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Dominant metric dimension of rooted product graph

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Abstract

Poverty is closely related to the problem of unequal distribution of basic needs across regions. To address this challenge, graph theory provides a rigorous framework for designing efficient distribution networks that ensure accessibility while minimizing cost. In this work, we examine the dominant metric dimension, explained as the smallest possible size of a set that is both resolving and dominating. The primary finding shows that for the graph formed via the rooted product $G \circ \boldsymbol{\mathcal{H}}$, this dimension aligns to the sum of the dominant metric dimensions of all graphs within the family $\boldsymbol{\mathcal{H}}$. When the sequence consists of complete

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graph, star graph, and complete bipartite graph with the rooted vertex at a pendant, the result should be reduced by *n*. These findings provide insight into how the number of distribution centers can be optimized in various network structures. From the perspective of poverty alleviation, this result implies that efficient placement of distribution hubs can reduce infrastructure costs while ensuring that every community is within reach of essential goods. This paper determines the dominant metric dimension of the specified product $G \circ \mathcal{H}$, represented as $Ddim(G \circ \mathcal{H})$. The findings indicate that the value of $Ddim(G \circ \mathcal{H})$ depends on the structural properties of the graphs within the sequence $\mathcal{H}$. When $\mathcal{H}$ consists of rooted cycle graphs, rooted star graphs, and rooted path graphs, the dominant metric dimension of $G \circ \mathcal{H}$ is equal to the total sum of the dominant metric dimensions of each element in $\mathcal{H}$. On the other hand, when H consists of rooted complete graph, $Ddim(G \circ \mathcal{H})$ is the sum of their individual dominant metric dimensions, each reduced by one.

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1. Introduction

As time goes by, the problems faced by humans also develop. With their intellectual abilities, humans continue to develop their knowledge to address the challenges they face. Likewise, in the field of mathematics, including graphs and combinatorics, there has been rapid development. One such development is the concept of metric dimension, which has evolved into the dominant metric dimension.

The concept of distance in graphs was developed to define a basis of a graph. A basis of a graph is a set with minimal cardinality that can distinguish every vertex in the graph. A basis of a graph distinguishes each vertex using the concept of ordered pairs, where the elements of an ordered pair contain the distances between the distinguished vertex and each vertex in the basis, respectively.

In graph theory, let $G = (V(G), E(G))$ be a connected graph. The distance separating the vertices of G and certain subset of $V(G)$ are defined as the resolving set, metric dimension and basis, as presented in [1]. The expression "metric dimension" was subsequently formalized by Harary and Melter [2]. Subsequent research has expanded upon these foundations through theoretical explorations [3], [4], [5] and investigations into graph operations [6], [7], [8], [9].

A separate concept is that of a dominant set. A set $S \subseteq V(G)$ is called a dominating set if every vertex not in S are connected to at least one vertex in S. Among all sets with this property, the one with the smallest cardinality defines the domination number $\gamma(G)$. Susilowati et al. [10] integrated these ideas by defining the dominant metric dimension, denoted $Ddim(G)$, which corresponds to the smallest possible cardinality of a set that fulfills both the resolving and dominating criteria.

This study analyzes graphs construct through the rooted product, focusing on their dominant metric dimension [11]. Specifically, we examine the rooted product of a connected graph G and a sequence of graphs $\mathcal{H}$, consisting of star, complete, path, cycle, and complete bipartite structures.

In addition, research on metric dimension in graph operations has attracted considerable attention. For instance, Yero, Kuziak, and Rodríguez-Velázquez [6], Saputro et al. [7], Saputro, Mardiana, and Purwasih [8], and Iswadi et al. [9] investigated metric-related parameters under various graph operations. In particular, Sardar et al. [12] studied opposition distance within certain families of rooted product graphs using the generalized inverse of the Laplacian matrix, thereby underscoring the significance of rooted product constructions for the analysis of distance-based graph invariants.

The properties of resolving set and dominant resolving set in graphs are presented below.

Lemma 1.1: [10]. *Consider a connected graph G and $W \subseteq V(G)$ that serves as a resolving. Therefore, the examples of any two distinct vertices within W are unique.*

Lemma 1.2: [10]. *If every subset of $V(G)$ with k elements is not resolving, any subset of size smaller than k cannot function as one.*

Lemma 1.3: [9]. *In a connected graph G, if a set is resolving, any of its supersets will also maintain that resolving property.*

Some research outcomes related to the dominant metric dimension of several special graphs are presented below.

Theorem 1.1: [10].

i. $Ddim(G) = n - 1$ *if G is a star graph whose order satisfies $n \geq 2$.*
ii. $Ddim(K_n) = Dim(K_n) = n - 1$ *whenever K_n denotes a complete graph of order $n \geq 2$.*
iii. $Ddim(P_n) = \gamma(P_n) = \left\lceil \frac{n}{3} \right\rceil$ *for a path graph P_n with $n > 4$.*
iv. $Ddim(C_n) = \gamma(C_n) = \left\lceil \frac{n}{3} \right\rceil$ *for a cycle graph C_n where $n \geq 7$.*
V. $\mathrm{Ddim}(K_{m,n}) = \mathrm{dim}(K_{m,n}) = m + n - 2$ when $K_{m,n}$ represents a complete bipartite graph with $m, n \geq 2$.

The dominant metric dimensions of the special graphs above are used to help find the metric dimensions of the rooted product graphs involving them. The following presents some of the terms used in this study

Definition 1.1: [9]. A vertex u of a graph G is a dominating vertex if it has an edge to any other vertex of G.

Definition 1.2: [13]. A vertex whose degree equals one is said to be a pendant vertex, while its adjacent edge is termed a pendant edge.

Definition 1.3: [14]. Consider G be a connected graph of order n, and $\mathcal{H} = H_1, H_2, H_3, \dots, H_n$ be a sequence of n rooted graphs such that, for each i, the vertex $v_{1i} \in V(H_i)$ is the designated root of H_i. $G \circ \mathcal{H}$ defined as the roots product of G and the sequence $\mathcal{H}$, is the graph formed byattaching each H_i to the i −th of the G through their respective roots.

Technically, the vertex set of $G \circ \mathcal{H}$ is obtained as follows. Consider G be a connected graph of order n, $\mathcal{H}$ consists of n rooted graph $H_1, H_2, H_3, \dots, H_n$ and $v_{i1} \in V(H_i)$ serves as the designated root of H_i. As for $G \circ \mathcal{H}$, the set vertices $V(G \circ \mathcal{H}) = \{(u_i, v_{ij}) | u_i \in V(G), v_{ij} \in V(\mathcal{H}), i = 1,2, 3, \dots, n;\ j = 1,2,3, \dots, |V(H_i)|\}$ and $(u_i, v_{ij})(u_k, v_{kl}) \in E(G \circ \mathcal{H})$, with $u_i = u_k$ exactly when $v_{ij}v_{kl} \in E(H_i)$ or $u_i \neq u_k$ exactly when $u_iu_k \in E(G)$. More precisely, $u_iu_k \in E(G)$ if and only if $v_{ij} = u_{i1}, v_{kl} = u_{k1}, i \neq k$.

This study employs the notation $\overset{i}{S}_r$, $\overset{i}{K}_m$, and $\overset{i}{K}_{p,q}$ to represent the i-th copy of rooted star graph (order r), complete bipartite graph (order $p + q$), and complete graph (order m), respectively. Some results $Ddim(G \circ \mathcal{H}) = \sum_{i=1}^{n} Ddim(S_r^i)$ and $Ddim(G \circ \mathcal{H}) = \sum_{i=1}^{n}(Ddim(K_m^i) - 1)$ presented in [15].

2. Dominant Resolving Set of Rooted Product Graph

The subsequent section examines the idea of a dominant resolving set in rooted product graphs, followed by the development of the dominant metric dimension for this graph construction. By establishing specific conditions under which vertices in different H_i can be distinguished using distances, we lay the groundwork for $G \circ \mathcal{H}$. If vertices u and v of graph G are connected by an edge e, then vertex u and v are to be adjacent, written as $u \sim v$. Henceforth, the graph used in the Lemma and Theorem below is a connected graph

Lemma 2.1: *Consider G be a connected graph with order* n *with vertex set $\{u_i | i = 1,2, \dots, n\}$ and let $\mathcal{H}$ be a sequence of any connected graphs H_i with vertex set $\{v_{ij} | j = 1,2, \dots, |V(H_i)|\}$ for $i = 1,2, \dots, n$ then $d((u_i, v_{i1}), (u_i, v_{ip})) \neq d((u_q, v_{q1}), (u_i, v_{ip}))$ where $i \neq q$.*

Proof: Let vertex set of G be $\{u_i | i = 1,2, \dots, n\}$ and let $V(H_i) = \{v_{ij} | j = 1,2, \dots, |V(H_i)|\}$ for all i and $V(G \circ \mathcal{H}) = \{(u_i, v_{ij}) | u_i \in V(G); v_{ij} \in V(H_i)\}$. Since $d((u_q, v_{q1}), (u_i, v_{ip})) = d((u_q, v_{q1}), (u_i, v_{i1})) + d((u_i, v_{i1}), (u_i, v_{ip}))$, therefore proved $d\left((u_i, v_{i1}), (u_i, v_{ip})\right) \neq d\left((u_q, v_{q1}), (u_i, v_{ip})\right)$

After obtaining **Lemma 2.1**, the next is to determine which particular vertices can be selected as a set capable of distinguishing other vertices in the graph. This is given in **Lemma 2.2.**

Lemma 2.2: *If $G \circ \mathcal{H}$ represents the rooted product of a connected graph G with n order, and $\mathcal{H} = \{H_1, H_2, \dots, H_n\}$, where each H_i is a path graph whose order exceeds four. If the root vertex of every H_i corresponds to a pendant vertex, then $W = \{(u_i, v_{i2}), (u_i, v_{i5})\}$ forms a resolving set of $G \circ \mathcal{H}$.*

Proof: Let $W = \{(u_i, v_{i2}), (u_i, v_{i5})\}$. Take any vertex $(u_i, v_{i1}), (u_q, v_{q1}) \in V(G \circ \mathcal{H})$, $i \neq q$, by **Lemma 2.1** we got that $r((u_i, v_{i1})|W) \neq r((u_q, v_{q1})|W)$. Then for $(u_i, v_{ip}) \in V(G \circ \mathcal{H})$,

$$d\left((u_i, v_{ip}), (u_i, v_{i2})\right) = \begin{cases} 1, p = 1 \\ p-2, p > 1 \end{cases}$$

$$d\left((u_i, v_{ip}), (u_i, v_{i5})\right) = \begin{cases} 5-p, p \leq 4 \\ p-5, p > 4 \end{cases}$$

Hereafter, for every $(u_i, v_{ip}), (u_q, v_{qr}) \in V(G \circ \mathcal{H})$ then $r((u_i, v_{ip})|W) \neq r((u_q, v_{qr})|W), i \neq q$ and $p \neq r$. Thus, W constitutes a resolving of $G \circ \mathcal{H}$

The existence of the distinguishing set, as shown in **Lemma 2.2**, provides an important foundation for further analysis. This result validates that a given set can uniquely distinguish every vertex in the graph. With this knowledge, we can proceed to verify the primary result of **Theorem 2.1.**

Theorem 2.1: *In a connected graph G with n order and $\mathcal{H} = \{H_1, H_2, \dots, H_n\}$ is the sequence of path graphs of order more than four. If the rooted vertex of path graph is the pendant vertex of the graph, then $Ddim(G \circ \mathcal{H}) = \sum_{i=1}^{n} Ddim(H_i)$.*

Proof: Let vertex set of G be $\{u_i | i = 1,2, \dots, n\}$ and let $V(H_i) = \{v_{ij} | j = 1,2, \dots, |V(H_i)|\}$ for all i. The rooted vertex of path graph is the pendant vertex of the graph, and v_{i1} is the rooted vertex of H_i, for all i. For $u_i \in V(G)$ to $v_{i1} \in V(H_i)$ so we obtained

$V(G \circ \mathcal{H}) = \{(u_i, v_{ij}) | u_i \in V(G); v_{ij} \in V(H_i)\}$. There are two cases for the path graph, they are when the $|V(H_i)|$ is a multiple of three and when the $|V(H_i)|$ is not multiple of three, then

1. For $|V(H_i)| = 3k, k \in N$, select $W_i = \{v_{i2}, v_{i5}, v_{i8}, \dots, v_{i(3j-1)}\}$ so that $W_i = \left\lceil \frac{|V(H_i)|}{3} \right\rceil$
2. For $|V(H_i)| \neq 3k, k \in N$, select $W_i = \{v_{i2}, v_{i5}, \dots, v_{i(3j-1)}, v_{i|V(H_i)|}\}$ so that
$$W_i = \left\lceil \frac{|V(H_i)|}{3} \right\rceil$$

WLOG, W_i is a dominant basis of the path graph H_i as explained **Theorem 1.1 iii.** Choose $W = \bigcup_{i=1}^{n} \{(u_i, v_{ij}) | v_{ij} \in W_i\}$ so that $|W| = \sum_{i=1}^{n} Ddim(H_i)$. Since $\{(u_i, v_{i2}), (u_i, u_{i5})\} \subseteq W$ and from **Lemmas 1.3** and **2.2** imply that W is, in addition, a resolving set. Now, we will prove that W is a dominant set of

$G \circ \mathcal{H}$. Since for $i = 1,2,\dots,n$, $E(H_i) = \{v_{ij}v_{i(j+1)} | j = 1,2,\dots,|V(H_i)| - 1\}$, we get that $(u_i, v_{i(3k-2)}) \sim (u_i, v_{i(3k-1)})$ and $(u_i, v_{i(3k-1)}) \sim (u_i, v_{i(3k)})$. It is follows that W forms dominant set of $G \circ \mathcal{H}$ and consequently W serves as a dominant resolving set for $G \circ \mathcal{H}$. We further aim to establish that W achieves the minimum cardinality among all dominant resolving sets. Take $S \subseteq V(G \circ \mathcal{H})$ with $|S| < |W|$. Let $|S| = |W| - 1$. Then there is an index i such that S comprises at most $|W_i| - 1$ members of the vertex set $V(H_i)$. Consequently there exists a vertex who is not a neighbor of anyone in S so that S is not dominant set of $G \circ \mathcal{H}$. From the previous analysis and by **Lemma 1.2**, we can concluded that $W = \bigcup_{i=1}^{n} \{(u_i, v_{ij}) | v_{ij} \in W_i\}$ is a dominant basis set of $G \circ \mathcal{H}$. Thus, it is proven that for all connected graphs of order n and a sequence of path graphs of order more than four. If the rooted vertex of path graph is the pendant vertex of the graph, then $Ddim(G \circ \mathcal{H}) = \sum_{i=1}^{n} Ddim(H_i)$

This lemma is aimed at determining the resolving set of $G \circ H$, where the graph $\mathcal{H}$ is a sequence of cycle graphs of order more than six.

Lemma 2.3: *If $G \circ \mathcal{H}$ is a graph formed by rooted product between a connected graph G with n order, and $\mathcal{H} = \{H_1, H_2, \dots, H_n\}$, where each H_i is a cycle of order exceeding six, then $W = \{(u_i, v_{i1}), (u_i, v_{i4})\}$ is a resolving set of $G \circ \mathcal{H}$.*

Proof: Let $W = \{(u_i, v_{i1}), (u_i, v_{i4})\}$. Take any vertex $(u_i, v_{i1}), (u_q, v_{q1}) \in V(G \circ \mathcal{H})$, $i \neq q$, by **Lemma 2.1** we got that $r((u_i, v_{i1})|W) \neq r((u_q, v_{q1})|W)$. If $|V(H_i)|$ is odd, then for every $(u_i, v_{ip}) \in V(G \circ \mathcal{H})$, imply

$$d\left((u_i, v_{ip}), (u_i, v_{i1})\right) = \begin{cases} p-1, p = 1,2,\dots,\left\lceil \frac{|V(H_i)|}{2} \right\rceil \\ \left\lfloor \frac{|V(H_i)|}{2} \right\rfloor, p = \left\lceil \frac{|V(H_i)|}{2} \right\rceil + 1 \\ |V(H_i)| + 1 - p, p = \left\lceil \frac{|V(H_i)|}{2} \right\rceil + 2, \left\lceil \frac{|V(H_i)|}{2} \right\rceil + 3, \dots, |V(H_i)| \end{cases}$$

$$d\left((u_i, v_{ip}), (u_i, v_{i4})\right) = \begin{cases} |4-p|, p = 1,2,\dots,\left\lceil \frac{|V(H_i)|}{2} \right\rceil + 3 \\ \left\lfloor \frac{|V(H_i)|}{2} \right\rfloor, p = \left\lceil \frac{|V(H_i)|}{2} \right\rceil + 4 \\ |V(H_i)| + 1 - p, p = \left\lceil \frac{|V(H_i)|}{2} \right\rceil + 5, \left\lceil \frac{|V(H_i)|}{2} \right\rceil + 6, \dots, |V(H_i)| \end{cases}$$

for even $|V(H_i)|$, we have

$$d\left((u_i, v_{ip}), (u_i, v_{i1})\right) \begin{cases} p-1, p = 1,2,\dots,\frac{|V(H_i)|}{2} + 1 \\ |V(H_i)| + 1 - p, p = \frac{|V(H_i)|}{2} + 2, \frac{|V(H_i)|}{2} + 3, \dots, |V(H_i)| \end{cases}$$

$$d\left((u_i, v_{ip}), (u_i, v_{i4})\right) \begin{cases} |4-p|, p = 1,2,3,\dots,\frac{|V(H_i)|}{2} + 4 \\ |V(H_i)| + 1 - p, p = \left\lceil \frac{|V(H_i)|}{2} \right\rceil + 5, \left\lceil \frac{|V(H_i)|}{2} \right\rceil + 6, \dots, |V(H_i)| \end{cases}$$

Moreover, for every $(u_i, v_{ip}), (u_q, v_{qr}) \in V(G \circ \mathcal{H}), r((u_i, v_{ip})|W) \neq r((u_q, v_{qr})|W), i \neq q$ and $p \neq r$. Thus, W constitutes a resolving of $G \circ \mathcal{H}$

Having obtained the resolving set as shown in **Lemma 2.3**, we have the necessary foundation to proceed to the proof of **Theorem 2.2.** Thus, this lemma serves as an integral part in connecting the basic result with the principal theorem in this section.

Theorem 2.2: *In a connected graph G with n order and $\mathcal{H} = \{H_1, H_2, \dots, H_n\}$ is the sequence where each H_i is a cycle of order exceeding six. Then $Ddim(G \circ \mathcal{H}) = \sum_{i=1}^{n} Ddim(H_i)$.*

Proof: Let vertex set of G be $\{u_i | i = 1,2, \dots, n\}$ and let $\mathcal{H} = \{H_i | i = 1,2, \dots, n\}$ be a sequence where each H_i is a cycle of order exceeding six with the set of vertices $V(H_i) = \{v_{ij} | i = 1,2, \dots, n; j = 1,2, \dots, |V(H_i)|\}$. Two possibilities arise for the cycle graph, they are when $|V(H_i)|$ is a multiple of three and when $|V(H_i)|$ is not a multiple of three, then

1. For $|V(H_i)| = 3k, k \in N$, select $W_i = \{v_{i1}, v_{i4}, \dots, v_{i(3j-2)}\}$ so that $W_i = \left\lceil \frac{|V(H_i)|}{3} \right\rceil$
2. For $|V(H_i)| \neq 3k, k \in N$, select $W_i = \{v_{i1}, v_{i4}, \dots, v_{i(3j-2)}, v_{i(|V(H_i)|-1)}\}$ so that $W_i = \left\lceil \frac{|V(H_i)|}{3} \right\rceil$

WLOG, W_i is a dominant basis of the cycle graph H_i as explained **Theorem 1.1 iv.** Choose $W = \bigcup_{i=1}^{n} \{(u_i, v_{ij}) | v_{ij} \in W_i\}$ so that $|W| = \sum_{i=1}^{n} Ddim(H_i)$. Since $\{(u_i, v_{i1}), (u_i, u_{i4})\} \subseteq W$ it follows from Lemmas 1.3 and 2.3 that W a resolving set for $G \circ \mathcal{H}$. Our next step is to establish that W is, in fact, a dominant set as well. Since $E(H_i) = \{v_{ij}v_{i(j+1)} | j = 1,2, \dots, |V(H_i)| - 1\} \cup \{v_{in}v_{i1}\}$ for $i = 1,2, \dots, n$, we get that $(u_i, v_{i(3k-2)}) \sim (u_i, v_{i(3k-1)})$ and $(u_i, v_{i(3k-1)}) \sim (u_i, v_{i(3k)})$. Consequently, W serves as a dominant set of $G \circ \mathcal{H}$, and at the same time, it functions as a dominant resolving set. Consider an arbitrary set $S \subseteq V(G \circ \mathcal{H})$ with $|S| < |W|$. Suppose $|S| = |W| - 1$. Then, there is an index i for which S comprises at most $|W_i| - 1$ members of the vertex set $V(H_i)$. Since S contains at most $|W_i| - 1$ vertices from $V(H_i)$, at least one vertex must occur that shares no adjacency with any vertex from S. Consequently, S fails to be a dominant set of $G \circ \mathcal{H}$. Having established that S is not a resolving dominant set for $G \circ \mathcal{H}$, we invoke **Lemma 1.2**. This lemma guarantees that the property of being a non-resolving set propagates downward; thus, any set with cardinality smaller than $|S|$ cannot be a resolving dominant set of G. Therefore, it has been shown that $W = \bigcup_{i=1}^{n} \{(u_i, v_{ij}) | v_{ij} \in W_i\}$ is a minimum resolving dominant set for $G \circ \mathcal{H}$. Moreover, since W_i is a dominant basis H_i with $|W_i| = Ddim(H_i)$. Thus

$$Ddim(G \circ \mathcal{H}) = \sum_{i=1}^{n} Ddim(H_i)$$

The next theorem extends this result to the case where $\mathcal{H}$ is a sequence of complete bipartite graphs. By considering complete bipartite graphs, we obtain a characterization that illustrates how the dominant metric dimension behaves under a different structural setting.

Theorem 2.3: *In a connected graph* G *with* n *order and* $\mathcal{H} = \{H_1, H_2, \dots, H_n\}$ *is the sequence of the complete bipartite graphs* $K_{p,q}$ *where* $p, q \in \{2,3, \dots, k\}$ *and* $k \in \mathbb{Z}^+$. *Then* $Ddim(G \circ \mathcal{H}) = \sum_{i=1}^{n}(Ddim(H_i) - 1)$.

Proof: Consider $V(G) = \{u_i | i = 1,2, \dots, n\}$ and let $\mathcal{H} = \{H_i | i = 1,2, \dots, n\}$ be a sequence of complete bipartite graph $K_{p,q}$ where $p, q \in \{2,3, \dots, k\}$ and $k \in \mathbb{Z}^+$ with the of vertices $V(H_i) = \{a_{ig} | g = 1,2, \dots, p\} \cup \{b_{ih} | h = 1,2, \dots, q\}$. By the understanding of rooted product, we obtained $V(G \circ \mathcal{H}) = \{(u_i, a_{ig}) | u_i \in V(G); a_{ig} \in V(H_i)\} \cup \{(u_i, b_{ih}) | u_i \in V(G); b_{ih} \in V(H_i)\}$. WLOG, $W_i = \{a_{ig} | g = 1,2, \dots, p-1\} \cup \{b_{ih} | h = 1,2, \dots, q-1\}$ is the dominant basis of complete bipartite graph as explained in **Theorem 1.1.v.** Choose $W = \bigcup_{i=1}^{n}\{(u_i, a_{ig}) | a_{ig} \in W_i\} \backslash \{(u_i, a_{i1})\} \bigcup_{i=1}^{n}\{(u_i, b_{ih}) | b_{ih} \in W_i\}$ so $|W| = \sum_{i=1}^{n}(Ddim(H_i) - 1)$. By **Lemma 1.1** we achieved that the representation every vertex in W is different. So, we can obtain

$V((G \circ \mathcal{H}) \backslash W) = \bigcup_{i=1}^{n}\{(u_i, a_{i1})\} \bigcup_{i=1}^{n}\{(u_i, a_{ip})\} \bigcup_{i=1}^{n}\{(u_i, b_{iq})\}$. There are three possibilities for each H_i which shown that every vertex has the different representation to W.

1. For $(u_i, a_{i1}), (u_i, a_{ip}) \in V((G \circ \mathcal{H}) \backslash W)$. It is $r((u_i, a_{i1}) | \{W_i\} \backslash \{(u_i, a_{i1})\}) = r((u_i, a_{ip}) | \{W_i\} \backslash \{(u_i, a_{i1})\})$. But since for every $i \neq j$ we have $d((u_i, a_{i1}), (u_j, a_{j1})) \neq d((u_i, a_{ip}), (u_j, a_{j1}))$, then for every $v \in \{W_i\} \backslash \{(u_i, a_{i1})\}$, we have $d(v, (u_i, a_{i1})) \neq (v, (u_i, a_{ip}))$ so that $r((u_i, a_{i1}) | \{W_i\} \backslash \{(u_i, a_{i1})\}) \neq r((u_i, a_{ip}) | \{W_i\} \backslash \{(u_i, a_{i1})\})$. Moreover, since $\{W_i\} \backslash \{(u_i, a_{i1})\} \subseteq W$, then we have $r((u_i, a_{i1}) | W \neq r((u_i, a_{ip}) | W)$.
2. For $(u_i, a_{i1}), (u_i, b_{iq}) \in V((G \circ \mathcal{H}) \backslash W)$. It is $r((u_i, a_{i1}) | \{W_i\} \backslash \{(u_i, a_{i1})\}) \neq r((u_i, b_{iq}) | \{W_i\} \backslash \{(u_i, a_{i1})\})$. Moreover since $\{W_i\} \backslash \{(u_i, a_{i1})\} \subseteq W$, next we have $r((u_i, a_{i1}) | W \neq r((u_i, b_{iq}) | W)$.
3. For $(u_i, a_{ip}), (u_i, b_{iq}) \in V((G \circ \mathcal{H}) \backslash W)$. It is $r((u_i, a_{ip}) | \{W_i\} \backslash \{(u_i, a_{i1})\}) \neq r((u_i, b_{iq}) | \{W_i\} \backslash \{(u_i, a_{i1})\})$. Moreover since $\{W_i\} \backslash \{(u_i, a_{i1})\} \subseteq W$, next we have $r((u_i, a_{ip}) | W \neq r((u_i, a_{iq}) | W)$.

Also by **Lemma 2.1** for every $u \in W_i$ and $v \in W_j$ with $i \neq j$ we can obtained that $d(u, (u_i, a_{i1})) \neq d(v, (u_i, a_{i1}))$ imply $d(u, (u_i, a_{i2})) \neq d(v, (u_i, a_{i2}))$ so that $r(u | \{(u_i, a_{i2})\}) \neq r(v, | \{(u_i, a_{i2})\})$. Since $\{(u_i, a_{i2})\} \subseteq$ W, then we have

$r(u | W \neq r(v | W)$ for each $u \in W_i$ and $v \in W_j$ and $i \neq j$. Therefore $W = \bigcup_{i=1}^{n}\{(u_i, a_{ig}) | a_{ig} \in W_i\} \{(u_i, a_{i1})\} \bigcup_{i=1}^{n}\{(u_i, b_{ih}) | b_{ih} \in W_i\}$ is a resolving set of $(G \circ \mathcal{H})$. Furthermore, every vertex $(u_i, a_{i1}), (u_i, a_{ip}) \in V(H_i)$ is adjacent to $(u_i, b_{i1}) \in W$ and every vertex $(u_i, b_{iq}) \in V(H_i)$ is adjacent to $(u_i, a_{i2}) \in W$. Therefore W is a dominant resolving set of $G \circ \mathcal{H}$. For any $S \subseteq V(G \circ \mathcal{H})$ having $|S| < |W|$, WLOG, we suppose that $|S| = |W| - 1$. Then there is an index i such that S has at most $|W_i| - 2$ member of the vertex set $V(H_i)$, so that S is not resolving set of $G \circ \mathcal{H}$. Furthermore, by **Lemma 1.2**, $W = \bigcup_{i=1}^{n}\{(u_i, a_{ig})|a_{ig} \in W_i\}\{(u_i, a_{i1})\} \bigcup_{i=1}^{n}\{(u_i, b_{ih})|b_{ih} \in W_i\}$ is a resolving dominant set with the minimum cardinality for the graph $G \circ \mathcal{H}$. Moreover, since W_i is a dominant basis H_i with $|W_i| = Ddim(H_i) - 1$. Thus, it is proven that $Ddim(G \circ \mathcal{H}) = \sum_{i=1}^{n}(Ddim(H_i) - 1)$

A graph has a maximum metric dimension when its metric dimension equals the graph's order less one. To put it differently, almost all vertices of the graph are required to constitute a resolving set.

Theorem 2.4: *In a connected graph G with n order and $\mathcal{H} = \{H_1, H_2, \dots, H_n\}$ is the sequence of any connected graphs. If H_i is a graph with maximum metric dimension, then $Ddim(G \circ \mathcal{H}) = \sum_{i=1}^{n}(Ddim(H_i) - 1) = |V(H_i)| - 1$.*

Proof: Let $V(G) = \{u_i|i = 1,2, \dots, n\}$. Let $\mathcal{H} = \{H_i|i = 1,2, \dots, n\}$ is a sequence of any connected graph with maximum metric dimension. Let $V(H_i) = \{v_{ij}|i = 1,2, \dots, n; j = 1,2, \dots, |V(H_i)|\}$, we identify $u_i \in V(G)$ to $v_{i1} \in V(H_i)$ so we $V(G \circ \mathcal{H}) = \{(u_i, v_{ij})|u_i \in V(G); v_{ij} \in V(H_i)\}$. Because H_i is a connected graph with maximum metric dimension, WLOG, $W_i = \{v_{ij}|j = 1,2,3, \dots, |V(H_i)| - 1\}$ forms a dominating set of H_i. Choose $W = \bigcup_{i=1}^{n}\{(u_i, v_{ij})|v_{ij} \in W_i\} \backslash \{(u_i, v_{i1})\}$ so $|W| = \sum_{i=1}^{n}(Ddim(H_i) - 1)$. By **Lemma 1.1** we achieved that the representation every vertex in W is different. Hence, we conclude $V((G \circ \mathcal{H})\backslash W) = \bigcup_{i=1}^{n}\{(u_i, v_{i|V(H_i)|})\} \bigcup_{i=1}^{n}\{(u_i, v_{i1})\}$. Select two arbitrary vertices $\{(u_p, v_{ps}), (u_q, v_{qr})\} \subseteq W$; $p \neq q$. Since $d((u_p, v_{p1}), (u_p, v_{ps})) \neq d((u_p, v_{p1}), (u_q, v_{qr}))$ and $d((u_p, v_{p|V(H_p)|}), (u_p, v_{ps})) \neq d((u_p, v_{p|V(H_p)|}), (u_q, v_{qr}))$, so $r((u_p, v_{p1})|\{(u_p, v_{ps}), (u_q, v_{qr})\}) \neq r((u_p, v_{p|V(H_p)|})|\{(u_p, v_{ps}), (u_q, v_{qr})\})$. Next, because $\{(u_p, v_{ps}), (u_q, v_{qr})\} \subseteq W$, then for $i = 1,2, \dots n, r((u_i, v_{i1})|W) \neq r((u_i, v_{i|V(H_i)|})|W)$. Therefore, $W = \bigcup_{i=1}^{n}\{(u_i, v_{ij})|v_{ij} \in W_i\} \backslash \{(u_i, v_{i1})\}$ is a resolving set. Because H_i is a connected graph with maximum cardinality so every vertex in H_i are adjacent so W. Furthermore, $W = \bigcup_{i=1}^{n}\{W_i\}\backslash(u_i, v_{i1})$ is a resolving dominant set of $G \circ \mathcal{H}$. Now, let $S \subseteq V(G \circ \mathcal{H}.)$ be an arbitrary set such that $|S| < |W|$. In the case where $|S| = |W| - 1$, it follows that for some index i, the number of vertices from $V(H_i)$ included in S is bounded above by $|W_i| - 1$. Consequently, there exist two distinct vertices with identical representations relative to S, which implies that S is not a resolving set for $G \circ \mathcal{H}$. The set S has been shown not to be a resolving dominant set of $G \circ \mathcal{H}$. Applying **Lemma 1.2**, which asserts that the non-resolving property propagates downward to all smaller sets, we conclude that no set with the cardinality less than $|W|$ can be a resolving dominant set of G.

Therefore, we can concluded that $W = \bigcup_{i=1}^{n}\{(u_i, v_{ij})|v_{ij} \in W_i\} \backslash \{(u_i, v_{i1})\}$ is a resolving dominant set with the minimal cardinality for the graph $G \circ \mathcal{H}$. Moreover, since W_i is a dominant basis H_i with $|W_i| = Ddim(H_i)$. Thus, we have shown that$Ddim(G \circ \mathcal{H}) = \sum_{i=1}^{n}(Ddim(H_i) - 1)$

The established lemmas and theorems indicate how the dominant resolving number of a rooted product graphs frequently corresponds to the dominant metric dimensions of its components. In particular, the rooted vertex tends to appear as a critical element in the resolving set. Based on this pattern, we propose the following conjecture

Conjecture: Consider G be connected graph of n order and $\mathcal{H} = \{H_1, H_2, \dots, H_n\}$ is the sequence of any connected graphs. If $Ddim(G \circ \mathcal{H}) = \sum_{i=1}^{n}(Ddim(H_i) - 1)$, then the rooted vertex of H_i is an element of the resolving set.

Verifying or disproving this conjecture remains an open problem for future research.

3. Conclusion and Suggestions

Consider G be a connected graph, and $\mathcal{H}$ be a sequence of graphs, each of which is either a bipartite graph, complete graph, path graph, or cycle graph. We examine the dominant metric dimension for the rooted graph $G \circ \mathcal{H}$. The results show that the dominant metric dimension of the graph resulting from the rooted product $G \circ \mathcal{H}$ can be written as the total of the graphs in the sequence $\mathcal{H}$'s dominant metric dimension. In the case where $\mathcal{H}$ sequence of complete graph, complete bipartite graph, and star graph with the rooted vertex at a pendant, the result must be reduced by n.

Future research may extend this study by considering other classes of graphs in the rooted product operation, such as grid graphs or Petersen graphs, which may better represent real-world distribution networks in urban and rural settings. Moreover, applying computational simulations based on actual geographical and demographic data could validate the practical implications of the dominant metric dimension in poverty-related distribution systems.

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