Journal of Discrete Mathematical Sciences & Cryptography
ISSN 0972-0529 (Print), ISSN 2169-0065 (Online)
Vol. 29 (2026), No. 5, pp. 2139–2146
DOI : 10.47974/JDMSC-2716

Unique $\mathcal{F}$-Closure Modules ($\mathcal{UFC}$ -module) and $\mathcal{F}$-$\mathcal{G}$-extending modules

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Abstract

In this research, we presented new types of modules, which are $\mathcal{F}$ -extending modules, $\mathcal{F}$-Goldie extending ($\mathcal{F}$-$\mathcal{G}$-extending module) and a unique $\mathcal{F}$-closure module ($\mathcal{UFC}$ -module), Which are generalizations for each of extending modules, Goldie extending module and a unique closure module.

In this work, we introduce a new type of modules that is ψ_F-G-extending modules and we define a module M is called ψ_F-G-Goldie-extending iff for all submodule $\mathcal{V}$ of M that containing F, where $\boldsymbol{\mathcal{H}} \leq_{F\oplus}$ M, F $\leq \boldsymbol{\mathcal{H}}$ satisfies and V ψ_F $\boldsymbol{\mathcal{H}}$. We provide many characteristics related to this kind. In a module M, we define the relationships on the group of submodules of M:

i. V$\psi_{\mathcal{F}}$ B iff $\mathcal{K} \leq$ M satisfies $\mathcal{V} \leq_{\mathcal{F}E} \mathcal{K}$ and $B \leq_{\mathcal{F}E} \mathcal{K}$.

ii. V$\varphi_{\mathcal{F}}$ B iff $\mathcal{V} \cap$ B $\leq_{FE}$ $\mathcal{V}$ and $\mathcal{V} \cap B \leq_{FE} B$.

A module $\mathcal{M}$ is called $\mathcal{F}$ -Goldie extending ($\mathcal{F}$-$\mathcal{G}$-extending module) if for every $\mathcal{V} \leq \mathcal{M}$ that contains $\mathcal{F}$, where a $\mathcal{F}$-direct summand $\boldsymbol{\mathcal{H}}$ of $\mathcal{M}$ satisfies $\mathcal{F} \leq \boldsymbol{\mathcal{H}}$ and $\mathcal{V}$ $\varphi_{\mathcal{F}}$ $\boldsymbol{\mathcal{H}}$ and we introduce some Characteristics and Features for this kind module.

Subject Classification: *03C30, 03C52, 03C50, 03C90.*

Keywords: *F-essential submodules, F-closed, F-G extending.*

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1. Introduction

All rings appearing in this work are understood to have identity, R be a ring and per modules are left R- modules has unity. In [1-3], for a module M, rename for relations on the group of submodules of M.

(1) $\mathcal{V}\alpha B$ iff $where\ \mathcal{H} \leq_e \mathcal{M}$ satisfies $\mathcal{V} \leq_e \mathcal{H}$ and $B \leq_e \mathcal{H}$.

(2) $\mathcal{V}\beta\mathcal{B}$ iff $\mathcal{V} \cap \mathcal{B} \leq_e \mathcal{V}$ and $\mathcal{V} \cap \mathcal{B} \leq_e \mathcal{B}$.

Recalled $\mathcal{M}$ is named a module extending if for all $\mathcal{V} \leq \mathcal{M}$ where $\mathcal{H} \leq_{\oplus} \mathcal{M}$ satisfies $\alpha\, \mathcal{H}$. A module $\mathcal{M}$ is name Goldie extending $\forall\, \mathcal{V} \leq \mathcal{M}$ one can find

$\mathcal{H} \leq_{\oplus} \mathcal{M}$ satisfies $\beta\, \mathcal{H}$ [4]. Recalled $\mathcal{V}\ is\ FE - small$ submodule of $\mathcal{M}$ ($\mathcal{V} \ll_{FE} \mathcal{M}$), if for all $\mathcal{K} \leq_e \mathcal{M}$ such that $F \subseteq \mathcal{V} + \mathcal{K}$ implies that $F \subseteq \mathcal{K}$ [5].

In this paper, we define new relations ψ_F, $\varphi_{\mathcal{F}}$ on modules and $\psi_F - G -$ extending modules, also we give many results on ψ_F, $\varphi_{\mathcal{F}}$.

In a module M, we define the relationships on the group of submodules of $\mathcal{M}$:

(i) $\mathcal{V}\psi_{\mathcal{F}}\mathcal{B}$ iff $\exists\, \mathcal{K} \leq M$ satisfies $\mathcal{V} \leq_{\mathcal{F}E} \mathcal{K}$ and $\mathcal{B} \leq_{\mathcal{F}E} \mathcal{K}$.

(ii) $\mathcal{V}\varphi_{\mathcal{F}}B$ iff $\mathcal{V} \cap B \leq_{FE} \mathcal{V}$ and $\mathcal{V} \cap B \leq_{FE} B$ [6-7].

We have introduced new types of modules based on these relationships.

A module $\mathcal{M}$ is called $\mathcal{F}$ -extending, whenever $\forall \mathcal{A} \in \mathcal{S}_{\mathcal{F}}$ there is $\mathcal{D} \leq_{\mathcal{F}\oplus} \mathcal{M}$ and $\mathcal{D} \in \mathcal{S}_{\mathcal{F}}$ satisfies $\mathcal{A} \leq_{\mathcal{F}E} \mathcal{D}$. A module $\mathcal{M}$ is named $\mathcal{F}$ -Goldie extending ($\mathcal{F}$-$\mathcal{G}$-extending module), whenever for every $\mathcal{V} \leq \mathcal{M}$ that contains $\mathcal{F}$, there is $\mathcal{H} \leq_{\mathcal{F}\oplus} \mathcal{M}$ satisfies $\mathcal{F} \leq \mathcal{H}$ and $\mathcal{V}\ \varphi_{\mathcal{F}}\ \mathcal{H}$. A module $\mathcal{M}$ is said to be a unique $\mathcal{F}$-closure module ($\mathcal{UFC}$ -module) whenever each $A \leq \mathcal{M}$ has a unique $\mathcal{F}$-closure in M. we introduce some characteristics and features therefore kind modules.

2. The Relation $\boldsymbol{\psi_{\mathcal{F}}}$

This section is devoted to examining the properties of submodules $\boldsymbol{\psi_{\mathcal{F}}}$equivalent to a direct summand.

Definition 2.1: Let $\mathcal{F} \leq \mathcal{M}$ and $\mathcal{S}_{\mathcal{F}}$ =$\{\ \mathcal{B} \leq \mathcal{M}\ :\ \mathcal{F} \leq \mathcal{B}\ \}$. Let $\mathcal{A}$ and $\mathcal{B} \in \mathcal{S}_{\mathcal{F}}$, we say $\mathcal{A}\psi_{\mathcal{F}}\mathcal{B}\ where$ a submodule $\mathcal{V}$ of $\mathcal{M}$ satisfies $\mathcal{A} \leq_{\mathcal{F}E} \mathcal{V}$ and $\mathcal{B} \leq_{\mathcal{F}E} \mathcal{V}$.

A module $\mathcal{M}$ is define $\mathcal{F} -$ extending, if $\forall \mathcal{A} \in \mathcal{S}_{\mathcal{F}}$, there is $\mathcal{H} \leq_{\oplus} \mathcal{M}$ and $\mathcal{H} \in \mathcal{S}_{\mathcal{F}}$ satisfies $\mathcal{A} \leq_{\mathcal{F}E} \mathcal{H}$.

$\big($**Notes** $\mathcal{H} \leq_{\mathcal{F}\oplus} \mathcal{M}$ $(\mathcal{M} = \mathcal{H} \oplus_{\mathcal{F}} \mathcal{H}_1)$ $\Rightarrow \mathcal{M} = \mathcal{H} + \mathcal{H}_1$ and $\mathcal{H} \cap \mathcal{H}_1 \leq \mathcal{F}\big)$

Examples 2.2:

(1) Let $\mathcal{Z}_4$ be as $\mathcal{Z}$-module and suppose $\mathcal{U} = \mathcal{Z}_4$, $\mathcal{V} = \{\overline{0}, \overline{2}\}$ and $\mathcal{F} = \{\overline{0}\}$. Since $\mathcal{V} \leq_{\mathcal{F}E} \mathcal{Z}_4$ and $\mathcal{U} \leq_{\mathcal{F}E} \mathcal{Z}_4$, then $\mathcal{V}\psi_{\mathcal{F}}\mathcal{U}$.

(2) Consider $\mathcal{Z}_{12}$ as $\mathcal{Z}$-module and let $\mathcal{U} = \{\overline{0}\,, \overline{3}, \overline{6}\,, \overline{9}\}$, $\mathcal{F}$=$\{\overline{0}\,, \overline{6}\}$ and

$\mathcal{V} = \{\overline{0}\,, \overline{2}\,, \overline{4}\,, \overline{6}, \overline{8}\,, \overline{10}\}$ and since $\mathcal{U} \leq_{\mathcal{F}E} \mathcal{Z}_{12}$, and as $\mathcal{V} \nleq_{\mathcal{F}E} \mathcal{Z}_{12}$, hence $\mathcal{V}$ is not relate to $\mathcal{U}$ by $\psi_{\mathcal{F}}$.

Proposition 2.3: Suppose $\phi \leq \mathcal{M}$. Then $\mathcal{V}\psi_\phi\mathcal{B}$ iff $\frac{\mathcal{V}}{\phi}\psi\frac{\mathcal{B}}{\phi}$ for all $\mathcal{V}$ and $\mathcal{B} \in \mathcal{S}_\phi$.

Proof: Suppose $\mathcal{V}\psi_\phi\mathcal{B}$. Then we can find $\mathcal{C} \leq \mathcal{M}$ satisfies $\mathcal{V} \leq_{\phi E} \mathcal{C}$ and $\mathcal{D} \leq_{\phi E} \mathcal{C}$. Then $\frac{\mathcal{V}}{\phi} \leq_e \frac{\mathcal{B}}{\phi}$ and $\frac{\mathcal{V}}{\phi} \leq_e \frac{\mathcal{C}}{\phi}$, then $\frac{\mathcal{V}}{\phi}\psi\frac{\mathcal{B}}{\phi}$

Conversely, $\frac{\mathcal{A}}{\phi}\psi\frac{\mathcal{B}}{\phi}$ then there is a submodule $\frac{\mathcal{C}}{\phi}$ of $\frac{\mathcal{M}}{\phi}$ such that $\frac{\mathcal{V}}{\phi} \leq_e \frac{\mathcal{C}}{\phi}$ and $\frac{\mathcal{B}}{\phi} \leq_e \frac{\mathcal{C}}{\phi}$. Then $\mathcal{V} \leq_{\phi E} \mathcal{C}$ and $\mathcal{B} \leq_{\phi E} \mathcal{C}$. Thus $\mathcal{V}\ \psi_\phi\mathcal{B}$.

Remark 2.4: The $\psi_\mathcal{F}$ relation is a symmetric relation and reflexive relation.

Proof: The proof follows directly from the definition.

Definition 2.5: Consider submodules $\mathcal{V}, \mathcal{F}$ and $\mathcal{U}$ of an module $\mathcal{N}$. Thus $\mathcal{U}$ is ***$\mathcal{F}$*-closure** of $\mathcal{V} + \mathcal{F}$ in $\mathcal{N}$ if $\mathcal{V} + \mathcal{F} \leq_{\mathcal{F}E} \mathcal{U}$ and $\mathcal{U}$ is a $\mathcal{F}$ -closed in $\mathcal{N}$.

A module $\mathcal{N}$ is said to be a **unique $\mathcal{F}$-closure module**($\mathcal{UFC}$-module) whenever for each submodule $\mathcal{A}$ of $\mathcal{N}$ has a unique $\mathcal{F}$ -closure in $\mathcal{N}$.

Examples 2.6:

(1) Consider Z_6as Z-module and let F=$\{\overline{0}\}$, since $\{\overline{0}, \overline{3}\}$ is F-essential in $\{\overline{0}, \overline{3}\}$and $\{\overline{0}, \overline{3}\}$ is not F-essential inZ_6, then $\{\overline{0}, 3\}$ is F-closed in Z_6. Now since $\{\overline{0}, \overline{2}, \overline{4}\}$ is F-essential in $\{\overline{0}, \overline{2}, \overline{4}\}$and $\{\overline{0}, \overline{2}, \overline{4}\}$is not F -essential in Z_6, then $\{\overline{0}, \overline{2}, \overline{4}\}$is F -closed in Z_6. And $Z_6$is F-essential and F-closed in Z_6. Since $\{\overline{0}\}$ is F-essential in $\{\overline{0}\}$and $\{\overline{0}\}$is not F-essential in Z_6, then $\{\overline{0}\}$ is F-closed in Z_6. Thus $Z_6$is UFC-module.

(2) The module as Z-module, where p is a prime integer. Let F = 0 and since the module is not UC-module, it follows that is not a UFC-module.

Proposition 2.7: Let $\mathcal{F} \leq \mathcal{M}$. Then $\psi_\mathcal{F}$ is a transitive iff $\mathcal{M}$ is $\mathcal{UFC}$-module.

Proof: Assume that $\psi_\mathcal{F}$ is a transitive and suppose $\mathcal{V} \in \mathcal{S}_\mathcal{F}$ such that $\mathcal{V} \leq_{\mathcal{F}E} \mathcal{U}$ and $\mathcal{V} \leq_{\mathcal{F}E} \mathcal{C}$, where $\mathcal{U}$ and $\mathcal{C}$ are $\mathcal{F}$-closed submodules of M. To show$\mathcal{U} = \mathcal{C}$. Since $\mathcal{V} \leq_{\mathcal{F}E} \mathcal{U}$ and $\mathcal{U} \leq_{\mathcal{F}E} \mathcal{U}$, then $\mathcal{V}\psi_\mathcal{F}\mathcal{U}$. Since $\mathcal{V} \leq_{\mathcal{F}E} \mathcal{C}$ and $\leq_{\mathcal{F}E} \mathcal{C}$, then $\mathcal{V}\psi_\mathcal{F}\mathcal{C}$. But $\psi_\mathcal{F}$ is transitive, therefore $\mathcal{U}\psi_\mathcal{F}\mathcal{C}$. Hence there exists a submodule $\mathcal{D} \in \mathcal{S}_\mathcal{F}$ satisfies $\mathcal{U} \leq_{\mathcal{F}E} \mathcal{D}$ and $\mathcal{C} \leq_{\mathcal{F}E} \mathcal{D}$. But each of $\mathcal{U}$ and C are $\mathcal{F}$-closed, therefore, $\mathcal{U} + \mathcal{F} = +\mathcal{F} = \mathcal{D}$. Hence $\mathcal{U} = \mathcal{C} = \mathcal{D}$. Hence $\mathcal{M}$ is $\mathcal{UFC}$ -module. Conversely, suppose that $\mathcal{M}$ is $\mathcal{UFC}$ -module and let $\mathcal{V}, \mathcal{U}$ and $\mathcal{C} \in \mathcal{S}_\mathcal{F}$ such that $\mathcal{V}\psi_\mathcal{F}\mathcal{U}$ and $\mathcal{U}\psi_\mathcal{F}\mathcal{C}$,we have to prove $\mathcal{V}\psi_\mathcal{F}\mathcal{C}$. Since $\mathcal{V}\psi_\mathcal{F}\mathcal{U}$ and$\mathcal{U}\psi_\mathcal{F}\mathcal{C}$, then there exists submodules $\mathcal{V}_1$ and $\mathcal{U}_1 \in \mathcal{S}_\mathcal{F}$ such that $\leq_{\mathcal{F}E} \mathcal{V}_1$,$\mathcal{U} \leq_{\mathcal{F}E} \mathcal{V}_1$, $\mathcal{U} \leq_{\mathcal{F}E} \mathcal{U}_1$ and $\mathcal{C} \leq_{\mathcal{F}E} \mathcal{U}_1$. Let $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$ are $\mathcal{F}$-closed submodules of M contain $\mathcal{F}$ satisfies $\mathcal{V} \leq_{\mathcal{F}E} \mathcal{H}_1$, $\mathcal{U} \leq_{\mathcal{F}E} \mathcal{H}_2$and $\mathcal{C} \leq_{\mathcal{F}E} \mathcal{H}_3$. Since $\mathcal{V} \cap \mathcal{U} \leq_{\mathcal{F}E} \mathcal{V}_1$ and $\mathcal{U} \cap \mathcal{C}$ $\leq_{\mathcal{F}E} \mathcal{U}_1$, hence $\mathcal{V} \cap \mathcal{U} \leq_{\mathcal{F}E} \mathcal{V} \leq_{\mathcal{F}E} \mathcal{H}_1$, $\mathcal{V} \cap \mathcal{U} \leq_{\mathcal{F}E} \mathcal{U} \leq_{\mathcal{F}E} \mathcal{H}_2$, $\mathcal{V} \cap \mathcal{C} \leq_{\mathcal{F}E} \mathcal{U}$ $\leq_{\mathcal{F}E} \mathcal{H}_2$ and $\mathcal{U} \cap \mathcal{C} \leq_{\mathcal{F}E} \mathcal{C} \leq_{\mathcal{F}E} \mathcal{H}_3$. But $\mathcal{M}$ is $\mathcal{UFC}$-module, therefore $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}_3$, hence $\mathcal{V}\psi_\mathcal{F}\mathcal{C}$. Thus $\psi_\mathcal{F}$ is transitive.

Theorem 2.8: *Let $\mathcal{F} \leq \mathcal{M}$. Hence $\mathcal{M}$ is $\mathcal{F}$-extending iff $\forall \mathcal{A} \in \mathcal{S}_{\mathcal{F}}$ there is $\mathcal{U} \leq_{\mathcal{F}\oplus} \mathcal{M}$ satisfies $\mathcal{A}\psi_{\mathcal{F}}\mathcal{U}$.*

Proof: $\Rightarrow$) Assume $\mathcal{M}$ is $\mathcal{F}$-extending and $\mathcal{A} \in \mathcal{S}_{\mathcal{F}}$. As $\mathcal{M}$ is $\mathcal{F}$-extending, therefore $where\ a\ \mathcal{U} \leq_{\mathcal{F}\oplus} \mathcal{M}$, $\mathcal{U} \in \mathcal{S}_{\mathcal{F}}$ satisfies $\mathcal{A} \leq_{\mathcal{F}E} \mathcal{U}$, since for every $\mathcal{U}$ a submodule of $\mathcal{M}$, then $\mathcal{U} \leq_{\mathcal{F}E} \mathcal{U}$, hence $\mathcal{A}\psi_{\mathcal{F}}\mathcal{U}$.

$\Leftarrow$) Let $\mathcal{A} \in \mathcal{S}_{\mathcal{F}}$, In our hypothesis, there is a $\mathcal{U} \leq_{\mathcal{F}\oplus} \mathcal{M}$, $\mathcal{U} \in \mathcal{S}_{\mathcal{F}}$ satisfies $\mathcal{A}\psi_{\mathcal{F}}\mathcal{U}$. Thus where a submodule $\mathcal{B} \in \mathcal{S}_{\mathcal{F}}$ satisfies $\mathcal{A} \leq_{\mathcal{F}E} \mathcal{B}$ and $\mathcal{U} \leq_{\mathcal{F}E} \mathcal{B}$. to show $B \leq_{\mathcal{F}\oplus} \mathcal{M}$. Let $\mathcal{M} = \mathcal{U}\oplus_{\mathcal{F}}\mathcal{U}_1$, where$\mathcal{U}_1 \leq \mathcal{M}$. Since $\mathcal{U} \leq \mathcal{B}$ thus
$\mathcal{M} = \mathcal{B} + \mathcal{U}_1$. Since $\mathcal{U} \cap \mathcal{U}_1 \leq \mathcal{F}$, then$(\mathcal{B} \cap \mathcal{U}) \cap \mathcal{U}_1 \leq \mathcal{F}$,therefore
$\mathcal{U} \cap (\mathcal{U}_1 \cap \mathcal{B}) \leq \mathcal{F}$. But $\mathcal{U} \leq_{\mathcal{F}E} \mathcal{B}$, therefore, $(\mathcal{U}_1 \cap \mathcal{B}) \leq \mathcal{F}$. Hence
$\mathcal{M} = \mathcal{B}\oplus_{\mathcal{F}}\mathcal{U}_1$, therefore $\mathcal{M}$ is a $\mathcal{F}$-extending module.

Proposition 2.9: Let $\mathcal{F}$ submodule of $\mathcal{M}$ and let $\mathcal{V}, \mathcal{U} \in \mathcal{S}_{\mathcal{F}}$. If $\mathcal{V}\ \psi_{\mathcal{F}}\mathcal{U}$, then there exist $\mathcal{C} \leq \mathcal{M}$ satisfies $\frac{\mathcal{C}}{\mathcal{V}}$ and $\frac{\mathcal{C}}{\mathcal{U}}$ are singular.

Proof: Suppose that $\mathcal{V}\psi_{\mathcal{F}}\mathcal{U}$, then there is $\mathcal{C} \in \mathcal{S}_{\mathcal{F}}$ satisfies $\mathcal{U} \leq_{\mathcal{F}E} \mathcal{C}$ and $\mathcal{V} \leq_{\mathcal{F}E} \mathcal{C}$, therefore $\frac{\mathcal{V}}{\mathcal{F}} \leq_{\mathcal{F}E} \frac{\mathcal{C}}{\mathcal{F}}$ and $\frac{\mathcal{U}}{\mathcal{F}} \leq_{\mathcal{F}E} \frac{\mathcal{C}}{\mathcal{F}}$. Now consider the following two short exact sequences:

$$0 \longrightarrow \frac{\mathcal{V}}{\mathcal{F}} \xrightarrow{g_1} \frac{\mathcal{C}}{\mathcal{F}} \xrightarrow{f_1} \frac{\mathcal{C}/\mathcal{F}}{\mathcal{V}/\mathcal{F}} \longrightarrow 0$$

$$0 \longrightarrow \frac{\mathcal{U}}{\mathcal{F}} \xrightarrow{g_2} \frac{\mathcal{C}}{\mathcal{F}} \xrightarrow{f_2} \frac{\mathcal{C}/\mathcal{F}}{\mathcal{U}/\mathcal{F}} \longrightarrow 0$$

Where g_1, g_2 are inclusion map and f_1 , f_2 are the natural epimorphisms. Because
$\frac{\mathcal{V}}{\mathcal{F}} \leq_e \frac{\mathcal{C}}{\mathcal{F}}$ and $\frac{\mathcal{U}}{\mathcal{F}} \leq_e \frac{\mathcal{C}}{\mathcal{F}}$, then $\frac{\mathcal{C}/\mathcal{F}}{\mathcal{V}/\mathcal{F}}$ and $\frac{\mathcal{C}/\mathcal{F}}{\mathcal{U}/\mathcal{F}}$ are singular. By the third isomorphism theorem, $\frac{\mathcal{C}/\mathcal{F}}{\mathcal{V}/\mathcal{F}} \cong \frac{\mathcal{C}}{\mathcal{V}}$ and $\frac{\mathcal{C}/\mathcal{F}}{\mathcal{U}/\mathcal{F}} \cong \frac{\mathcal{C}}{\mathcal{U}}$. Thus $\frac{\mathcal{C}}{\mathcal{V}}$ and $\frac{\mathcal{C}}{\mathcal{U}}$ are singular.

Definition 2.10: Suppose that $\mathcal{F}$, $\mathcal{V}$ and $\mathcal{U}$ be submodules of $\mathcal{M}$. Then $\mathcal{V}\psi'_{\mathcal{F}}\mathcal{U}$ where $\mathcal{S} \leq \mathcal{M}$ satisfies $(\mathcal{V} + \mathcal{F}) \leq_{\mathcal{F}E} \mathcal{S}$ and $(\mathcal{U} + \mathcal{F}) \leq_{\mathcal{F}E} \mathcal{S}$.

Examples 2.11: Let $\mathcal{Z}_{12}$ as $\mathcal{Z}$-module, $\mathcal{U} = \mathcal{Z}_{12}$, $\mathcal{F} = \{\overline{0}, \overline{6}\}$ and $\mathcal{V} = \{\overline{0}, \overline{4}, \overline{8}\}$. Since Z_{12} is a $\mathcal{F}$-essential in $\mathcal{Z}_{12}$. But $\mathcal{V} + \mathcal{F} = \{\overline{0}, \overline{2}, \overline{4}, \overline{6}, \overline{8}, \overline{10}\}$ is not $\mathcal{F}$ -essential in $\mathcal{Z}_{12}$, therefore $\mathcal{V}$ is not related to $\mathcal{Z}_{12}$ by $\psi'_{\mathcal{F}}$.

Remark 2.12: Let P, V and U be submodules of M, then $\mathcal{V}\psi'_{\mathcal{P}}\mathcal{U}$ iff $\frac{\mathcal{V}+\mathcal{P}}{\mathcal{P}}\ \psi\ \frac{\mathcal{U}+\mathcal{P}}{\mathcal{P}}$.

Proposition 2.13: Assume $\mathcal{F} \leq \mathcal{M}$. Then $\psi'_{\mathcal{F}}$ is transitive iff $\mathcal{M}$ is $\mathcal{UFC}$-module.

Proof: we can prove by the same argument of the proof of (2.7).

3. The Relation $\boldsymbol{\varphi_{\mathcal{F}}}$:

This section is devoted to examining the properties of submodules $\boldsymbol{\varphi_{\mathcal{F}}}$ equivalent to a direct summand.

Definition 3.1: Assume $\mathcal{M}$ be a module , $\mathcal{F} \leq \mathcal{M}$ and $\mathcal{V}, \mathcal{U} \in \mathcal{S}_{\mathcal{F}}$, then $\mathcal{V}\varphi_{\mathcal{F}}\mathcal{U}$ if $(\mathcal{V} \cap \mathcal{U}) \leq_{\mathcal{F}E} \mathcal{A}$ and $(\mathcal{V} \cap \mathcal{U}) \leq_{\mathcal{F}E} \mathcal{U}$.

A module $\mathcal{M}$ is name $\mathcal{F}$ -Goldie extending ($\mathcal{F}$-$\mathcal{G}$-extending module) whenever for all $\mathcal{V} \leq \mathcal{M}$ that containing $\mathcal{F}$, one can find $\mathcal{H} \leq_{\mathcal{F}\oplus} \mathcal{M}$ satisfies $\mathcal{F} \leq \mathcal{H}$ and $\mathcal{V}\ \varphi_{\mathcal{F}}\ \mathcal{H}$.

Cleary, if $\mathcal{F} = 0$ then ($\mathcal{F}$-$\mathcal{G}$-extending module) iff $\mathcal{G}$-extending module.

Example 3.2: consider $\mathcal{Z}_{12}$as $\mathcal{Z}$-module and let $= \{\bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{8}, \overline{10}\}$,

$\mathcal{F} = \{\bar{0}, \bar{6}\}$ and $\mathcal{B} = \mathcal{Z}_{12}$, thus $\mathcal{A} \cap \mathcal{B} = \mathcal{V} \leq_{\mathcal{F}E} \mathcal{B}$. As $\mathcal{V}$ is not $\mathcal{F}$-essential in $\mathcal{Z}_{12}$. Hence $\mathcal{V}$ is not related to $\mathcal{Z}_{12}$ by $\varphi_{\mathcal{F}}$

Theorem 3.3: *Let $\mathcal{F}, \mathcal{V}$ and $\mathcal{H}$ be a submodule of $\mathcal{M}$ satisfies $\mathcal{V}, \mathcal{H} \in \mathcal{S}_{\mathcal{F}}$. Then $\mathcal{V}\varphi_{\mathcal{F}}\mathcal{H}$ iff $\frac{\mathcal{V}}{\mathcal{F}}\varphi_{\mathcal{F}}\frac{\mathcal{H}}{\mathcal{F}}$.*

Proof: ⇒) assume $\mathcal{V}\varphi_{\mathcal{F}}\mathcal{H}$,then $\mathcal{V} \cap \mathcal{H} \leq_{\mathcal{F}E} \mathcal{V}$ and $\mathcal{V} \cap \mathcal{H} \leq_{\mathcal{F}E} \mathcal{H}$.Then $\frac{\mathcal{V}\cap\mathcal{H}}{\mathcal{F}} \leq_e \frac{\mathcal{V}}{\mathcal{F}}$ and $\frac{\mathcal{V}\cap\mathcal{H}}{\mathcal{F}} \leq_e \frac{\mathcal{H}}{\mathcal{F}}$, then $\frac{\mathcal{V}}{\mathcal{F}}\varphi_{\mathcal{F}}\frac{\mathcal{H}}{\mathcal{F}}$.

⇐) *Let* $\frac{\mathcal{V}}{\mathcal{F}}\varphi_{\mathcal{F}}\frac{\mathcal{H}}{\mathcal{F}}$, then $\frac{\mathcal{V}\cap\mathcal{H}}{\mathcal{F}} \leq_e \frac{\mathcal{V}}{\mathcal{F}}$ and $\frac{\mathcal{V}\cap\mathcal{H}}{\mathcal{F}} \leq_e \frac{\mathcal{H}}{\mathcal{F}}$. Then $\mathcal{V} \cap \mathcal{H} \leq_{\mathcal{F}E} \mathcal{V}$ and $\mathcal{V} \cap \mathcal{H} \leq_{\mathcal{F}E} \mathcal{H}$. Thus $\mathcal{V}\varphi_{\mathcal{F}}\mathcal{H}$.

Remark 3.4: The $\varphi_{\mathcal{F}}$ relation is an equivalence relation.

Proposition 3.5: Suppose $\mathcal{P}, \mathcal{V}$ and $\mathcal{U}$ be submodules of $\mathcal{M}$ for where $\mathcal{V}$ and $\mathcal{U} \in \mathcal{S}_{\mathcal{P}}$. Then $\mathcal{V}\varphi_{\mathcal{P}}\mathcal{U}$ iff $\mathcal{V} \cap \mathcal{C} \leq \mathcal{P}$ which implies $\mathcal{U} \cap \mathcal{C} \leq \mathcal{P}$ and $\mathcal{U} \cap \mathcal{D} \leq \mathcal{P}$ consequently $\mathcal{V} \cap \mathcal{D} \leq \mathcal{P}$, for all submodules $\mathcal{C}$ and $\mathcal{D}$ of $\mathcal{M}$.

Proof: ⇒) Suppose that $\mathcal{V}\varphi_{\mathcal{P}}\mathcal{U}$ and let $\mathcal{C} \leq \mathcal{M}$ satisfies $(\mathcal{V} \cap \mathcal{C}) \leq \mathcal{P}$. Then $\mathcal{V} \cap (\mathcal{U} \cap \mathcal{C}) \leq \mathcal{P}$, hence $(\mathcal{U} \cap \mathcal{C}) \cap (\mathcal{V} \cap \mathcal{U}) \leq \mathcal{P}$. But$(\mathcal{V} \cap \mathcal{U}) \leq_{\mathcal{P}E} \mathcal{U}$, therefore, $(\mathcal{U} \cap \mathcal{C}) \leq \mathcal{P}$. Now let $(\mathcal{U} \cap \mathcal{D}) \leq \mathcal{P}$,where $\mathcal{D} \leq \mathcal{M}$. Then $(\mathcal{V} \cap \mathcal{U}) \cap \mathcal{D} \leq \mathcal{P}$ and then $(\mathcal{V} \cap \mathcal{U}) \cap (\mathcal{V} \cap \mathcal{D}) \leq \mathcal{P}$. But $(\mathcal{V} \cap \mathcal{U}) \leq_{\mathcal{P}E} \mathcal{V}$, therefore, $(\mathcal{V} \cap \mathcal{D}) \leq \mathcal{P}$

⇐) To prove $\mathcal{V}\varphi_{\mathcal{P}}\mathcal{U}$. Let L be a submodule of $\mathcal{V}$ satisfies $\mathcal{V} \cap \mathcal{U} \cap \mathcal{L} \leq \mathcal{P}$. As $\mathcal{V} \cap (\mathcal{U} \cap \mathcal{L}) \leq \mathcal{P}$, thus by our assumption $\mathcal{V} \cap (\mathcal{U} \cap \mathcal{L}) = \mathcal{U} \cap (\mathcal{U} \cap \mathcal{L}) \leq \mathcal{P}$. Hence $(\mathcal{V} \cap \mathcal{L}) \leq \mathcal{P}$. But $\mathcal{L} \leq \mathcal{A}$, hence $\mathcal{L} \leq \mathcal{P}$. Similarly, suppose that $\mathcal{K} \leq \mathcal{U}$. Satisfies $(\mathcal{V} \cap \mathcal{U}) \cap \mathcal{K} \leq \mathcal{P}$. By our assumption therefore $\mathcal{V} \cap \mathcal{K} = \mathcal{V} \cap$

$(\mathcal{V} \cap \mathcal{K}) \leq \mathcal{P}$. Hence $\mathcal{U} \cap \mathcal{K} \leq \mathcal{P}$. But $\mathcal{K} \leq \mathcal{U}$, therefore, $\mathcal{K} \leq \mathcal{P}$. Thus $\mathcal{V}\varphi_{\mathcal{F}}\mathcal{U}$.

Proposition 3.6: Assume $\mathcal{A}_1, \mathcal{A}_2$,$\mathcal{B}_2$,$\mathcal{B}_1$ and $\mathcal{F}$ are submodules of a module $\mathcal{M}$ satisfies $\mathcal{B}_1, \mathcal{B}_2$, $\mathcal{A}_1$ and $\mathcal{A}_2$, $\in \mathcal{S}_{\mathcal{F}}$. If $\mathcal{A}_1\varphi_{\mathcal{F}}\mathcal{B}_1$ *and* $\mathcal{A}_2\varphi_{\mathcal{F}}\mathcal{B}_2$, then $(\mathcal{A}_1 \cap \mathcal{A}_2)$ $\varphi_{\mathcal{F}}$ $(\mathcal{B}_1 \cap \mathcal{B}_2)$.

Easy

Proposition 3.7: Let $f: \mathcal{M} \to \mathcal{N}$ be an homomorphism and onto and $\mathcal{F}, \mathcal{V}$ and $\mathcal{U}$ be submodules of $\mathcal{N}$ satisfies $\mathcal{V}, and\ \mathcal{U} \in \mathcal{S}_{\mathcal{F}}$. If $\mathcal{V}\varphi_{\mathcal{F}}\mathcal{U}$, then $f^{-1}(\mathcal{V})\ \varphi_{f^{-1}(\mathcal{F})} f^{-1}(\mathcal{U})$.

Proof: Let $\mathcal{V}$ and $\mathcal{U}$ be submodules of $\mathcal{N}$ satisfies $\mathcal{V}\varphi_{\mathcal{F}}\mathcal{U}$, then $\mathcal{V} \cap \mathcal{U} \leq_{\mathcal{F}E} \mathcal{V}$ and $\mathcal{V} \cap \mathcal{U} \leq_{\mathcal{F}E} \mathcal{U}$. then $f^{-1}(\mathcal{V} \cap \mathcal{U}) \leq_{f^{-1}(\mathcal{F})E} f^{-1}(\mathcal{V})$, implies that $f^{-1}(\mathcal{V}) \cap f^{-1}(\mathcal{U}) \leq_{f^{-1}(\mathcal{F})E} f^{-1}(\mathcal{V})$. Since $\mathcal{V} \cap \mathcal{U} \leq_{\mathcal{F}E} \mathcal{U}$, then $f^{-1}(\mathcal{V} \cap \mathcal{U}) \leq_{f^{-1}(\mathcal{F})E} f^{-1}(\mathcal{U})$, implies that $f^{-1}(\mathcal{V}) \cap f^{-1}(\mathcal{U}) \leq_{f^{-1}(\mathcal{F})E} f^{-1}(\mathcal{U})$. Thus $f^{-1}(\mathcal{V})\varphi_{f^{-1}(\mathcal{F})} f^{-1}(\mathcal{U})$.

Definition 3.8: Let $\mathcal{F}, \mathcal{V}$ and $\mathcal{U}$ be submodules of $\mathcal{M}$. Then $\mathcal{V}\varphi^*_{\mathcal{F}}\mathcal{U}$ if $(\mathcal{V} + \mathcal{F}) \cap (\mathcal{U} + \mathcal{F}) \leq_{\mathcal{P}E} \mathcal{V} + \mathcal{F}\ and\ (\mathcal{V} + \mathcal{F}) \cap (\mathcal{U} + \mathcal{F}) \leq_{\mathcal{P}E} \mathcal{U} + \mathcal{F}$.

Clearly the $\varphi^*_{\mathcal{F}}$ relation is an equivalence relation.

Examples 3.9: Let $\mathcal{Z}_6\ as\ \mathcal{Z}$ -module. Let $\mathcal{F} = \{\overline{0}, \overline{3}\}$, $\mathcal{A} = \{\overline{0}, \overline{2}, \overline{4}\}$ and $\mathcal{U} = \{\overline{0}\}$, thus $\mathcal{A} + \mathcal{F} = \mathcal{Z}_6$ and $\mathcal{U} + \mathcal{F} = \{\overline{0}, \overline{3}\}$. Since $(\mathcal{A} + \mathcal{F}) \cap (\mathcal{U} + \mathcal{F}) = \{\overline{0}, \overline{3}\}$ and $\{\overline{0}, \overline{2}, \overline{4}\} \leq \mathcal{Z}_6$, then $\{\overline{0}, \overline{3}\} \cap \{\overline{0}, \overline{2}, \overline{4}\} = \{\overline{0}\} \leq \mathcal{F}$. But $\{\overline{0}, \overline{2}, \overline{4}\} \nleq \mathcal{F}$, then $(\mathcal{A} + \mathcal{F}) \cap (\mathcal{U} + \mathcal{F})$ is not $\mathcal{F}$-essential in $\mathcal{A} + \mathcal{F}$. Thus $\mathcal{A}$ is not related to B by $\varphi_{\mathcal{F}}$.

Proposition 3.10: Let $\mathcal{K}, \mathcal{W}$ and U be a submodules of $\mathcal{M}$ satisfies $\mathcal{W}, \mathcal{U} \in \mathcal{S}_{\mathcal{K}}$. Then $\mathcal{W}\ \varphi^*_{\mathcal{K}}\mathcal{U}$ iff $\frac{\mathcal{W}+\mathcal{K}}{\mathcal{K}}\varphi\frac{\mathcal{U}+\mathcal{K}}{\mathcal{K}}$.

Proof: $\Rightarrow$) Suppose that $\mathcal{W}\ \varphi^*_{\mathcal{K}}\mathcal{U}$, therefore $(\mathcal{W} + \mathcal{K}) \cap (\mathcal{U} + \mathcal{K}) \leq_{\mathcal{F}E} \mathcal{V} + \mathcal{K}\ and\ (\mathcal{W} + \mathcal{K}) \cap (\mathcal{U} + \mathcal{K}) \leq_{\mathcal{K}E} \mathcal{U} + \mathcal{K}$. Then $\frac{(\mathcal{W}+\mathcal{K}) \cap (\mathcal{U}+\mathcal{K})}{\mathcal{K}} \leq_e \frac{\mathcal{W}+\mathcal{K}}{\mathcal{K}}$ and $\frac{(\mathcal{W}+\mathcal{K}) \cap (\mathcal{U}+\mathcal{K})}{\mathcal{K}} \leq_e \frac{\mathcal{U}+\mathcal{K}}{\mathcal{K}}$. Thus $\frac{\mathcal{W}+\mathcal{K}}{\mathcal{K}}\varphi\frac{\mathcal{U}+\mathcal{K}}{\mathcal{K}}$

Conversely, let $\frac{\mathcal{V}+\mathcal{K}}{\mathcal{K}}\varphi\frac{\mathcal{U}+\mathcal{K}}{\mathcal{K}}$, then $\frac{(\mathcal{W}+\mathcal{K}) \cap (\mathcal{U}+\mathcal{K})}{\mathcal{K}} \leq_e \frac{\mathcal{W}+\mathcal{K}}{\mathcal{K}}$ and $\frac{(\mathcal{W}+\mathcal{K}) \cap (\mathcal{U}+\mathcal{K})}{\mathcal{K}} \leq_e \frac{\mathcal{U}+\mathcal{K}}{\mathcal{K}}$ Hence $(\mathcal{W} + \mathcal{K}) \cap (\mathcal{U} + \mathcal{K}) \leq_{\mathcal{K}E} \mathcal{W} + \mathcal{K}\ and\ (\mathcal{W} + \mathcal{K}) \cap (\mathcal{U} + \mathcal{K}) \leq_{\mathcal{K}E} \mathcal{U} + \mathcal{K}$. Then $\mathcal{W}\varphi^*_{\mathcal{K}}\mathcal{U}$.

Theorem 3.11: *Assume $\mathcal{M}$ be a distributive module, therefore $\mathcal{M}$ is $\mathcal{K}$-$\mathcal{G}$-extending module iff*

$\mathcal{A} \leq \mathcal{M}$ contains $\mathcal{K}$, there is $\mathcal{U} \leq_{\mathcal{F}\oplus} \mathcal{M}$ contains $\mathcal{K}$ satisfies $\mathcal{A}\psi'_{\mathcal{F}}\mathcal{U}$ and

$\mathcal{U}_1 + \mathcal{K}$ is $\mathcal{K}$-complement for $\mathcal{A}$, where $\mathcal{M} = \mathcal{U}\oplus_{\mathcal{F}}(\mathcal{U}_1 + \mathcal{K})$, where $\mathcal{U}_1 \leq \mathcal{M}$.

Proof: $\Leftarrow$) Clear.

$\Rightarrow$) Assume that $\mathcal{M}$ is $\mathcal{K}$-$\mathcal{G}$-extending and $\mathcal{A} \leq \mathcal{M}$ satisfying $\mathcal{K} \leq \mathcal{A}$. Hence there is $\mathcal{U} \leq_{\mathcal{K}\oplus} \mathcal{M}$ satisfies $\mathcal{K} \leq \mathcal{U}$ and $\mathcal{A}\varphi_{\mathcal{K}}\mathcal{U}$. Let $\mathcal{M} = \mathcal{U}\oplus_{\mathcal{K}}(\mathcal{U}_1 + \mathcal{K})$, $\mathcal{U}_1$ is a submodule of $\mathcal{M}$. Since $\mathcal{A} \cap \mathcal{U} \leq_{\mathcal{K}E} \mathcal{A}$ and $\mathcal{U}_1 + \mathcal{K} \leq \mathcal{U}_1 + \mathcal{K}$, then $\frac{\mathcal{A}\cap\mathcal{U}}{\mathcal{K}} \leq_E \frac{\mathcal{A}}{\mathcal{K}}$ and $\frac{\mathcal{U}_1+\mathcal{K}}{\mathcal{K}} \leq_E \frac{\mathcal{U}_1+\mathcal{K}}{\mathcal{K}}$. So $\left(\frac{\mathcal{A}\cap\mathcal{U}}{\mathcal{K}}\right) \cap \left(\frac{\mathcal{U}_1+\mathcal{K}}{\mathcal{K}}\right) \leq_E \left(\frac{\mathcal{A}}{\mathcal{K}}\right) \cap \left(\frac{\mathcal{U}_1+\mathcal{K}}{\mathcal{K}}\right)$.
As M is a distributive, then

$\left(\frac{\mathcal{A}\cap\mathcal{U}}{\mathcal{K}}\right)\cap\left(\frac{\mathcal{U}_1+\mathcal{K}}{\mathcal{K}}\right)=\frac{(\mathcal{A}\cap\mathcal{U})\cap(\mathcal{U}_1+\mathcal{K})}{\mathcal{K}}=\frac{(\mathcal{A}\cap\mathcal{U})\cap\mathcal{U}_1+(\mathcal{A}\cap\mathcal{U})\cap\mathcal{K}}{\mathcal{K}}=\frac{[(\mathcal{A}\cap\mathcal{U})\cap\mathcal{U}_1]+\mathcal{K}}{\mathcal{K}}\leq\frac{[\mathcal{K}\cap\mathcal{U}_1]+\mathcal{K}}{\mathcal{K}}\leq\frac{\mathcal{K}}{\mathcal{K}}=0$. Hence $\left(\frac{\mathcal{A}}{\mathcal{K}}\right)\cap\left(\frac{\mathcal{U}_1+\mathcal{K}}{\mathcal{K}}\right)=0$, implies that $\mathcal{A}\cap(\mathcal{H}_1+\mathcal{K})=\mathcal{K}$. Now, suppose $\mathcal{B}\leq\mathcal{M}$ satisfies $\mathcal{U}_1+\mathcal{K}\leq\mathcal{B}$ and $\mathcal{A}\cap\mathcal{B}\leq\mathcal{K}$. Since $(\mathcal{A}\cap\mathcal{U})\leq_{\mathcal{F}E}\mathcal{U}$ and $\mathcal{B}\leq_{\mathcal{K}E}\mathcal{B}$, then $\frac{(\mathcal{A}\cap\mathcal{U})}{\mathcal{K}}\leq_E\frac{\mathcal{U}}{\mathcal{K}}$ and $\frac{\mathcal{B}}{\mathcal{K}}\leq_E\frac{\mathcal{B}}{\mathcal{K}}$.

Theorem 3.12: *Let $\frac{\mathcal{M}}{\mathcal{K}}$ be a nonsingular module. Then $\mathcal{M}$ is $\mathcal{K}$-$\mathcal{G}$-extending if and only if $\mathcal{M}$ is $\mathcal{K}$ -extending.*

Proof: Suppose that $\mathcal{M}$ is $\mathcal{K}$-$\mathcal{G}$-extending module and let $\mathcal{A}$ be a submodule of $\mathcal{M}$ satisfies $\mathcal{K}\leq\mathcal{A}$. Then there exists a $\mathcal{K}$-direct summand $\mathcal{D}$ of $\mathcal{M}$ satisfies $\mathcal{K}\leq\mathcal{D}$ and $\mathcal{A}\psi'_{\mathcal{K}}\mathcal{D}$. Implies that $(\mathcal{A}\cap\mathcal{D})\leq_{\mathcal{K}E}\mathcal{A}$ and $(\mathcal{A}\cap\mathcal{D})\leq_{\mathcal{K}E}\mathcal{D}$. It is enough to show $\mathcal{A}\leq\mathcal{D}$. As $\frac{\mathcal{A}+\mathcal{D}}{\mathcal{D}}$ is singular. Now by the third isomorphism theorem we have $\frac{\mathcal{A}+\mathcal{D}}{\mathcal{D}}\cong\frac{(\mathcal{A}+\mathcal{D})/\mathcal{K}}{\mathcal{D}/\mathcal{K}}\leq\frac{\mathcal{M}/\mathcal{K}}{\mathcal{D}/\mathcal{K}}=\frac{(\mathcal{D}\oplus(\mathcal{D}_1+\mathcal{F})/\mathcal{K}}{\mathcal{D}/\mathcal{K}}$, then $\frac{(\mathcal{D}\oplus(\mathcal{D}_1+\mathcal{K})/\mathcal{K}}{\mathcal{D}/\mathcal{K}}=\frac{(\mathcal{D}_1+\mathcal{K})}{(\mathcal{D}\cap(\mathcal{D}_1+\mathcal{K})}=\frac{(\mathcal{D}_1+\mathcal{K})}{\mathcal{K}}\leq\frac{\mathcal{M}}{\mathcal{K}}$ which is nonsingular where $\mathcal{D}_1$ is a submodule of $\mathcal{M}$,But the only submodule which is singular and nonsingular is the zero submodule therefore $\mathrm{A}+\mathcal{D}=\mathcal{D}$. Hence $\mathcal{A}\leq\mathcal{D}$. Thus $\mathcal{M}$ is $\mathcal{K}$-extending module. The converse is obvious.

4. Discussion and Conclusion

We defined and developed the concept of $\mathcal{G}$-extending modules and established fundamental properties and characterizations of this class. The introduced relations provide useful tools for studying submodule structures and extending properties in modules. Several results connecting $\mathcal{G}$-extending modules to extending, nonsingular, and distributive modules were obtained. These results open the door for further research on this type of module and its applications in module theory and ring theory.

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Received January, 2026