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Advanced computational methods for compact mean labeling in star graphs

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Abstract

Graph labeling involves assigning unique integer values to the vertices and edges of a graph, starting from zero or a higher value. A star graph, denoted as $K_{(1,\,r)}$, consists of a central vertex connected to r peripheral vertices, forming a complete bipartite graph. In this research, the notion of compact mean labeling using star graphs is explored. Typically, a star graph has $p < q$ edges. However, for compact mean labeling, the graph should have $p \leq q$. To resolve this inconsistency, the vertex labeling of the graph has been determined from zero to 3q-p. This proposed extended compact mean labeling technique is verified for the bi-star graphs, the join sum of the star graph, and the union of the star graph. Our findings

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offer new insights into the characteristics and application of star graphs in extended compact mean labeling, particularly in comparing the strong and weak ties within a network.

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Keywords: *Extended compact mean labeling, Graph labeling, Join sum of graph, Star graph, Union of star graph.*

1. Introduction

Graph labeling, a significant area of graph theory, was first introduced in 1967 by Rosa [3] through the Kotzig-Ringel conjecture method. Rosa [3] referred to this as α-, β-, and ρ-labeling, with β- labeling later becoming known as Graceful Labeling, a term coined by Golomb [4]. In 1976, Bloom expanded the scope of graph labeling by correlating these concepts with various practical applications, including radar, astronomy, communication design and network analysis. Kumar et al. [1], gave some rules for estimating computer network security. Furthermore in detail Sunita et al. [2] gave a cost-efficient real-time application using mathematical model with the help of computing environment.

Over the past 56 years, graph labeling has evolved significantly, with nearly 200 different strategies being researched and documented. This extensive body of work highlights the versatility and importance of graph labeling in both theoretical research and practical applications.

2. Literature Review

The following research articles were studied for the core research work.

Every graph taken into consideration here is finite, simple and connected. In 1967, Rosa [3] introduced the system labeling of graphs. Golomb [4] later termed Graceful Labeling for β− valuation introduced by Rosa which came into existence henceforth. Kay, Bondy, and Murty [5] explained about the application of graph theory. Wang, Dueñas-Osorio, and Padgett [6] offered a new mutually reinforcing network node and link ranking method. El-Zanat, Fu, and Shiue [7] analysed the α-labeling number of bipartite graphs. Kumar et al. [8] talked how we can enhance security for credit card transaction within smart cities for cyber security trials.

Arumugam et al., [9] present several open problems with graph labeling. Maheswari, Ramesh and Balaji [10], [11] defend the results of skolem and relaxed mean labeling. Furthermore, Maheswari et al. [12], gave more detailed results about relaxed mean labeling in her next paper. Shruthy and Maheswari [13] gave the double encryption and decryption process using the enhanced vigenere cipher and graph labeling. Difference modulo labeling was discussed by Rekha and Maheswari [14]. Sarmitha, Vidyanandini, and Ranjan [15] paper is about square difference labeling co-secure domine in a middle graph. Barrientos and Youssef [16] discussed about Relaxing the injectivity constraint on arithmetic and harmonious labelings. Deepa, Maheswari, and Balaji [17]

discusses about ciphertext and decipher using graph labeling techniques. Subramaniyan et al. [18] gave the notion of Compact Mean Labeling came up. Kaneria et al. [19] discussed a few results related to graceful labeling for complete bipartite graph. Ezhilarasi and Kavitha [20] gave a brief study on labeling of star graph. Janaki, Maheswari, and Balaji [21] briefed about a method for secure encoding known as difference divisor labeling for graph-based cryptographic applications. Nandhini, Maheswari, and Balaji [22] gave the possibilities of edge repetition using double mean labeling.

3. Preliminiries

Definition 3.1: "A star graph $K_{1,r}$ is a complete bipartite graph with a single vertex connected to one set and the rest of the vertices connected to the other set."

Definition 3.2: "A "relaxed mean graph" is described as a graph $G = (V, E)$ with vertices 'p' and edges 'q' that has a function "f" with the vertex set as G to $\{1,2,\dots,q-1,q+1\}$ and relaxing 'q' such that " f*" be the resulted map from the edge set of 'G' to $\{1,2,\dots,q\}$. It is provided according to

$$f^*(e = uv) = \begin{cases} \frac{f(u)+f(v)}{2}, \text{if } f(u) + f(v) \text{ is even} \\ \frac{f(u)+f(v)+1}{2}, \text{if } f(u) + f(v) \text{ is odd} \end{cases}$$

So, the stated edges and the vertices remain distinct" [23].

Definition 3.3: "Each edge is labeled with $\frac{f(u)+f(v)}{2}$ if $f(u) + f(v)$ is even vertex and $f(u) + f(v)$ odd as $\frac{f(u)+f(v)+1}{2}$ with vertex p and edge q, have a distinct integers with the vertex labeled as {0,1,2,…., 2q - p}, then it is said to be compact mean labeling. Then the edge label $\{1, 2, \dots, q\}$ is distinct" [24].

Definition 3.4: "Each edge is labeled with $\frac{f(u)+f(v)}{2}$ if $f(u) + f(v)$ is even vertex and $f(u) + f(v)$ odd as $\frac{f(u)+f(v)+1}{2}$ with vertex p and edge q, have a distinct integers with the vertex labeled as {0,1,2,…., 3q - p}, then it is said to be Extended Compact Mean Labeling (ECML). Then the edge label $\{1, 2, \dots, q\}$ is distinct. The obtained labeled graph using this definition is known as ECML graph."

Definition 3.5: Let a Graph consists of l copies. Then the Graph G =< G_1, G_2,… G_r > is formed by connecting of two graph G_i and G_{i+1} by a vertex, i.e $1 \leq i \leq r-1$ then it is known as join sum of graphs.

4. Main Results

Theorem 4.1: *Every star graph* $\boldsymbol{K_{1,r}}$ *is a ECML graph if* $r \geq 1$.

Proof: Consider the graph G = $K_{1,r}$.

By defining a map f: V (G) → {0,1,2, … 3q - p}, so that $p = q + 1$ and $q = r$ as follows:

$$f(u) = 0$$

$$f(v_i) = \{2(q - i) + 1\}; i = 1,2, \ldots r.$$

The corresponding edge labels is given as $uv_i = q - i + 1, \forall\ 1 \le i \le r$,

Thus, the resultant edge labels of G is distinct from $\{1,2 \ldots, q\}$.

Hence $K_{1,r}$ is a ECML graph.

Corollary 4.2: *If* $r = 4$. $K_{1,r}$ *is a ECML graph.*

Proof: By theorem 4.1 V(G) → {0, 1, …, 7} such that $p = 5$ and $q = 4$.

Thus, the corresponding edge labels is {1, 2, 3, 4}.

Hence $K_{1,4}$ is a ECML graph.

Theorem 4.3: *A bi-star graph* $\boldsymbol{B_{(r,s)}}$ *is an ECML graph.*

Proof: Consider a graph G be the bi-star graph $B_{r,s}$,G = ($V_1 \cup V_2$, E) i.e., $V_1$and V_2 are stable sets of V(G).

Then $V_1 \to \{u_1, u_2, \ldots, u_i\}$ and $V_2 \to \{v_1, v_2, \ldots, v_j\}$, where u_1,v_1,…, u_i,v_j, ∈ ℕ.

The labeling function can be defined as such f: V → {0,1,2, … 3q - p}, such that $p = q + 1$ and $q = r + s + 1$ as follows:

$f(u_0) = 1$

$$f(u_i) = \{2i + 1\}; i = 1,2, \ldots s.$$

$$f(v_0) = 0$$

$$f(v_j) = \{2(s + i) + 1\}; i = 1,2, \ldots s.$$

The corresponding edge labels is given as $u_0v_0 = 1$; and u_0u_i is $2 + (i - 1)$, $\forall\ 1 \le i \le s$,

The corresponding edge labels of v_0v_j is given as $s + i + 1$, $\forall\ 1 \le j \le s$.

Therefore, the resultant edge labels of G are distinct from $\{1,2, \ldots, q\}$.

Hence G of $B_{r,s}$ is a ECML graph.

Corollary 4.4: *If* $r = 5$ *and* $s = 5$. $B_{(r,s)}$ *is a ECML graph.*

Proof: By theorem 4.3 V(G) → {0, 1, …, 21} such that $p = 12$ and $q = 4$.

Thus, the corresponding edge labels is {1, 2, … 11}.

Hence $B_{(5,5)}$ is a ECML graph.

Theorem 4.5: *The Join sum of the star graph* $< K_{1,r_1}; \ldots; K_{1,r_t} >$ *is a ECML graph.*

Proof: Consider a graph G represent join sum of the star graph $K_{1,r_1}; \ldots; K_{1,r_t}$, where $r_1, r_2, \ldots, r_t \in N$.

Let $V_1 \rightarrow \{u_{1,1}, u_{1,2}, \ldots, u_{1,j}, u_{2,1} \ldots, u_{i,j}\}$, $V_2 \rightarrow \{v_{1,0}, v_{2,0}, \ldots, v_{i,0}\}$ and the join sum of vertices for star graph be $w_1, w_2, \ldots, w_{t-1}$ where $u_1, v_1, w_1 \ldots, u_i, v_i, w_{t-1} \in \mathbb{N}$.

The labeling function can be defined as such f: V → {0,1,2, … 3q - p}, such that $p = q + 1$ and $q = 3r + 4$ as follows:

$$f(u_{1,i}) = \{2(i-1)\}\ \forall\ i = 1,2, \ldots r_l.$$

$$f(u_{j,i}) = 2\{\sum_{l=1}^{j-1} r_l + (i-1)\}\ \forall\ i = 1,2, \ldots, r_l.$$

$$f(v_{i,0}) = \{4i-3\}\ \forall\ i = 1,2, \ldots r_l.$$

$$f(w_i) = \{4i-1\}\ \forall\ i = 1,2, \ldots t-1.$$

Whereas $j = 2,3, \ldots\ t$.

Therefore, the resultant edge labels of G are distinct from {1,2, … , q}.

Hence $< K_{1,r_1}; \ldots; K_{1,r_t} >$ is a ECML graph.

Theorem 4.7: *Union of the star graph* $< K_{1,r} \cup C_s \cup K_{1,r} >$ *is an ECML graph if* $r \leq 1$.

Proof: Consider a graph $K_{1,r}$,where $r \leq 1$.

By using the definition of the join sum of the graph by connecting two graphs $K_{1,r}$ using a new vertex w_i. Likewise, when each vertex is connected using two new vertices then a cycle C_s is formed.

Let $V_1 \rightarrow \{u_1, u_2, \ldots, u_i\}$, $V_2 \rightarrow \{v_1, v_2, \ldots, v_i\}$ and the join sum of vertices of a star graph be $w_1, w_2, \ldots, w_i$, which form a cycle C_s.

Let $u_1, v_1, w_1 \ldots, u_i, v_i, w_i \in \mathbb{N}$. The labeling function can be defined as such f: V → {0,1,2, … 3q - p}, such that $p = q + 1$ and $q = 2r + s$ as follows:

$$f(u_{1,i}) = \{2(i-1)\ \forall\ i = 1,2, \ldots r_l.$$

$$f(u_{j,i}) = 2\{\sum_{l=1}^{j-1} r_l + (i-1) + 3\}\ \forall\ i = 1,2, \ldots, r_l.$$

$$f(v_{i,0}) = \{4i-3\}\ \forall\ i = 1,2, \ldots r_l.$$

$$f(w_i) = 2\{r + i - 1\}\ \forall\ i = 1,2, \ldots t.$$

Whereas $j = 2,3, \ldots\ r+1$.

The relevant edge labels are the following:

$u_{1,i}v_{1,0} = i$; and $u_{2,i}v_{2,0}$ is $2r + i - 1$, $\forall\ 1 \leq i \leq r$,

Thus, the corresponding edge labels of the cycle C_s is given as, $v_{1,0}w_i$ is $r + i$ and $v_{2,0}w_i = r + i + 2$, $\forall\ 1 \leq i \leq t$.

Thus, the resultant edge labels of G are distinct from {1,2, … , q}.

Hence $< K_{1,r} \cup C_s \cup K_{1,r} >$ is a ECML graph.

5. Analyzing Market Dynamics through Star Graph Structures

Star graphs are the most effective graphs for network analysis, especially when depicting complicated networks like logistic systems, decision-making processes, and communications networks. This study seeks to investigate the use of star graphs in marketing, especially for determining market potential and client preferences across regions.

The technique of extended compact mean labeling can significantly enhance the analysis of marketing network represented through star graph. While introducing a new product or when advertising a product in a new country, it is possible to view the target market as a graph. In this graph, people are considered as nodes while the arcs are the relations extracted from the analysis of sales data. This strategy is useful for defining the potential for launching the product in the chosen location by identifying the customers' preference and adapting the product to their needs. The ECML technique also aids in organizing and distinguishing market sectors effectively.

Knowledge of the market and its environment is essential in forecasting product sales and improving the coverage of the market by products. As we together subdivide people by language, by culture, and by financial means, bipartite network can be arranged such that it resembles a star graph. In this case, vertices are people, while edges are the connections between people. It is also important to know the markets' strong and weak vertices to be effective in marketing.

In this context, if a vertex has high degree (strong vertex), it means it has strong connections, on the other hand if it has low degree (weak vertex), then it implies the vertex has weak connections. It is evident that strong vertices are more powerful and can help promote products more effectively, thus making the process faster. On the other hand, there is a need to examine the characteristics of weak ties to support the market growth. Thus, weak ties introduce new ideas and identify deficiencies of products, which enables developing customers' confidence and expanding a business.

Therefore, it is possible to conclude that ECML technique effectively uses star graphs to study strong and weak ties which is a helpful instrument to consider customer choices and the potential of markets. By combining ECML with star graph based marketing model we not only enhance how we organise data but also boost our strategic insights, making it essential tool for today's market analysis.

6. Conclusion

This study was mainly concerned with the labeling of graphs where $p \geq q$. Compact labeling has been commonly applied with graphs in which $p \leq q$, but in this research, the extended compact mean labeling is utilized with the graphs in which $p \geq q$. Then, using this extended compact mean labeling, we assigned values to the vertices of G which led to inducing new values to the edges. Thus, our results show that this method of labeling works effectively in specific graph structures, including bistars, the join sum of star graphs, and

cycles of star graphs. Therefore, these results can be considered as the contribution to the further research and advancements in the field of graph labeling and can be applied to different fields and areas of study.

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