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A novel degree based topological index and its application in decision making

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Abstract

Topological indices quantify graph structure, aiding complex decision-making. Fuzzy graphs incorporate uncertainty, while neutrosophic graphs address indeterminacy and inconsistency. This work introduces a novel degree based topological index, establishes its bounds for general fuzzy graphs, and demonstrates its utility in neutrosophic decision making frameworks.

Subject Classification: *05C72.*

Keywords: *Fuzzy graphs, Topological index, Neutrosophic graphs, Decision making, Laplacian energy.*

1. Introduction

Zadeh [11] foundational 1965 work on fuzzy sets laid the groundwork for fuzzy graph theory. Building on this, Rosenfeld [12] advanced the field by exploring fuzzy relations on fuzzy sets and introducing key concepts like fuzzy subgraphs and fuzzy paths [7].

Various forms of fuzzy graphs have been developed and applied across diverse domains, demonstrating the growing relevance of fuzzy graph theory in addressing complex real-world problems [6]. Smarandache [13] first proposed single valued neutrosophic sets in 1995. These sets are useful in managing inconsistent, ambiguous data, which is helpful for social network analysis, pattern detection, and decision-making [3].

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Degree based topological indices are numerical metrics used to describe the structure of graphs based on the degrees of their vertices, with significant applications in chemistry [2]. Recent developments are highlighted in [1] and [8]. In this article, we have outlined its application in decision making.

2. Fundamentals

The preliminary concepts necessary for further progress are outlined in this section.

Definition 2.1: [4], [5] "A graph is an ordered pair $G = (V, E)$ with a non empty vertex set $V(G)$ and the edge set $E(G)$ respectively.

A fuzzy graph $\mathcal{G} = (\mathcal{V}, \eta, \tau)$ corresponding to the graph $\mathcal{G}$ is a non empty set $\mathcal{V}$ together with a pair of functions $\eta: V \to [0,1]$ and $\tau: \mathcal{V} \times \mathcal{V} \to [0,1]$ such that for all $a, b \in V$, $\tau(a,b) \leq min\{\eta(a), \eta(b)\}$ where $\eta(a)$ and $\tau(a,b)$ represent the membership values of the vertex a and the edge (a,b) in $\mathcal{G}$ respectively. $(\mathcal{V}, \eta^*, \tau^*)$ represents the fuzzy graph $\mathcal{G}$ where η^* and τ^* represents the non zero membership values of the vertices and the edges respectively."

Definition 2.2: [4], [5] "The order of a fuzzy graph $\mathcal{G}$ is represented as $O(\mathcal{G})$ is the sum of all membership values of all vertices and is denoted as $O(\mathcal{G}) = \sum_{a \in \eta^*} \eta(a)$. The size of the fuzzy graph $\mathcal{G}$ denoted by $S(\mathcal{G})$ is the sum of all membership values of all edges and is denoted by $S(\mathcal{G}) = \sum_{(a,b) \in \eta^* \times \eta^*} \tau(a,b)$"

Definition 2.3: [3] "Let $\mathcal{G} = (\eta, \tau)$ be a single valued neutrosophic graph, where η is a Neutrosophic set defined on the vertex set V and τ is a Neutrosophic set defined on the edge set E. The graph satisfies the following conditions:

$$T_\tau(a,b) \leq \min(T_\eta(a), T_\eta(b)), \qquad I_\tau(a,b) \geq \max(I_\eta(a), I_\eta(b)),$$
$$F_\tau(a,b) \geq \max(F_\eta(a), F_\eta(b)),$$

where a and b are vertices of G, and $(a,b) \in E$ is an edge in G."

Definition 2.4: [3] "Let $\mathcal{G} = (\eta, \tau)$ be a single valued neutrosophic graph, and let $a \in V$ be a vertex of G. The degree of the vertex a, denoted by $d(a)$, is defined as

$$d_{\mathcal{G}}(a) = (d_T(a), d_I(a), d_F(a)) =$$
$$\left(\sum_{\substack{b \in V \\ b \neq a}} T_\tau(a,b), \sum_{\substack{b \in V \\ b \neq a}} I_\tau(a,b), \sum_{\substack{b \in V \\ b \neq a}} F_\tau(a,b) \right)$$

where $d_T(a), d_I(a), d_F(a)$ represents the sum of the truth membership values, sum of the indeterminacy membership values and sum of the falsity membership values respectively."

3. Degree Product Difference Index of fuzzy graphs

In this section, we propose a novel degree-based topological index and explore its applicability across different fuzzy graphs. We have established bounds for specific graphs.

Definition 3.1: (Type 1 Degree Product Difference Index 1).

Consider the fuzzy graph $\mathcal{G} = (\mathcal{V}, \eta, \tau)$. The Degree Product Difference Index of the fuzzy graph, denoted by $\mathrm{FDPDI}(\mathcal{G})$, is defined as

$$\mathrm{FDPDI}(\mathcal{G}) = \sum_{ab\in\tau^*(\mathcal{G})} \frac{\eta(a)\cdot d_{\mathcal{G}}(a)\cdot\eta(b)\cdot d_{\mathcal{G}}(b)}{|\eta(a)\cdot d_{\mathcal{G}}(a)-\eta(b)\cdot d_{\mathcal{G}}(b)|+1}$$

The Type 1 Degree Product Difference Index is also known as the Fuzzy Degree Product Difference Index.

Example 1: To calculate the fuzzy degree product difference index (FDPDI) of the fuzzy graph $\mathcal{G}$ in the figure given below. The fuzzy graph under consideration is shown in Figure 1.

$d(a_1) = 0.7, d(a_2) = 0.5, d(a_3) = 0.4, d(a_4) = 0.3, d(a_5) = 0.7$ are the degrees of the vertices in $\mathcal{G}$

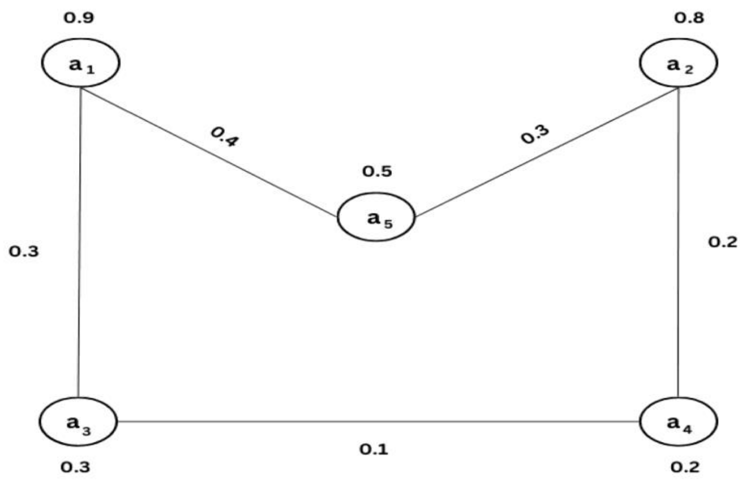

Figure 1
Fuzzy graph $\mathcal{G}$

$FDPDI(\mathcal{G})$ = 0.050066 + 0.01791 + 0.172265 + 0.1333 + 0.00679 = 0.380331

Theorem 3.1: *The fuzzy path graph $P_N = (\mathcal{V}, \eta^*, \tau^*), N \geq 3$ where $\eta^* = \{a_1, a_2, \dots, a_N\}$is the set of vertices, and $\tau^* = \{a_1a_2, \dots, a_{N-1}a_N\}$is the set of edges. The upper bound for the Fuzzy Degree Product Difference Index (FDPDI) of P_N is $FDPDI(P_N) \leq 4N - 10$.*

Proof: Consider the fuzzy path graph $P_N = (\mathcal{V}, \eta^*, \tau^*)$. Then, $d_{P_N}(a_1) = \tau_1, d_{P_N}(a_2) = \tau_1 + \tau_2, \ldots, d_{P_N}(a_{N-1}) = \tau_{N-2} + \tau_{N-1}, d_{P_N}(a_N) = \tau_{N-1}$. Since $\eta(a_i) \leq 1$ and $d_{P_N}(a_i) \leq 1$, the graph can be partitioned into two: one for the first and last vertices, and another for the intermediate vertices. Given that both the vertex weight and the edge weight are bounded by 1, we derive the following inequality

$$\mathrm{FDPDI}(P_N) \leq 2\left(\frac{2}{2}\right) + \sum_{i=2}^{N-2} 4 = 4n - 10$$

Theorem 3.2: *Let $C_N = (\mathcal{V}, \eta^*, \tau^*)$ represent a fuzzy cycle graph, where $\eta^* = \{a_1, a_2, \ldots, a_N\}$ is the set of vertices, and N is the order of the graph, with $N \geq 3$. The edge set is given by $\tau^* = \{a_1a_2, a_2a_3, \ldots, a_{N-1}a_N, a_Na_1\}$, which contains N edges. The upper bound for the Fuzzy Degree Product Difference Index (FDPDI) of the fuzzy cycle graph C_N is $FDPDI(C_N) \leq 4N$.*

Proof: Consider the fuzzy cycle graph $C_N = (\mathcal{V}, \eta^*, \tau^*)$. The degree of each vertex in this graph is defined as follows: $d_{C_N}(a_1) = \tau_1 + \tau_N, d_{C_N}(a_2) = \tau_1 + \tau_2, \ldots, d_{C_N}(a_{N-1}) = \tau_{N-2} + \tau_{N-1}, d_{C_N}(a_N) = \tau_{N-1} + \tau_N$. Given that $\eta(a_i) \leq 1$ for all vertices a_i, and that the edge weights τ_i are also bounded by 1, the FDPDI for the fuzzy cycle graph C_N is similarly bounded. The upper bound is given by

$$\mathrm{FDPDI}(C_N) \leq N\left(\frac{2\cdot 1\cdot 2\cdot 1}{|2-2|+1}\right) = 4N.$$

Theorem 3.3: *Let $S_N = (\mathcal{V}, \eta^*, \tau^*)$ denote a star graph, where $\eta^* = \{a_1, a_2, \ldots, a_N\}$ consists of N vertices. The edge set is given by $\tau^* = \{a_1a_s, a_2a_s, \ldots, a_{N-1}a_s\}$, with $a_N = a_s$ being the central vertex. The upper bound for the Fuzzy Degree Product Difference Index (FDPDI) of the star graph S_N is $FDPDI(S_N) \leq N - 1$.*

Proof: Each vertex a_i in S_N, for $1 \leq i \leq N-1$, is connected to a central vertex a_s. The degree of the central vertex a_s is $N-1$, while each peripheral vertex a_i has a degree of 1. Since the star graph S_N contains $N-1$ edges and the vertex weights $\eta(a_s)$ and $\eta(a_i)$ are both bounded by 1, the FDPDI for S_N can be expressed as $\mathrm{FDPDI}(S_N) \leq \sum_{i=1}^{N-1}\left(\frac{N-1}{N-1}\right) \leq N-1$.

Theorem 3.4: *The FDPDI for a wheel graph W_N with N peripheral vertices, where $N \geq 4$ is given as $FDPDI(W_N) \leq 3(N-1)\left(3 + \frac{N-1}{N-3}\right)$.*

Proof: In the wheel graph, W_N, a_h is the central hub vertex, and a_i and a_{i+1} are adjacent vertices on the rim of the wheel. Since the degree of the central vertex a_h is $d_{W_N}(a_h) = N-1$ and $\eta(a_h) \leq 1$, the contribution to the FDPDI from the hub vertex is $\frac{3(N-1)^2}{N-3}$.

The peripheral vertices form a cycle C_N with $N-1$ vertices, and assuming the maximum membership value $\eta(a_i)=1$ and degree $d_{W_N}(a_i)=3$ for the rim vertices, the FDPDI for the $N-1$ edges of the rim is $9(N-1)$. Therefore, $\mathrm{FDPDI}(W_N) \leq \left(\frac{9(N-1)+3(N-1)^2}{N-3}\right)$, which simplifies to the upper bound desired.

Theorem 3.5: *The complete graph $K_N=(\mathcal{V},\eta^*,\tau^*)$ of order N consists of the vertex $a_i \in \eta^*$ which is connected to every vertex a_j and $a_i \neq a_j$. Then the Fuzzy Degree Product Difference Index (FDPDI) for K_N is given as $FDPDI(K_N) \leq \frac{N\cdot(N-1)^3}{2}$.*

Proof: The degree of each vertex a_i in K_N is $N-1$. The FDPDI for K_N considers the interactions between each pair of vertices. For any given pair of vertices (a_i,a_j), the contribution to the FDPDI is expressed as $\frac{\eta(a_i)\cdot(N-1)^2\cdot\eta(a_j)}{|\eta(a_i)\cdot(N-1)-(N-1)\cdot\eta(a_j)|+1}$. Also, $\eta(a_i)\leq 1$ and there are $\binom{N}{2}$ edges in K_N, the upper bound can be determined as stated.

Theorem 3.6: *For two sets of vertices U_1 and U_2 in the complete bipartite graph K_{u_1,u_2}, where $|U_1|=u_1$ and $|U_2|=u_2$, the upper bound for the Fuzzy Degree Product Difference Index (FDPDI) is given by $FDPDI(K_{u_1,u_2}) \leq \frac{u_1^2\cdot u_2^2}{|u_1-u_2|+1}$.*

Proof: In the complete bipartite graph K_{u_1,u_2}, each vertex in the set U_1 has degree u_2, while each vertex in the set U_2 has degree u_1. Since there are u_1u_2 edges in K_{u_1,u_2}, linking vertices in U_1 to those in U_2, and the membership values are bounded by 1 we derive the desired upper bound.

4. Degree Product Difference Index (DPDI) in Neutrosophic Graphs

In this section, we have proposed the DPDI for single valued neutrosophic graphs.

Definition 4.1: (Type 2 Degree Product Difference Index).

The Type 2 Degree Product Difference index of the neutrosophic graph $\mathcal{G}=(\eta,\tau)$ is defined by,

$$\mathrm{DPDI}(\mathcal{G}) = \sum_{a_ia_j\in\tau^*((\mathcal{G})} \frac{\eta(a_i)\cdot d_{\mathcal{G}}(a_i)\cdot\eta(a_j)\cdot d_{\mathcal{G}}(a_j)}{|\eta(a_i)\cdot d_{\mathcal{G}}(a_i)-\eta(a_j)\cdot d_{\mathcal{G}}(a_j)|+1}\cdot|SC_P(a_i,a_j)|$$

The strength of the connection between any pair a_i,a_j of vertices is

$$\mathrm{SC}_P(a_i,a_j)=\left(\min_{e\in P_{a_ia_j}}T_\tau(e),\max_{e\in P_{a_ia_j}}I_\tau(e),\max_{e\in P_{a_ia_j}}F_\tau(e)\right),$$

where $P_{a_ia_j}$ is the path between a_i and a_j. The magnitude of this connection is given by:

$$|\mathrm{SC}_P(a_i, a_j)| = 3\left(\min_{e\in P_{a_i a_j}} T_\tau(e)\right) - 2\left(\max_{e\in P_{a_i a_j}} I_\tau(e)\right) - \max_{e\in P_{a_i a_j}} F_\tau(e),$$

and the overall strength of connection between a_i and a_j in the graph $\mathcal{G}$ is $\mathrm{SC}_{\mathcal{G}}(a_i, a_j) = \max_P\{|\mathrm{SC}_P(a_i, a_j)|\}$.

Theorem 4.1: *Let $\mathcal{G}$ and $\mathcal{H}$ represent two neutrosophic graphs. If $\mathcal{G}$ and $\mathcal{H}$ are isomorphic, then the degree product difference index values of both neutrosophic graphs are identical.*

Proof: Let $\mathcal{G} = (\eta_1, \tau_1)$ and $\mathcal{H} = (\eta_2, \tau_2)$ denote two neutrosophic graphs. Consider an identity function $i_{\mathcal{G}}: \eta_1(a_1) \to \eta_2(a_2)$ for all $a_1 \in \eta_1$, such that there exists a corresponding $a_2 \in \eta_2$. Furthermore, let $i_{\mathcal{H}}: \tau_1(a_1, b_1) \to \tau_2(a_2, b_2)$, ensuring that each vertex of $\mathcal{G}$ is mapped to a vertex in $\mathcal{H}$ with the same membership values and corresponding edges. Therefore, while the structural arrangement of the neutrosophic graphs may vary, they possess the same sets of vertices and edges, resulting in an identical degree product difference index.

5. Decision Making in the Metabolic Syndrome

Metabolic syndrome is modeled using decision making technique whose algorithm is elaborated below. The overall decision-making process is illustrated in Figure 2.

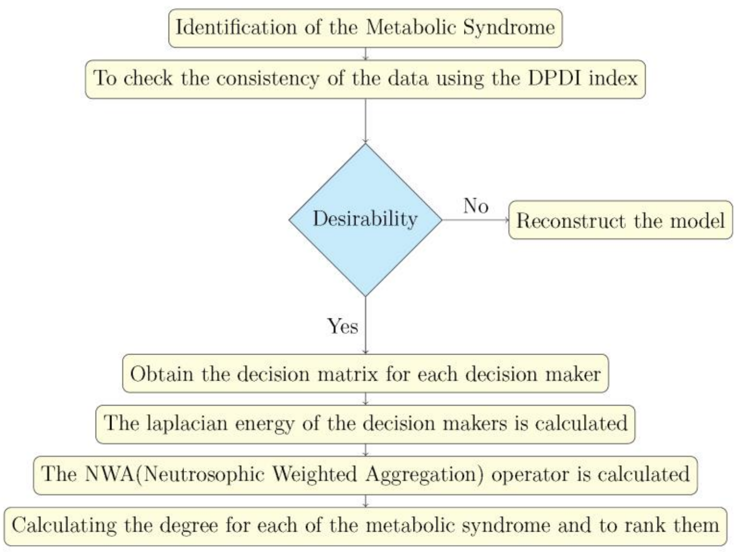

Figure 2
Sequential flow diagram

The procedure for identifying the most commonly linked metabolic syndrome from the algorithm described above involves to calculating the laplacian energy, weights of the decision makers and the NWA operator [7]. The metabolic syndrome includes central obesity (CO), raised blood pressure (BP), high blood triglycerides (HBTGL), low levels of HDL (LLHDL), and impaired fasting glucose (IFG) [9].

Step 1: The metabolic syndrome are ranked based on the importance of causing several diseases by the decision makers. The ranking and corresponding neutrosophic values of the metabolic syndrome are presented in Table 1. Let us calculate the degree product difference index to see if the data is good. The consistency of the data is verified using the DPDI index as shown in Figure 3.

Table 1

Metabolic Syndrome

Metabolic Syndrome	Rank	Single Valued Neutrosophic Numbers [10]
Central Obesity (v_1)	2	(.75,.25,.20)
Raised BP (v_2)	2	(.75,.25,.20)
High blood triglycerides (v_3)	3	(.50,.50,.50)
Low level of HDL (v_4)	3	(.50,.50,.50)
Impaired fasting glucose (v_5)	1	(.90,.10,.10)

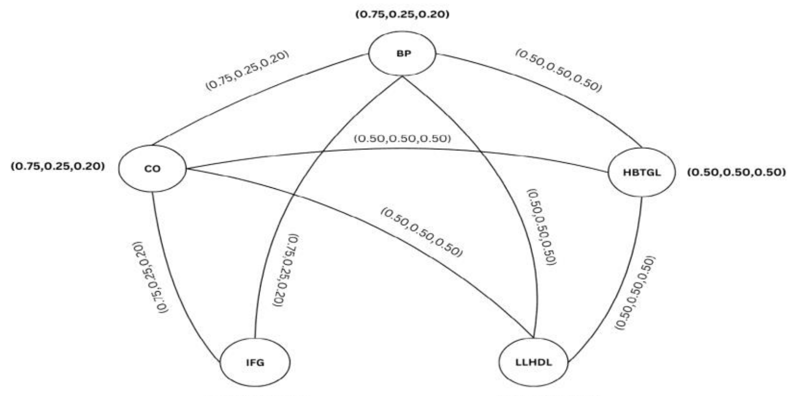

Figure 3

Consistency of the data

A positive value of the Type 2 Degree Product Difference Index indicates the reliability of the data which allows us to proceed to subsequent calculations.

Step 2: The decision matrices of the first decision maker and the second decision maker are obtained. The decision matrices of the first and second decision makers are depicted in Figure 4 and Figure 5 respectively.

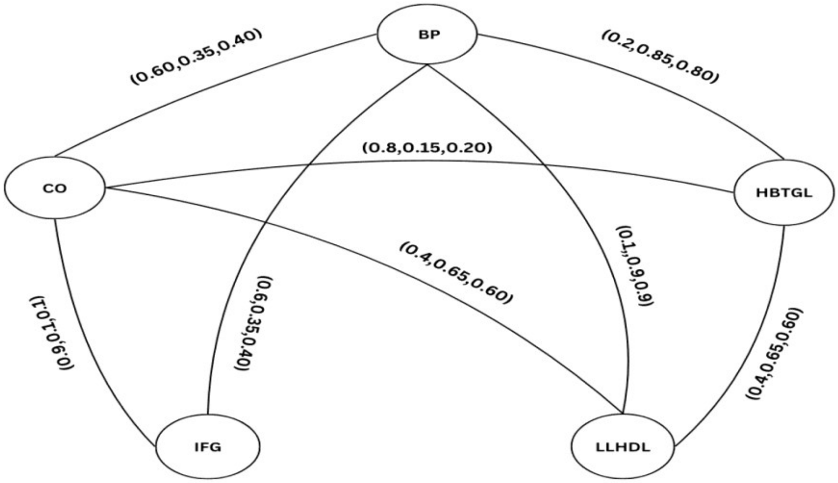

Figure 4
Decision maker 1

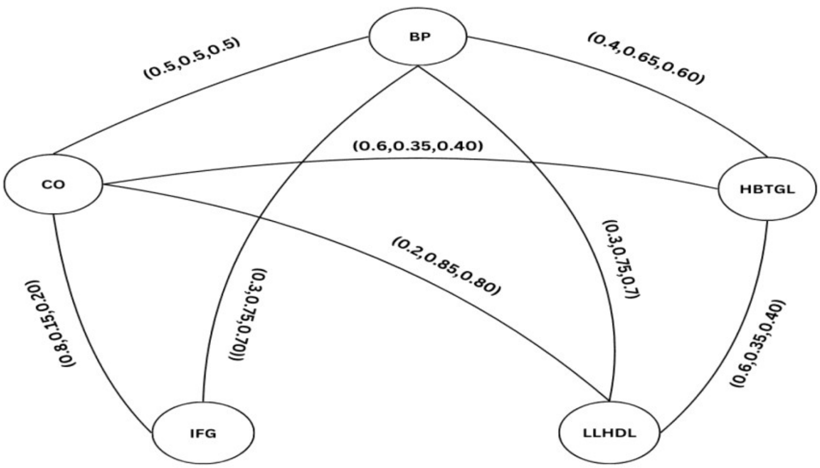

Figure 5
Decision maker 2

Step 3: The next step in the algorithm is to proceed with the laplacian energy of the decision makers based on the matrices R_1 and R_2.

$LE(R_1) = (8.5517, 8.2257, 8.2271)$ and $LE(R_2) = (7.3999, 8.6999, 8.6000)$ The weights of the decision makers are as follows:

$$w_1(R_1) = \left(\frac{8.5517}{15.9516}, \frac{8.2257}{16.9256}, \frac{8.2271}{16.8272}\right) = (0.5361, 0.4859, 0.4889)$$

$$w_2(R_2) = \left(\frac{7.3999}{15.9516}, \frac{8.6999}{16.9256}, \frac{8.6000}{16.8272}\right) = (0.4638, 0.5140, 0.5110)$$

Step 4: The final decision matrix is given as follows by calculating NWA operator:

$$R = \begin{bmatrix} (0,0,0) & (0.93,0.42,0.45) & (0.98,0.23,0.29) & (0.79,0.74,0.69) & (0.99,0.12,0.14) \\ (0.93,0.42,0.45) & (0,0,0) & (0.80,0.74,0.69) & (0.69,0.81,0.79) & (0.89,0.51,0.53) \\ (0.98,0.23,0.29) & (0.80,0.74,0.69) & (0,0,0) & (0.91,0.47,0.48) & (0,0,0) \\ (0.79,0.74,0.69) & (0.69,0.81,0.79) & (0.91,0.47,0.48) & (0,0,0) & (0,0,0) \\ (0.99,0.12,0.14) & (0.89,0.51,0.53) & (0,0,0) & (0,0,0) & (0,0,0) \end{bmatrix}$$

Step 5: The degree for each of the metabolic syndrome is elaborated for the final analysis using the matrix R.

$$d(v_1) = (3.700,1.521,1.570), d(v_2) = (3.328,2.498,2.463)$$
$$d(v_3) = (2.694,1.445,1.434)\ d(v_4) = (2.410,2.038,1.974)$$
$$d(v_5) = (1.892,0.641,0.675)$$

Thus we can say that $v_1 > v_2 > v_3 > v_4 > v_5$. Thus we have obtained central obesity as the most commonly linked metabolic syndrome.

Conclusion

The bounds of the fuzzy graphs are obtained for the index. Its nature is also examined in the instance of neutrosophic groups.

The integration of topological indices with fuzzy and neutrosophic graphs optimizes outcomes and addresses modern decision-making challenges. Other characterizations of the index and its implications in various applications can be dealt further.

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