Journal of Discrete Mathematical Sciences & Cryptography
ISSN 0972-0529 (Print), ISSN 2169-0065 (Online)
Vol. 29 (2026), No. 6, pp. 2421–2430
DOI : 10.47974/JDMSC-2702

A generalization of the extending property for semimodules via weakly supplement subsemimodules

Shimaa M. Dawood *
Saad A. Al-Saadi †
Department of Mathematics
College of Science
Mustansiriyah University
Baghdad
Iraq

Abstract

This work introduces and investigates a generalization of extending semimodules via weak-supplement subsemimodules. An S-semimodule L is called weakly-supplement extending (shortly, $\mathcal{wse}$—semimodule) if every subsemimodule of L is essential in a weakly-supplement subsemimodule of L. We asserted that the class of $\mathcal{wse}$-semimodules properly contains both extending and lifting semimodules. Analogously to extending semimodules, this paper gives many characterizations of $\mathcal{wse}$-semimodules. We showed that L is wse-semimodule if and only if each closed subsemimodule of L is a weakly supplement subsemimodule of L. Moreover, we discussed the relation of wse-semimodule and its relation with other known classes of semimodule, such as $\oplus$-supplemented, lifting, and uniform semimodules. We inspected whether the $\mathcal{wse}$-semimodule property is satisfied under direct sum, subsemimodule, epimorphic image, and inverse image.

Subject Classification (2020): *16Y60, 16Y80.*

Keywords: *Semiring, Extending semimodule, Weakly-supplement subsemimodule, Lifting semimodule.*

1. Introduction

As algebraic structures, semirings and their applications in various fields have motivated many authors to explore new concepts in semirings and

* *ORCID-ID:* `0000-0003-4238-6565`
† *ORCID-ID:* `0000-0002-6369-8173`

* *E-mail:* `Shamaa_84@uomustansiriyah.edu.iq (Corresponding Author)`
† *E-mail:* `saalsaadi@uomustansiriyah.edu.iq`

© Taru Publications

semimodules. In his book, Golan [1] presented interdisciplinary applications of semirings. The concept of injective semimodules over semirings has been studied extensively (for details, see [2]). Following [3], extending semimodules were introduced as a generalization of injective semimodules. Recently, many papers have appeared providing new characterizations, generalizations, and related work on extending semimodules (cf. [4], [5], [6],[7]). The main focus of this paper is devoted to introducing and studying extending semimodules via weakly-supplement subsemimodules.

For the sake of wholeness, we give some definitions and results that are relevant to our work. Let L be a semimodule over a semiring S. A subsemimodule N of L is called subtractive if $n_1 + n_2 \in N$ and $n_1 \in N$ imply that $n_2 \in N$ [1]. A subsemimodule Y of L is said to be small if for any subsemimodule Y^* of L such that $Y + Y^* = L$ implies$Y^* = L$, denoted by $Y \ll L$ [8]. An epimorphism $f: A \longrightarrow B$ (where A and B are S-semimodules) is called small if $\ker(f) \ll A$ [1]. An S- semimodule L is called the direct sum of subsemimodules T_1, and T_2 of L, if each element $t \in T$ can be uniquely written as $t = t_1 + t_2$ where $t_1 \in T_1$ and $t_2 \in T_2$ [8] and denoted by $T = T_1 \oplus T_2$. Also, T_1 is said to be the summand of T if there is a subsemimodule T_2 of T such that $T = T_1 \oplus T_2$. A semimodule L is called lifting if for every subsemimodule $N \leq L$ of L there exist subsemimodules X and Ysuch that $L = X \oplus Y$ with $X \leq N$ and $N \cap Y \ll L$ [9]. Let N and K are subsemimodule of a semimodule L. Then N is called a supplement of K in L if $N + K = L$ and $N \cap K \ll L$. A subsemimodule Y is called a supplement subseminodule in L if it is a supplement of some subsemimodule of L. A semimodule L is said to be supplemented if every subsemimodule of L has a supplement [10]. Moreover, a subtractive subsemimodule N of a semimodule L is supplement if and only if it is minimal with the propriety $N + K = L$ [11]. Let X and Y be subsemimodules of a semimodule L such that $X + Y = L$, $X \cap Y \ll Y$ then we say that Y is a weak supplement of X in L [10]. A semimodule L is called weakly supplemented if every subsemimodule of N has a weak supplement in L[10]. It is clear from the definitions that each direct summand is a supplement subsemimodule and every supplement subsemimdoule is weakly supplement. A non-zero subsemimodule J of L is called an essential subsemimodule of L, if $J \cap B \neq (0)$ for each nonzero subsemimodule B of L and dented by $J \leq^e L$ [12]. A semimodule L is called extending if every subsemimodule of L is essential in a direct summand of L [3]. The left annihilator of x in S is defined by $ann(x) = \{s \in S \mid sx = 0\}$ and the set $Z(L) = \{x \in L | ann(x) is\ essential\ in\ S\}$. It is easy to see that $\mathrm{Z}(L)$ is a subsemimodule of L and we will call it the singular subsemimodule of L. If $\mathbb{Z}(L) = \mathrm{L}$ then L is called singular, and if $Z(L) = 0$, then L is called nonsingular [1]. A semimodule L is said to be uniform if any subsemimodule of L is essential [7]. A semimodule L is referred to as strongly extending if closed subsemimodules of L are invariant summands [13]. A semimodule L is said to be semisimple if it is a direct sum of its simple subsemimodules [14]. Let K be a subsemimodule of a semimodule L, then we say L/K is a quotient semimodule of L by K , defined by $L/K = \{[x] |\ x\ \epsilon L\}$ [10]. A subsemimodule N of L is said

to be fully invariant if $f(N) \sqsubseteq N$ for each endomorphism f on L [14]. A semimodule is said to be duo if each subsemimodule of L is fully invariant [14],[18]. A semimodule C is said to be injective if, for any S-semimodules D and K, any S-monomorphism $\alpha: D \rightarrow K$, and any S-homomorphism $\phi: D \rightarrow C$, there is an S-homomorphism $\beta: K \rightarrow C$ in which $\alpha = \beta \circ \phi$ [15]. If E is an injective S- semimodule and an essential extension of a semimodule C, then E is said to be an injective hull of C, denoted by $E(C)$ [3]. The Jacobson radical of a semimodule L, denoted by $\mathrm{J}(L)$, is the intersection of all maximal subsemimodules of L [16]. A semimodule L is referred to as stable-extending whenever a stable subsemimodule of L is contained essentially in a summand of L [16]. A semimodule L is called distributive if, for all subsemimodules $A, \mathcal{B}$, and $\mathcal{C}$ of L, the following equality holds $\mathcal{A} \cap (\mathcal{B} + \mathcal{C}) = (\mathcal{A} \cap \mathcal{B}) + (\mathcal{A} \cap \mathcal{C})$ [17],[19].

2. Weakly Supplement Extending Semimodules

Firstly, we introduce the following concept:

Definition 2.1: An *S-semimodule L* is called *weakly-supplement extending* (abbreviated as $\mathcal{wse}$ −semimodule) if every *subsemimodule* of L is essential in a weakly-supplement *subsemimodule* of L.

The next result provides as a characterization of $\mathcal{wse}$ −semimodule.

Proposition 2.2: An *S-semimodule L* is a *$\mathcal{wse}$ −semimodule* if and only if every closed *subsemimodule* of L is weakly-supplement *subsemimodule* in L.

Proof: Let A be a closed subsemimodule of L. Since L is a $\mathcal{wse}$ −semimodule, then $A \leq^{e} T$, where T is a weakly-supplement subsemimodule of L. However, A is a closed subsemimodule of L (that is, A has no proper essential extension in L), hence $A = T$. So A is a weakly-supplement subsemimodule in L. Conversely, let A be a subsemimodule of L and, by [5], A is essential in a closed subsemimodule T of L. By the assumption, T is a weakly-supplement subsemimodule in L, and so L is a $\mathcal{wse}$ −semimodule.

Remarks and Examples (2.3):

1. Every weakly supplemented semimodule, and hence every supplemented semimodule, is a $\mathcal{wse}$ −semimodule. The reverse direction is not always true. The $\mathbb{Z}$ as $\mathbb{Z}^{+}$-semimodule is not weakly-supplemented (and hence is not supplemented). For example, the subsemimodule $2\mathbb{Z}$ of $\mathbb{Z}_{\mathbb{Z}^{+}}$ has no weakly-supplement in $\mathbb{Z}_{\mathbb{Z}^{+}}$. But $\mathbb{Z}$ as $\mathbb{Z}^{+}$-semimodule is $\mathcal{wse}$ −semimodule.
2. Every extending semimodule is a $\mathcal{wse}$-semimodule, while the opposite is not generally true. Actually, $\mathbb{Z}_8 \oplus \mathbb{Z}_2$ as an $\mathbb{Z}^{+}$-semimodules is a $\mathcal{wse}$ −semimodule (since it is weakly supplemented) which is not an extending semimodule [5].

3. Every semi-simple semimodule is a $\mathcal{wse}$ −semimodule. The converse is not true generally. For example, $\mathbb{N}$ as $\mathbb{N}$-semimodule is $\mathcal{wse}$ −semimodule but is not semi-simple.
4. Every uniform semimodule is a $\mathcal{wse}$ −semimodule, while the opposite is not always true. For example, $\mathbb{Z}_{10}$ as $\mathbb{Z}^+$ semimodule is $\mathcal{wse}$ −semimodule, but it is not uniform.
5. Extending semimodules and lifting semimodules are incomparable; however, every lifting semimodule is a $\mathcal{wse}$ −semimodule. Moreover, $\mathbb{Z}$ as $\mathbb{Z}^+$-semimodule is $\mathcal{wse}$ −semimodule, and it is not lifting.
6. Every $\oplus$-supplemented semimodule is a $\mathcal{wse}$ −semimodule, while the converse is not always true. For example, $\mathbb{Z}$ as $\mathbb{Z}^+$-semimodule is $\mathcal{wse}$ −semimodule and is not $\oplus$-supplemented.

The next result provides a sufficient condition for $\mathcal{wse}$ −semimodules.

Proposition 2.4: An S-semimodule L is a $\mathcal{wse}$ −semimodule if and only if every subtractive subsemimodule of L is essential in a weakly-supplement subsemimodule of L.

Proof: The necessary part follows directly from the definition of $\mathcal{wse}$ −semimodule. Conversely, let A be a subsemimodule of L. By [5], there is a closed subtractive subsemimodule K of L such that $A \leq^e K$. By the assumption, there exists a weakly-supplement N of L such that $K \leq^e N$. Hence, $A \leq^e N$. Therefore, L is $\mathcal{wse}$ −semimodule.□

Proposition 2.5: Let L be a subtractive S-semimodule which has an injective envelope. Then L is $\mathcal{wse}$ −semimodule if and only if $D \cap L$ is weakly-supplement subsemimodule in L for each summand D of $E(L)$.

Proof: Let D be a summand of the injective hull $E(L)$ of L. Put $E(L) = D + N$, where N is subsemimodule of $E(L)$.Suppose that $D \cap L \leq^e T$, where T is a subsemimodule of L. Let $t \in T$, then $t = x + y$ for some $x \in D$ and $y \in N$. Since D is a summand of the injective hull $E(L)$, by [18, corollary 3.2], D is subtractive. Now, if $t = x + y \in D$ and $x \in D$ then $y \in D$, and this contradicts $D \cap N = 0$. Thus $t \notin D$ and so, $y \neq 0$. Since L is essential in $E(L)$ (by [19, Lemma 2]) then there is $s \in S$ such that $0 \neq sy \in C$. Now $st = sx + sy \in D \cap L \subseteq T$. Again since D is a subtractive and $sx, sx + sy \in D$, then $sy \in D$ and $sy \in T \cap N$. But $N \cap L$ is essential in T, so $(0) = (D \cap L) \cap N \leq^e T \cap N = 0$. Then $sy = 0$, which is a contradiction. Thus $D \cap L$ is a closed subsemimodule of L.By Proposition (2.2), $D \cap L$ is a weakly-supplement of L. Conversely, let N be a subsemimodule of L and Let C be a complement of N in L. By [20, Definition 2.1], C is a subtractive subsemimodule of L. Also by [20, proposition 2.4], then $N \oplus C$ is an essential subsemimodule of L.and by [21, lemma 2.2.3], $E(L) = E(N) \oplus E(C) = E(N \oplus C) = E(L)$. Then, $E(C)$ is a summand of $E(L)$,

and by hypothesis, $E(C) \cap L$ is a weakly-supplement of L. But N is essential in $E(N)$ and L is essential in L, then by [12, Proposition 3.3] $N = N \cap L$ is essential in the weakly-supplement $E(N) \cap L$ in L. Then L is $\mathcal{we}$ −semimodule.

Lemma 2.6: *Let $\alpha: X \to Y$ be an epimorphism image of semimodules X and Y. If N is a closed subsemimodule of Y, then $\alpha^{-1}(N)$ is a closed subsemimodule of X.*

Proof: Let $\alpha: X \to Y$ be an epimorphism, and N be a closed subsemimodule of Y. By the first fundamental theorem of homomorphism for semimodules [22, Theorem 4.2], ${}^{X}/_{\ker(\alpha)} \cong Y$. Let $K = \alpha^{-1}(N)$, which is a subsemimodule of X, then $N \cong {}^{K}/_{\ker(\alpha)}$. Suppose that $\ker(\alpha) \leq K \leq_e T \leq X$, then $T/K \cong T/\ker(\alpha)/K/\ker(\alpha)$. By [6, proposition 4.2] T/K is a singular semimodule, and since Y is non-singular, then $K/\ker(\alpha) \leq_e T/\ker(\alpha)$. Thus, $K/\ker(\alpha) = T/\ker(\alpha)$ by [12, proposition 4.2], hence $K = T$, which means that $K = \alpha^{-1}(N)$ is a closed semimodule of X.□

Proposition 2.7: Let X be a $\mathcal{we}$ −semimodule, then every non-singular image of X is a $\mathcal{we}$ −semimodule.

Proof: Let $\alpha: X \to Y$ be an epimorphism of semimodules with X $\mathcal{we}$ – semimodule and Y a non-singular image of X. Let N be a closed subsemimodule of Y, then by Lemma 2.6, $\alpha^{-1}(N)$ is a closed subsemimodule of X. By proposition (2.2), since X is a $\mathcal{we}$ −semimodule, then $\alpha^{-1}(N)$ is a weakly supplement subsemimodule of X, so there is a subsemimodule H of X such that $X = H + \alpha^{-1}(N)$ and $H \cap \alpha^{-1}(N) \ll X$. then we have $Y = \alpha(H + \alpha^{-1}(N)) = \alpha(H) + \alpha(\alpha^{-1}(N)) = \alpha(H) + N$ and by [15] and since α is epimorphism then $ker(\alpha) \leq \alpha^{-1}(N)$ we have by [16] $\alpha(H \cap \alpha^{-1}(N)) = \alpha(H) \cap \alpha(\alpha^{-1}(N)) = \alpha(H) \cap N \ll Y$ since $(H \cap \alpha^{-1}(N) \ll X$. Then N is a weakly-supplement subsemimodule of Y, and so Y is a $\mathcal{we}$ −semimodule of L**.**

The next result asserts that a $\mathcal{we}$ −semimodule is closed under the isomorphic property.

Proposition 2.8: Let Y_1 and Y_2 are S-semimodule. If Y_1 is a $\mathcal{we}$ −semimodule and $Y_1 \cong Y_2$, then Y_2 is a $\mathcal{we}$ −semimodule .

Proof: Let $\alpha : Y_1 \to Y_2$ be an isomorphism and W be a subsemimodule of Y_2 , Then $\alpha^{-1}(W)$ is a subsemimodule of Y_1 Since Y_1 is $\mathcal{we}$ −semimodule of L. Then $\alpha^{-1}(W)$ is essential in a weakly-supplement subsemimodule K of Y_1, But $\alpha(\alpha^{-1}(W)) = W$ is an essential in $\alpha(K)$. Since K is weakly-supplement subsemimodule in Y_1 then there is H is a subsemimodule of Y_1 such that $Y_1 = K + H$, and $K \cap H \ll Y_1$.Then by [16] $\alpha(Y_1) = \alpha(K) + f(H)$. And so $Y_2 = \alpha(K) + \alpha(H)$, since $K \cap H \ll Y_1$ and $\alpha : Y_1 \to Y_2$ homomorphism, then $\alpha(K \cap H) \ll Y_2$ [16], since α is isomorphism then α is monomorphism and

$ker\,\alpha = 0$, and we have $ker\,\alpha \subseteq H$, then by [16], $\alpha\,(K \cap H) = \alpha\,(K) \cap \alpha\,(H) \ll Y_2$, then $\alpha(K)$ is weakly-supplement subsemimodule of Y_2 , hence Y_2 is a $\mathcal{wse}$ −semimodule.

Proposition 2.9: Let $\alpha: L \to K$ be a small epimorphism and K be a $\mathcal{wse}$ −semimodule, such that for every nonzero closed subsemimodule N of L with $ker\ \alpha \sqsubseteq N$. Then L is $\mathcal{wse}$ −semimodule.

Proof: Let K be a $\mathcal{wse}$ −semimodule, and $\gamma: L \to K$ be a small epimorphism, and N be a closed subsemimodule of L, since $\ker\gamma \leq N$ by [16] and γ is an epimorphism. Then by the first fundamental theorem of homomorphism for semimodules [7, Theorem 2.4], $\gamma\,(N) \cong N/\,ker\,\gamma$ is closed in $L/\,ker\,\gamma \cong K$, then $L/ker\gamma$ is a $\mathcal{wse}$ −semimodule, so $\gamma(N)$ is a weakly-supplement subsemimodule of K, then N is a weakly-supplement subsemimodule of L. Then L is $\mathcal{wse}$ −semimodule.□

Proposition 2.10: Let N be a weakly-supplement subsemimodule of $\mathcal{wse}$ −semimodule L such that the intersection of any two weakly-supplement subsemimodule in L is weakly-supplement in N. Then N is $\mathcal{wse}$ −semimodule.

Proof: Let N be a weakly-supplement subsemimodule in L and let F be a subsemimodule in N. Since L is a $\mathcal{wse}$ −semimodule, F is essential in a weakly-supplement subsemimodule H of L. By hypothesis, $N \cap H$ is a weakly-supplement subsemimodule in N since F is essential in H and N is essential in N. Then we have $F \cap N$ is essential in $H \cap N$ and $F \cap N = F$ is essential in $H \cap N$. Then N is $\mathcal{wse}$ −semimodule.□

Proposition 2.11: Every direct summand of a $\mathcal{wse}$ −semimodule is a $\mathcal{wse}$ −semimodule.

Proof: Let K be a direct summand of $\mathcal{wse}$ −semimodule L, and H be a closed subsemimodule of K. By [5], K is a closed subsemimodule of L. Since L is a $\mathcal{wse}$ −semimodule and H is a closed subsemimodule of L, H is a weakly supplement subsemimodule of L. Let N be a subsemimodule of L such that $L = H + N\ and\ H \cap N << L$. Since every closed subsemimodule is subtractive [5], then by using the modular law for semimodules [20], then $K = K \cap L = K \cap (H + N) = H + (K \cap N)$ and $H \cap (K \cap N) = (H \cap K) \cap N$. Since $(H \cap N) \cap K \subseteq H \cap N$ and $H \cap N \ll L$, so, $H \cap (K \cap N) \ll L$. Since K is a direct summand of L, then by [22], $H \cap (K \cap N) \ll K$ and so H is a weakly-supplement subsemimodule of K. Therefore, K is a $\mathcal{wse}$ −semimodule.□

Proposition 2.12: If K_1 and K_2 are semimodules such that $K = K_1 \oplus K_2$ is a distributive semimodule. Then $K_1 \oplus K_2$ is a $\mathcal{wse}$ −semimodule iff K_1 and K_2 are $\mathcal{wse}$ −semimodules.

Proof: The necessary part is directly by proposition (2.11). Conversely, let D be a closed subsemimodule of $K_1 \oplus K_2$. Since $K_1 \oplus K_2$ is distributive semimodule, then $D = D \cap (K_1 \oplus K_2) = (D \cap K_1) \oplus (D \cap K_2)$. So, $D \cap K_1$ and $D \cap K_2$ are closed subsemimodules of K_1 and K_2 respectively. But K_1 and K_2 are $\mathcal{wse}$ −semimodule. Thus there is subsemimodules N_1 of K_1 and N_2 of K_2 so $K_1 \oplus K_2 = (N_1 + (D \cap K_1)) \oplus (N_2 + (D \cap K_2))$ such that $(N_1 + N_2) \oplus \big((D \cap K_1) \oplus (D \cap K_2)\big) = (N_1 + N_2) + D$, since $(N_1 \cap D) \ll K_1$ and $(N_2 \cap D) \ll K_2$ are $(N_1 \cap D) + (N_2 \cap D) \ll K_1 \oplus K_2$, Hence D is weakly-supplement subsemimodules of $K_1 \oplus K_2$, therefore $K_1 \oplus K_2$ is $\mathcal{wse}$ −semimodule.□

Proposition 2.13: Let $L = \oplus_{i \in I} K_i$ be a semimodule such that every closed subsemimodule in L is fully invariant. Then L is $\mathcal{wse}$ −semimodule iff each K_i for each $i \in I$ is $\mathcal{wse}$ −semimodule .

Proof: The necessary part is directly by proposition (2.11). Conversely, let H be a closed subsemimodule of L and since L is fully invariant then there is a mapping $\alpha: Y \to Y_i$ define by $\alpha_i(x) = y_i$, where $y_i \in Y_i$ and $x = \sum_{i \in I} y_i$ such that $\alpha_i(H) \sqsubseteq H$ and $\alpha_i(H) \sqsubseteq Y_i$ hence $L = \oplus_{i \in I} (H \cap K_i)$. Since $H \cap K_i$ is direct summand then by [5, proposition 2.1.14] $H \cap K_i$ is closed subsemimodules of L. But $H \cap K_i \subseteq K_i \subseteq \mathrm{L}$ then $H \cap K_i$ is closed in K_i. Therefore $H \cap K_i$ is a weakly-supplement subsemimodule of K_i. Thus $L = \oplus_{i \in I} (H \cap K_i)$ is $\mathcal{wse}$ −semimodule , Then L is $\mathcal{wse}$ −semimodule.□

Proposition 2.14: Every weakly-supplement subsemimodule is a direct summand. Then L is $\mathcal{wse}$ −semimodule if and only if L is extending.□

Lemma 2.15: [5] *A subsemimodule of a semimodule is a complement if and only if it is closed.*

Lemma 2.16: *Let L be an S- semimodule and $N \sqsubseteq L$, then N is closed if and only if $[N:L]$ is a closed ideal in S*

Proof: Let N be a closed subsemimodule of a semimodule L, then there exists a subsemimodule H of L such that N is a relative complement of N if $H \cap N = 0$, where H is the maximal subsemimodule satisfies the condition and we get $[N : L]$ is relative complement of $[H : L]$ in S , Also we suppose that $[N : L] \leq^e A$. Where A is ideal in S and since $[H : L] \cap A \sqsubseteq A$ and $(0) = (A \cap [H : LI] \cap [N:L])$ hence $(0) = A \cap [H : L]$. But $[N:L]$ is ideal in S satisfies the relation $[N:L] \cap [H\ L] = 0$, hence $[N:L] \sqsubseteq A$, therefore $[N:L] = A$. conversely suppose that $H \sqsubseteq L$ such that $N \leq^e H$ and so $[N:L] \leq^e [H : L]$ and A is ideal in S such that $A \sqsubseteq [H : L]$ and $[N:L] \cap A = 0$, So $N \cap AL = 0$, But $AL \sqsubseteq H$ and $N \leq^e H$ hence $AL = 0$ when $A = 0$, therefore $[N:L] \leq^e [H:L]$ but $[N:L]$ is closed ideal in S, we get $[N:L] = [H:L]$ hence $N = K$ and therefore N be a closed subsemimodule of a semimodule of L.□

Lemma 2.17: *Let L be an S-semimodule and $N \sqsubseteq L$ if $N \leq^{e} L$ then $[N:(x)] \leq^{e} S, \forall\, x \in L$.*

Proof: Let $x \in L$ and $A \neq 0$ is an ideal in S if $AS_x = 0$ then $ann(x) \sqsubseteq A$. But, $ann(x) \sqsubseteq [N:(x)]$. So $[N:(x)] \sqsubseteq A$ then $A \cap [N:(x)] \neq 0$, and if $AS_x \neq 0$ then $N \cap AS_x \neq 0$. Hence $a \in A$ such that $Ax \neq 0$ then $a \in [N:(x)]$, so $A \cap [N:(x)] \neq 0$ and hence $[N:(x)] \leq^{e} S, \forall\, x \in L$.□

Proposition 2.18: Every finitely generated faithful multiplication S-semimodule over a commutative $\mathcal{wse}-$ semiring S is a $\mathcal{wse}-$semimodule.

Proof: Let N and L be subsemimodules of S, we set $[N : L] = \{s \in S: sL \subseteq N\}$. It is clear that $[N:L]$ is an ideal of S [21]. Let N be a closed subsemimodule of a semimodule L, Since N is a closed subsemimodule in L, $[H:L]$ is a closed ideal in S by Proposition 2.16. Thus, by hypothesis, $[N:L]$ is weakly-supplement ideal in S. Then there exists an ideal I of S such that $[N:L] + I = S$ and $[N:L] \cap I \ll S$. Now by proposition 2.16, we have $L = SL = ([N:L] + I)L = [N:L]L + IL$. Let $([N:L]L) \cap IL) + KL = L$, since L is a multiplication semimodule [23] so $[([N:L] \cap I) + K]L = L$. But L is a finitely generated faithful multiplication S-semimodule. And $L = SL = [([N:L] \cap I) + K]L$, then $S = ([N:L] \cap I) + K$ and since $[N:L] \cap I \ll S$, then we have $S = K$ then $L = KL$ is $\mathcal{wse}-$semimodule.

Acknowledgments

The authors would like to thank Mustansiriyah University (http://www.uomustansiriyah.edu.iq), Baghdad, Iraq, for its moral support in the present work.

References

[1] J. S. Golan, Theory of Semirings with Applications in Mathematics and Theoretical Computer Science, Pitman Monogr. Surveys Pure Appl. Math., Longman Scientific and Technical, Harlow (1992).

[2] K. Głazek, A Guide to the Literature on Semirings and Their Applications in Mathematics and Information Sciences: With Complete Bibliography, Dordrecht: Kluwer Academic Publishers (2002).

[3] S. A. Alhashemi, "Extending Semimodules and Some of Their Generalizations," (Ph.D. thesis), University of Babylon, Babil, Iraq (2022).

[4] A. H. A. Khaleq and A. M. A. Alhossaini, "Distributive semimodules: Theory, properties, and extensions," *Journal of Kufa for Mathematics and Computer*, vol. 10, no. 2, pp. 90–96 (2023).

[5] A. J. Naeemah and S. A. Al-Saadi, "Strongly extending semimodules over a semiring," *Journal of Discrete Mathematical Sciences and Cryptography*, vol. 27, no. 5, pp. 1677–1687 (2024).

[6] K. S. H. Aljebory and A. M. A. Alhossaini, "Almost injective semimodules," *Iraqi Journal of Science*, vol. 64, no. 9, pp. 4634–4643 (2023).

[7] A. Faris and S. A. Al-Saadi, "Fully stable semirings and related concepts," *Journal of Discrete Mathematical Sciences and Cryptography,* vol. 28, no. 4-A, pp. 1025–1030 (2025).

[8] N. X. H. Tuyen and H. X. Thang, "On superfluous subsemimodules," *Georgian Mathematical Journal,* vol. 10, no. 4, pp. 763–770 (2003).

[9] M. Sall, L. Fall, and D. Sow, "On S-lifting semimodules over semirings," in Scientific Days of Doctoral School of Mathematics and Computer Sciences International Conference, Cham, Switzerland: Springer Nature, pp. 147–160 (2023).

[10] M. M. T. Altaee and A. M. A. Alhossaini, "Supplemented and π-projective semimodules," *Iraqi Journal of Science*, vol. 61, no. 6, pp. 1479–1487 (2020).

[11] M. Sall, M. Barry, and L. Fall, "On small subtractive and projective semimodules over zero-sum semiring," *JP Journal of Algebra, Number Theory and Applications*, vol. 61, no. 2, pp. 117–134 (2023).

[12] K. Pawar, "A note on essential subsemimodules," *New Trends in Mathematical Sciences*, vol. 1, no. 2, pp. 18–21 (2013).

[13] A. J. Naeemah and S. A. Al-Saadi, "The class of semimodules with direct summands are stable," *Journal of Discrete Mathematical Sciences and Cryptography*, vol. 28, no. 4-B, pp. 1281–1289 (2025).

[14] A. M. A. Alhossaini and Z. A. H. Aljebory, "On p-duo semimodules," *Journal of University of Babylon, Pure and Applied Sciences*, vol. 26, no. 4 (2018).

[15] S. N. Il'in, "On injective envelopes of semimodules over semirings," *J. Algebra Appl.,* vol. 15, no. 7, Art. no. 1650122 (2016).

[16] S. H. Alsaebari and A. M. Alhossaini, "Nearly injective semimodules," *Journal of University of Babylon for Pure and Applied Sciences*, vol. 27, no. 1, pp. 11–31 (2019).

[17] A. J. Naeemah and S. A. Al-Saadi, "Stable-extending semimodules over semirings," in *AIP Conference Proceedings,* vol. 3264, no. 1, p. 050110, AIP Publishing LLC (Mar. 2025).

[18] J. N. Chaudhari and D. R. Bonde, "On direct sum of partitioning subsemimodules of semimodules over semirings," *Journal of Advanced Research in Pure Mathematics*, vol. 4, no. 1, pp. 81–88 (2012).

[19] E. D. W. Diop, "On essential subsemimodules and weakly co-Hopfian semimodules," *European Journal of Pure and Applied Mathematics*, vol. 9, no. 3, pp. 250–265 (2016).

[20] S. E. Atani and F. E. K. Saraei, "On coatomic semimodules over commutative semirings," *Cankaya University Journal of Science and Engineering*, vol. 8, no. 2 (2011).

[21] M. M. Altaee, π-injective, π-projective semimodules and other related concepts (Ph.D. thesis), University of Babylon, Babil, Iraq (2019).

[22] R. E. Atani, "Prime subsemimodules of semimodules," *International Journal of Algebra*, vol. 4, no. 26, pp. 1299–1306 (2010).

[23] R. R. Nazari and S. Ghalandarzadeh, "Multiplication semimodules," Acta Univ. Sapientiae, Mathematica, vol. 11, no. 1, pp. 172–185 (2019).

Received November, 2025