 2q - 4n - 2i - 2, \\ f(u_4 v_i) &= 4n - q + 2i + 3, i = 1, 2, \dots, n, \\ f(u_1) &= 6n + 6, f(u_2) = 4n + 4, f(u_3) = 2n + 2 \end{aligned}$$

So, the edge labeling of the above graph contains a set of even integers ranging from 2 to $2q$. Hence the graph is Even-Even graceful.

Illustration 2.1: Theorem 2.1 is explained in Figure 1 as follows for $n = 6$.

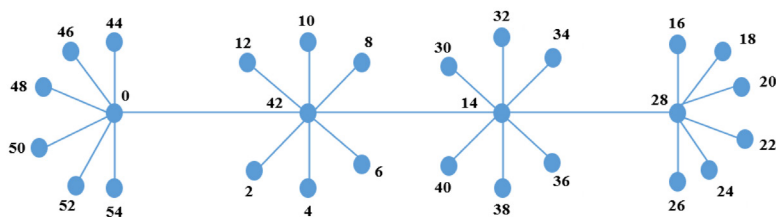


Figure 1

$S_n \oplus S_{n+2} \oplus S_{n+2} \oplus S_n$ with Even-Even graceful labeling.

Theorem 2.2: The 5-star $S_n \oplus S_{n+2} \oplus S_{n+2} \oplus S_{n+2} \oplus S_n$ is classified as Even-Even graceful for $n \geq 2$.

Proof: Suppose $S_n \oplus S_{n+2} \oplus S_{n+2} \oplus S_{n+2} \oplus S_n$ be a 5-star graph with central path P_5 such that both ends u_1, u_5 of the central path are joined to n number of vertices, while each internal vertices u_2, u_3, u_4 are joined to $n+2$ vertices. Total number of vertices and edges are $5n+5$ and $5n+4(=q)$ respectively. Suppose x_i, y_i, z_i, v_i and $w_i, i=1, 2, 3, \dots, n$ are joined with u_1, u_2, u_3, u_4 and u_5 respectively. The vertex labeling is given by

$$\begin{aligned} f(x_i) &= 2q - 2i + 2, f(y_i) = q - 3n - 2i - 2, f(z_i) = 2q - 2n - 2i, \\ f(v_i) &= q - n - 2i, f(w_i) = q + n - 2i + 2, i = 1, 2, \dots, n, \\ f(u_1) &= 0, f(u_2) = 8n + 8, f(u_3) = 2n + 2, f(u_4) = 6n + 6, f(u_5) = 4n + 4 \end{aligned}$$

And the edge labeling is given by

$$\begin{aligned} f(u_1 x_i) &= 2q - 2i + 2, f(u_2 y_i) = 11n - q + 2i + 10, f(u_3 z_i) = 2q - 4n - 2i - 2, \\ f(u_4 v_i) &= 7n - q + 2i + 6, f(u_5 w_i) = q - 3n - 2i - 2, i = 1, 2, \dots, n, \\ f(u_1 u_2) &= 8n + 8, f(u_2 u_3) = 6n + 6, f(u_3 u_4) = 4n + 4, f(u_4 u_5) = 2n + 2 \end{aligned}$$

So, the edge labeling of the above graph contains a set of even integers ranging from 2 to $2q$. Hence the graph is Even-Even graceful.

Illustration 2.2: Theorem 2.2 is explained in Figure 2 as follows for $n = 6$.

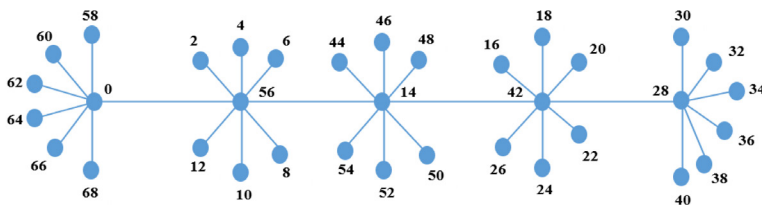


Figure 2

$S_n \oplus S_{n+2} \oplus S_{n+2} \oplus S_{n+2} \oplus S_n$ with Even-Even graceful labeling.

Theorem 2.3: The 6-star $S_n \oplus S_{n+2} \oplus S_{n+2} \oplus S_{n+2} \oplus S_{n+2} \oplus S_n$ is classified as Even-Even graceful for $n \geq 2$.

Proof: Suppose $S_n \oplus S_{n+2} \oplus S_{n+2} \oplus S_{n+2} \oplus S_{n+2} \oplus S_n$ be a 6-star graph with central path P_6 such that both ends u_1, u_6 of the central path are joined to n number of vertices, while each internal vertices u_2, u_3, u_4, u_5 are joined to $n+2$ vertices. Total number of vertices and edges are $6n+6$ and

$6n+5(=q)$ respectively. Suppose x_i, y_i, z_i, v_i, w_i and $t_i, i=1,2,3,\dots,n$ be the vertices joined with vertices u_1, u_2, u_3, u_4, u_5 and u_6 respectively. The vertex labeling is given by

$$\begin{aligned} f(x_i) &= 2q - 2i + 2, f(y_i) = q - 4n - 2i - 3, f(z_i) = 2q - 2n - 2i, \\ f(v_i) &= q - 2n - 2i - 1, f(w_i) = q + 2n - 2i + 3, f(t_i) = q - 2i + 1, i = 1, 2, \dots, n, \\ f(u_1) &= 0, f(u_2) = 10n + 10, f(u_3) = 2n + 2, f(u_4) = 8n + 8, \\ f(u_5) &= 4n + 4, f(u_6) = 6n + 6 \end{aligned}$$

And the edge labeling is given by

$$\begin{aligned} f(u_1x_i) &= 2q - 2i + 2, f(u_2y_i) = 14n - q + 2i + 13, f(u_3z_i) = 2q - 4n - 2i - 2, \\ f(u_4v_i) &= 10n - q + 2i + 9, f(u_5w_i) = q - 2n - 2i - 1, \\ f(u_6t_i) &= 6n - q + 2i + 5, i = 1, 2, \dots, n, \\ f(u_1u_2) &= 10n + 10, f(u_2u_3) = 8n + 8, f(u_3u_4) = 6n + 6, f(u_4u_5) = 4n + 4, \\ f(u_5u_6) &= 2n + 2 \end{aligned}$$

So, the edge labeling of the above graph contains a set of even integers ranging from 2 to $2q$. Hence the graph is Even-Even graceful.

Illustration 2.3: Theorem 2.3 is explained in Figure 3 as follows for $n = 6$.

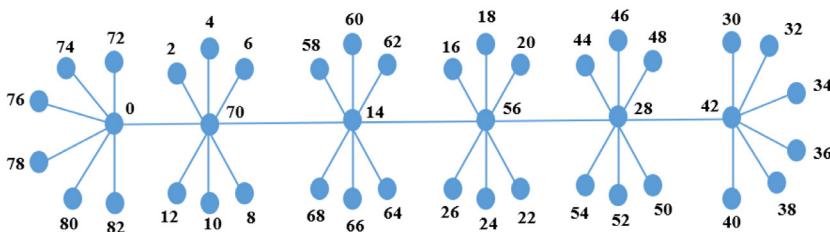


Figure 3

$S_n \oplus S_{n+2} \oplus S_{n+2} \oplus S_{n+2} \oplus S_{n+2} \oplus S_n$ with Even-Even graceful labeling.

Theorem 2.4: $K_{1,n,n}$ is Even-Even graceful for all odd natural numbers n .

Proof: Consider the set of vertices as $\{u, u_i, v_i\}$, where the root vertex u is extended upto v_i through u_i for $i = 1, 2, \dots, n$. Let $E(K_{1,n,n}) = \{uu_i, u_i v_i\}$ be edge set. Total number of vertices and edges are $2n+1$ and $2n(=q)$ respectively.

The vertex labeling is $f(u) = 0$,

$$f(u_i) = \begin{cases} 4n - 2i + 2, & i = \text{odd} \\ 2i, & i = \text{even} \end{cases}, f(v_i) = \begin{cases} 2i, & i = \text{odd} \\ 4n - 2i + 2, & i = \text{even} \end{cases}.$$

And the edge labeling is

$$f(uu_i) = \begin{cases} 4n - 2i + 2, & i = \text{odd} \\ 2i, & i = \text{even} \end{cases}, f(u_i v_i) = |4n - 4i + 2|.$$

So, the edge labeling of the above graph contains a set of even integers ranging from 2 to $2q$. Hence the graph is Even-Even graceful.

Illustration 2.4: Theorem 2.4 is explained in Figure 4 as follows for $n = 5$.

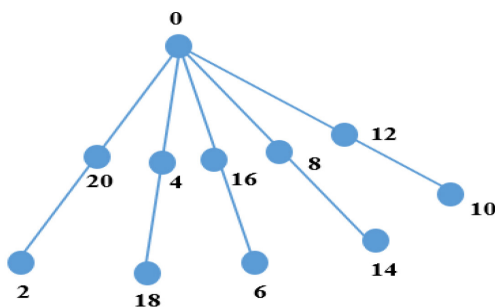


Figure 4

$K_{1,n,n}$ with Even-Even graceful labeling.

Theorem 2.5: $K_{1,n,n,n}$ is Even-Even graceful for all odd natural numbers n .

Proof: Consider the set of vertices as $\{u, u_i, v_i, w_i\}$, where the root vertex u is extended upto w_i through $u - u_i - v_i - w_i$ for $i = 1, 2, \dots, n$. Total number of vertices and edges are $3n + 1$ and $3n (= q)$ respectively. Set of edges is defined as $E(K_{1,n,n,n}) = \{uu_i, u_i v_i, v_i w_i\}$.

The vertex labeling is $f(u) = 0$,

$$f(u_i) = \begin{cases} 6n - i + 1, & i = \text{odd} \\ i, & i = \text{even} \end{cases}, f(v_i) = \begin{cases} n + i, & i = \text{odd} \\ 5n - i + 1, & i = \text{even} \end{cases}$$

$$f(w_i) = \begin{cases} 4n - i + 1, & i = \text{odd} \\ 2n + i, & i = \text{even} \end{cases}.$$

And the edge labeling is

$$f(uu_i) = \begin{cases} 6n-i+1, & i = \text{odd} \\ i, & i = \text{even} \end{cases}$$

$$f(u_i v_i) = |5n-2i+1|, f(v_i w_i) = |3n-2i+1|, i = 1, 2, \dots, n.$$

So, edge labeling of above graph contains a set of even integers ranging from 2 to $2q$. Hence the graph is Even-Even graceful.

Illustration 2.5: Theorem 2.5 is explained in Figure 5 as follows for $n = 5$.

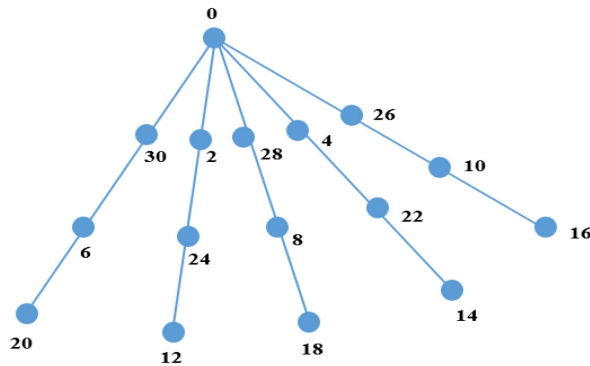


Figure 5

$K_{1,n,n,n}$ with Even-Even graceful labeling.

Theorem 2.6: $K_{1,n,n,n,n}$ is Even-Even graceful for all odd natural numbers n .

Proof: Consider the set of vertices as $\{u, u_i, v_i, w_i, x_i\}$, where the root vertex u is extended upto x_i through $u-u_i-v_i-w_i-x_i$ for $i=1, 2, \dots, n$. Total number of vertices and edges are $4n+1$ and $4n(=q)$ respectively.

Set of edges is defined as $E(K_{1,n,n,n,n}) = \{uu_i, u_i v_i, v_i w_i, w_i x_i\}$.

Vertex labeling is $f(u) = 0$,

$$f(u_i) = \begin{cases} 8n-i+1, & i = \text{odd} \\ i, & i = \text{even} \end{cases}, f(v_i) = \begin{cases} n+i, & i = \text{odd} \\ 7n-i+1, & i = \text{even} \end{cases}$$

$$f(w_i) = \begin{cases} 6n-i+1, & i = \text{odd} \\ 2n+i, & i = \text{even} \end{cases}, f(x_i) = \begin{cases} 3n+i, & i = \text{odd} \\ 5n-i+1, & i = \text{even} \end{cases}$$

Edge labeling is

$$f(uu_i) = \begin{cases} 8n - i + 1, & i = \text{odd} \\ i, & i = \text{even} \end{cases}$$

$$f(u_i v_i) = |7n - 2i + 1|, f(v_i w_i) = |5n - 2i + 1|, f(w_i x_i) = |3n - 2i + 1|, i = 1, 2, \dots, n$$

So, edge labeling of above graph contains a set of even integers ranging from 2 to $2q$. Hence the graph is Even-Even graceful.

Illustration 2.6: Theorem 2.6 is explained in Figure 6 as follows for $n = 5$.

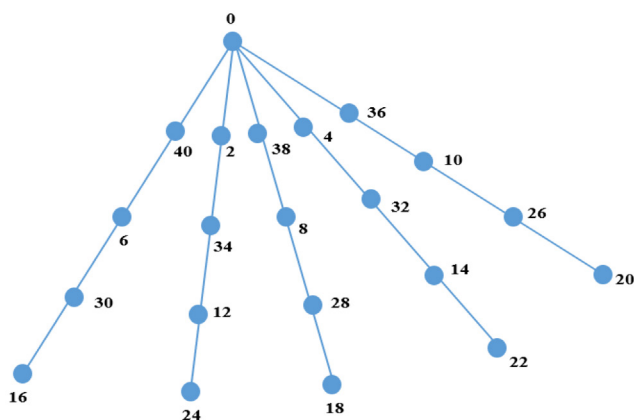


Figure 6

$K_{1,n,n,n,n}$ with Even-Even graceful labeling.

3. Discussion and Conclusion

This paper justifies that 4-star, 5-star, and 6-star graphs are Even-Even graceful. Furthermore, double, triple, and quadruple extended star graphs are Even-Even graceful for all odd natural numbers n . One may extend Even-Even gracefulness of n -star graph for $n \geq 7$. Similarly, one may proceed to show Even-Even gracefulness of n extended star graphs for $n \geq 6$.

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