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Z-small primary submodules

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Abstract

Assume G is the commutative rings with identity and Q is the left G-modules. An aim for this research is to present and study a new concept (Z-small primary submodules), which is a generalization of a previously studied concept (primary submodules). Some of the properties and characteristics of this kind of submodule have been deduced and demonstrated, along with examples. The relationship between it and previous concepts was studied, and conditions of equivalence between them were given. We show that every primary submodule qualifies as a z-small primary submodule. When a module is a z- hollow (hollow), then every z-small primary submodule qualifies as a primary submodule. We also prove each small primary submodules be the z-small primary submodules. When a module is nonsingular, every z-small primary submodule is small primary. Also, we prove that [E: C] is a primary ideal of G, $\forall$ C $\ll_Z$ Z V and C $\supsetneq$ E, if E is a Z-small primary submodule of V.

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Keywords: *Primary submodules, Z-small sub modules, Z-small primary submodules, Small primary submodule.*

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1. Introduction

In [1] and [2], a concept called primary submodules was introduced and studied. Where "a non-zero submodule E of Q is called a primary if whenever $a \in G$ and $e \in Q$ with $ae \in E$ implies that $a \in \sqrt{[E:Q]}$ or $e \in E$". This concepts have been generalized by many researchers and have studied this generalizations for examples; In [2], the author studied the notions of quasi-primary submodules as a submodule E of Q is called "a quasi-primary if primary if $[E:K]$ is a primary ideal of G, for each submodule K of Q". In [3], authors introduced the concept a small primary submodule; Where the "submodule P of Q small primary submodule if whenever $i \in G, e \in Q, < e >$ is small in Q and $ie \in E$, then either $e \in E$ or $i \in \sqrt{[E:Q]}$". Where "a submodule E of Q is called small (notationally, $E \ll Q$) if $E + W = Q$ for all submodules W of Q implies $W = Q$", [4], [5]. We introduce the generalization for concept a Z- small primary submodules as follows; When $y \in Y$ or $q \in \sqrt{[Y:Q]}$ occurs whenever $q \in G$, $y \in Q$ and $< y >$ is a Z-small in Q, $qy \in Y$; Y is a Z-small primary and if a non-zero ideal g of G be the Z-small primary submodules for the G -modules G, so g be the Z-small primary in the G. When "a submodule V of Q is called Z-small ($briefly$ $V \ll_Z Q$) if $V + H = Q$, such that $H \supsetneqq Z_2(Y)$, then $H = Q$", when $Z_2\,(Y)$ be second singular submodules defined by $Z\left(\frac{Y}{Z(Y)}\right) = \frac{Z_2(Y)}{Z(Y)}$, the G-module Q named Z-hollow modules where each non-zero submodules for the Q be the Z-small in Q [4]. The G-modules Q named hollow modules where each non-zero submodules for the Q be the small in Q, [5-6]. Regarding this work, we outline and examine the following generalization of the notion of a Z-small primary sub module: In this research, we presented Z-small primary submodules notion. We discuss the relations among them as well as some kinds for previous introduced submodule.

2. The Results

Definition 2.1: A submodule Y is a Z- small primary submodule when $y \in Y$ or $q \in \sqrt{[Y:Q]}$ occurs whenever $q \in G$, $y \in Q$ and $< y >$ is a Z-small in Q, $qy \in Y$. If a non-zero ideals g for the G be the Z-small primary submodules for the G - modules G, so g be the Z-small primary for the G.

Remarks 2.2:

1. All primary submodules are Z-small primary components.

Proof: Assume that $qy \in Y$, where $q \in G$ and $<y> \ll_Z Q$.

Y is a primary in Q thus either $y \in Y$ or $q \in \sqrt{[Y:Q]}$. As a result, Y's a Z-small primary submodule. But not conversely; put $Q = Z$ to be a Z-module, $Y = 30\,Z$ is the Z-small primary since the only $< 0 > \ll_Z Z$. But it is not primary [2].

2. Every Z-small primary submodules Y for the modules Q be the primary submodules where Q be the Z—hollow (hollow, nonsingular) G—modules. When " G-modules Q named nonsingular modules if Z(Q) =0.

Proof: Assume that $qy \in Y$, where $q \in G$ *and* $y \in Q$, as proof. However, since Q is a Z—hollow (hollow, nonsingular), $<y> \ll_Z Q$ by [5]. Y is a Z-small primary in Q, thus either $y \in Y$ or $q \in \sqrt{[Y:Q]}$. As a result, Y Is Q a primary submodule.

3. All small primary submodules are Z-small primary components.

Proof: Assume that $qy \in Y$, where $q \in G$ and $<y> \ll_Z Q$.Y is a small primary in Q thus either $y \in Y$ or $q \in \sqrt{[Y:Q]}$, [3] As a result, Y's a Z-small primary submodules. But not conversely, put $Q = Z$ to be a Z-modules, $Y = 30Z$ is the Z-small primary (since the only $< 0 > \ll_Z Z$. But it is not a small primary [3].

Theorem 2.3: *Assume E is the nonzero submodules for the modules Q. In such a case, the following are comparable:*

I/ *The Z-small primary submodule E.*

II/ *$\forall, c \in G, W \ll_Z Q$ in such a way that $cW \subsetneq E$, denotes either $c \in \sqrt{[E:Q]}$ or $\subsetneq E$.*

Proof: (I) $\longrightarrow$ (II): Let $cW \subsetneq E$, If $W \nsubseteq E$, then $w \in W$ such that $w \notin E$. Since $w \in W$ and $W \ll_Z Q$ by [6], $(w) \ll_Z Q$ follows., Currently, $cw \in E$. Therefore, $c \in \sqrt{[E:Q]}$

II) $\longrightarrow$(I): Assume $cq \in E$, where $c \in G$, $q \in Q$ and $< q > \ll_Z Q$. Next, $< c >< q > \subsetneq E$. Thus, $c \in \sqrt{[E:Q]}$ by (II) or $< q > \subsetneq E$. As a result, $c \in \sqrt{[E:Q]}$ or $\in E$. Therefore a Z-small primary is E.

Theorem 2.4: *Assume that module Q has a non-zero submodule called L. In such case, the following are comparable:*

1. *The Z-small primary submodule L.*
2. *A Z-small primary is a submodule $[L:_Q B]$, where $B \subsetneq G$.*
3. *A Z-small primary is a $[L:_Q i]$, $\forall, i \in G$.*

Proof: 1/ $\Longrightarrow$2/: Given $cq \in [L:_Q B]$, $c \in G$ and $(q) \ll_Z L$, we cane deduce that $cqB \subsetneq L$ and, consequently, $cqi \in L$, $\forall i \in B$. Given that $(q) \ll_Z Q$, $(qd) \ll_Z Q$ follows [6]. Since L is Z-small primary, either $c \in \sqrt{[L:Q]} \subsetneq \sqrt{[[L:_Q B] : Q]}$ or $iq \in L, \forall i \in B$. Therefore, $c \in \sqrt{[[L:_Q B] : Q]}$ or $qB \subsetneq$

L.That means $c \in \sqrt{[[L:_Q \ B] : Q]}$ or $q \in [L:_Q \ B]$. Therefore the Z-small primary is $[L:_Q \ B]$.

2/ $\Longrightarrow$3/: It is evident. And 3/ $\Longrightarrow$1/: By assuming $= \ 1$, it is straightforward to follow.

Theorem 2.5: *Given a module Q, where Y is a non-zero submodule. In such case, the following are comparable:*

$1 -$ *Z-small primary is a submodule Y.*

$2 - \sqrt{([Y\colon Q])} \ = \ \sqrt{([Y\colon i])}$ *where (i) is Z-small in Q and $i \ \in Q\backslash Y$.*

$3 - \sqrt{[Y\colon Q]} \ = \ \sqrt{[Y\colon C]}$ *, where C is Z-small in Q and* $\forall, Y \ \subsetneq C < Q$.

Proof: 1/ $\Longrightarrow 2/$ Assum that i$\notin Y$ and c $\in \sqrt{[Y\colon i]}$. For some $k \in Z_+$, $c^k i \ \in Y$, and $c \ \in \sqrt{[Y\colon Q]}$ since Y is Z-small primary and i $\notin Y$. Hence $\sqrt{[Y\colon i]} \subsetneq \sqrt{[Y\colon Q]}$. If $a \in \sqrt{[Y\colon Q]}$, then for every k $\in Z_+$ $, a^k \in [Y\colon Q]$ and hence $a^k \ Q \subsetneq Y$. But $a^k \ i \ \subsetneq \ a^k \ Q \ \subsetneq Y$. Thus, $a^k \ \in [Y$:i] for some $\in Z_+$. Hence $a \ \in \sqrt{[Y\colon i]}$ and $\sqrt{[Y\colon Q]} \ \subsetneq \ \sqrt{[\ Y\colon i]}$. As a result, $\sqrt{[Y\colon Q]} = \sqrt{[Y\colon i]}$.

2/$\Longrightarrow 3/$ Assume $c \in \ \sqrt{[Y\colon C]}$, then $c^k \ \in [Y\colon C]$, and $c^k C \ \subsetneq \ Y$. Because $Y \ \subsetneq C$, there is a possibility that $i \ \in C$ and $i \ \notin Y$. However, since C is Z-small, (e) is also Z-small according to [6]. Then c^k i $\ \in Y \ and \ \ c^k \in [\ Y\colon i] \subsetneq \sqrt{[\ Y\colon e]} = \ \sqrt{[Y\colon Q]}$ by (2). This implies that $(c^k)^z \in [Y\colon Q]$ for some $z \ in \ Z_+$ and thus $c \ \in \sqrt{[\ Y\colon Q]}$, and thus $\sqrt{[Y\colon C]} \ \subsetneq \ \sqrt{[\ Y\colon Q]}$. But $[Y\colon Q] \ \subsetneq [Y\colon C]$, so $\sqrt{[Y\colon Q]} \ \subsetneq \ \sqrt{[Y\colon C]}$. Thus, $\sqrt{[Y\colon Q]} = \sqrt{[Y\colon C]}$.

$3/\Longrightarrow 1/$ Assume $c \in G, q \ \in Q$ and $(q) \ll_Z \ Q$ with $cq \ \in Y$. Assuming that $q \ \notin Y$, $C = Y + $(q) implying Y $\subsetneq C$. If $q \in C$, then $c \in [\ Y\colon C] \subsetneq \ \sqrt{[Y\colon C]}$ $= \sqrt{[Y\colon Q]}$. As a result, Y is Z-small primary.

Examples 2.6:

1- $< 0 >$ *is a* Z-small primary of Z_4 . Since $< \bar{2} > \ll_z Z_4$, and $2Z = \sqrt{4Z} = \sqrt{ann \ Z_4} \ = \ \sqrt{ann < \bar{2} >} = \sqrt{2Z} = 2Z$.

2- $< 0 >$ *is not* Z-small primary of Z_{12}. Since $< \bar{6} > \ll_z Z_{12}$, and $6Z = \sqrt{12Z} = \sqrt{ann \ Z_{12}} \ \neq \ \sqrt{ann < \bar{6} >} = \sqrt{2Z} = 2Z$.

Theorem 2.7: *Assume that Q be G-modules & A is ideals for the G with $A \ \subsetneq \ ann \ Q$. Then* (0) *is Z-small primary G-submodules for the Q iff* (0) *be the Z-small primary $G/A \ -$ submodules for the Q.*

Proof: If $(a + A)q = 0,$ (a+A) $\ \in G/A, q \in Q$ with $0 \neq \ (q) \ll_Z Q$. Then $(a + A)q = aq = 0$. Since (0) is a Z-small primary $G/A \ -$ submodule and $(q) \ll_Z Q$, so $a^m \ \in \ ann_G \ Q$. Thus, $a^m \ q = (a^m + A)q = \ (a + A)^m$ q $= \ 0$.

Hence $(a+A)^m \in ann_{G/A}\, Q$. As a result (0) is a Z-small primary $G/A-$ submodule. $\Leftarrow$) By analogous evidence.

Proposition 2.8: Let $\sqrt{[E:Q]}\, equal \sqrt{[H:Q]}$ and let E and H be Z-small primary submodules of module Q. Q' s Z-small primary submodule is then $\cap\, H$.

Proof: Assume that $cq \in E\, \cap H$ and that $\in G, q \in Q$, and $(q) \ll_Z Q$. And $cq\, \in E\, , cq \in H$. As a result, either $q \in E$ or $c \in \sqrt{[E:Q]}$ and either $q \in H\, or\, c \in \sqrt{[H:Q]}$. Thus, either $(q \in E$ and $q \in H)$ or $(c \in \sqrt{[\mathrm{E:Q}]} = \sqrt{[\mathrm{H:Q}]}$. This means either $(q \in E\, \cap H)$ or $(c \in \sqrt{[\mathrm{E} \cap H : \mathrm{Q}]}$. Therefore a Z-small primary is $E \cap H$.

Proposition 2.9: Let W, E be two submodules of an $G-$module V such that E is not contained in W. let W be the Z-small primary submodules for the V, then $E \cap W$ be the Z-small primary submodules for the E.

Proof: Since $E \nsubseteq W$, then $E \cap W$ is a proper submodule of E. Let $r \in G, e \in E$ and $e \ll_Z E$ such that $re \in E \cap W$, then $re \in W$. But $x \ll_Z E$, so $x \ll_Z V[4]$, so $x \in W$ or $e^n\, V \subseteq W\, n \in Z_+$. Hence either $x \in W \cap E$ or $e^n V \subseteq W \cap E$. But, $W \cap E$ is a Z-small primary of E.

Proposition 2.10: Assume that module V has a non-zero submodule called E. $[E:C]$ is a primary ideal of G, $\forall\, C \ll_Z V$ and $C \supsetneqq E$, if E be the Z-small primary submodules for the V.

Proof: Let $ci \in [E:C]$ and $c, i \in G$. Assume for every $q \in C$ and $ciq \in E, iq \notin E$. Since $C \ll_Z V$, [7] gives us $(iq) \ll_Z$ V. $c \in \sqrt{[E:V]} \subseteq \sqrt{[E:C]}$, although E is Z-small primary. $[E:C]$ is hence a primary ideal of $G, \forall\, C \ll_Z V$ and $C \supsetneqq E$.

Notes 2.11: The opposite of (2.10) is false. $< \overline{12} >$ is not a Z-small primary submodule of Z_{24}. However, $< \overline{6} > \ll_Z Z_{24}$ and $< \overline{12} > \subsetneqq < \overline{6} >, [< \overline{12} > : < \overline{6} >] = 2Z$ is a primary ideal of Z.

Proposition 2.12: Let $\alpha: V \longrightarrow V'$ is the $G-$epimorphism and E be the Z-small primary submodules for the $G-$modules V', so we have $\alpha^{-1}(E)$ be the Z-small primary submodules for the V.

Proof: To shows $\alpha^{-1}(E)$ be the proper submodules for the V. We assume $\alpha^{-1}(E) = V$, then $\alpha(V) \subseteq E$, which a contradiction to the assumption. Let $a\, \in G, e\, \in V$ and $(e) \ll_Z V$ such that $ae\, \in \alpha^{-1}(E)$. Then$a\, \alpha\, (e) \in E$. But $(e) \ll_Z V$, so $\alpha(e) \ll_Z V\,'$ by [4] and as E is a Z-small pimary submodule of V', then either $\alpha(e)\, \in E$ or $a^n\, V' \subseteq E$ for some $n \in Z_+$. If $\alpha\, (e) \in E$, then

$e \in \alpha^{-1}(E)$. If $a^n V' \subseteq E$, then $a^n \alpha(V) \subseteq E$ since $\alpha(V) = V'$. This implies $a^n V \subseteq \alpha^{-1}(E)$ $n \in Z_+$. Thus, $\alpha^{-1}(E)$ be Z-small primary submodules for V.

Proposition 2.13: Let $\alpha: V \longrightarrow V'$ *is* epimorphism and E is Z-small primary submodules for G −modules V containing $Ker\alpha$, then $\alpha(E)$ be Z-small primary submodules for V'.

Proof: $\alpha(E)$ is a proper submodule of V'. If not suppose $\alpha(E) = V'$, let $m \in V$ such that $\alpha(m) \in V' = \alpha(E), \exists\, n \in E$ such that $\alpha(n) = \alpha(m)$ hence $\alpha(n-m) = 0$, then $n - m \in ker\alpha \subseteq E$, then $m \in E$, hence $E = V$ (contradiction), since $E \subsetneq V$. Let $r \in G, m' \in V'$ such that $r\,m' \in \alpha(E)$ and $m' \ll_Z V'$, we want to show that either $m' \in \alpha(E)$ or $r \in [\alpha(E): V']$. Suppose $m' \notin \alpha(E)$. Since α is an epimorphism and $m' \in V'$, then there exists $m \in V$ such that $\alpha(m) = m'$. Now, $m \notin E$, so $r\,m' = r\,\alpha(m) = \alpha(r\,m) \in \alpha(E)$, then $\exists\, y \in E$ such that $\alpha(y) = \alpha(r\,m)$, hence $\alpha(r\,m - y) = 0$, then $(r\,m - y) \in ker\,\alpha \subseteq E$, then $r\,m \in E$. But $m' = \alpha(m) \ll_Z V'$, so $\alpha^{-1}(m') = m \ll_Z V$ and E is Z-small primary submodule in V, and $m \notin E$, therefore $r^n \in [E: V]$ and hence $r^n V \subseteq E$, which implies that $r^n \alpha(V) \subseteq \alpha(E)$. But α is an epimorphism, so $r^n \alpha(V) \subseteq \alpha(E)$, thus $r^n V' \subseteq \alpha(E)$. Thus $r^n \in [\alpha(E): V']$. Which implies that $\alpha(E)$ is Z-small primary submodule of V'.

Proposition 2.14: Suppose that V_1, V_2 be two G-modules and let $V = V_1 \oplus V_2$. If $E = E_1 \oplus E_2$ be Z-small primary submodules for the V, so we have E_1 & E_2 are Z-small primary for the V_1 & V_2 respective.

Proof: To prove E_1 is a Z-small primary of V_1. Let $a \in G, x \in V_1, (x) \ll_Z V_1$ such that $ax \in E$, then $a(x, 0) \in E \oplus E_2$. But $(x) \ll_Z V_1$ and $(0) \ll_Z V_2$, so $(x, 0) \ll_Z V_1 \oplus V_2$ by [6]. But $E_1 \oplus E_2$ is a Z-small primary submodule of V. Hence either $(x, 0) \in E_1 \oplus E_2$ or $a^n \in [E_1 \oplus E_2 : V_1 \oplus V_2]$ $n \in Z_+$. Thus, $x \in E_1$ or $a^n \in [E_1: V_1]$ $n \in Z_+$. Therefore E_1 be the Z-small primary of V_1. By similar proof, E_2 be the Z-small primary for the V_2 [6-7].

3. Discussion and Conclusion:

The most important findings we have achieved in this research are that all primary submodules are cap Z-small primary components, and every Z-small primary submodules Y for the modules Q be the primary submodules where Q be the Z-hollow G—modules. if E is the nonzero submodules for the modules Q, then Z-small primary submodules E if and only if $\forall$, $c \in G, W \ll_Z Q$ in such a way that $cW \subsetneq E$, denotes either $c \in \sqrt{[E: Q]}$ or $W \subsetneq E$.

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