Journal of Discrete Mathematical Sciences & Cryptography
ISSN 0972-0529 (Print), ISSN 2169-0065 (Online)
Vol. 29 (2026), No. 5, pp. 1919–1925
DOI : 10.47974/JDMSC-2353

Semiprime submodules relative to an ideal and some related concepts

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Abstract

In this research, a new concept is introduced and studied: semiprime submodules related to ideals, semiprime modules relative to ideals, and weakly semiprime submodules relative to ideals. Many results concerning these concepts are given.

Subject Classification: *13C05, 13A15, 13C13.*

Keywords: *Prime submodule, Semiprime submodule, Semiprime submodule relative to an ideal, Semiprime module relative to an ideal, Weakly semiprime submodule relative to an ideal.*

1. Introduction

In this work assume R is a commutative ring unity and W is an R-module. It is known that a proper submodule P of an R-module is named prime if whenever $a \in, x \in W, ax \in P$ yields $x \in P$ or $a \in (P:W)$ [1]. Numerous generalizations of prime submodule represent a crucial area of study in commutative algebra, see [1-6]. Beside these see [7]. A proper submodule P of

This paper was submitted to University of Al-Qadisiyah before submission to this journal.

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W is called semiprime submodule (shortly, sp.s) if $a^2x \in P(where \ \in R, x \in W), then\ ax \in P$. The authors in [8] introduced the concept of prime submodules relative to ideal I of $R(I < R)$, where $P < W$ is named prime submodule relative to I(P.s.r. to I) if whenever $a \in R, x \in W, ax \in P$ implies $\in P + IW$ or $a \in (P: W) + I$. Obviously every prime submodule is S.P.r to I, for any $I < R$, but the converse may be not true, see [10]. Moreover, the two concept are indicial if $I \leq (P: W)$.

In this paper, the concept of semiprime submodule relative to $I < R$ is explored. A proper submodule P of $W(P < W)$ is considered as semiprime submodule relative to I (sp.s.r.I) if $a^2x \in P(where \ \in R, x \in W), then\ ax \in P + IW$.

This paper has three sections. In S.2, a comprehensive study of Sp.s. to IAn ideal is presented, where many properties of sp.s are generalized to sp.s.r. to an ideal. In S.3, we introduce the concept of a semiprime module relative to an ideal, where a module is a semiprime module relative to an ideal I if the zero submodule is an SP.s.r. to an ideal I. Many properties of this type of module are given.

2. Semiprime submodule relative to an ideal

Definition 2.1: Let $\mathrm{P} < \mathrm{W}, \mathrm{I} < \mathrm{R}$. P is called a semiprime submodule relative to I(sp.s.r. to I) if $a^2x \in P$ (where $a \in R, x \in W$), implies $ax \in P + IW$.

An ideal J is a semiprime ideal relative to I(sp.s.r. to I) if $a^2 \in J$ implies $a \in J + I$.

Remarks and Examples 2.2: Assume W be an R-module, $A < W$ and $I < R$.

If $IW = W$, A is a sp.s.r. to I.

If W is a divisible R-module, the every $A < W$ is a sp.s.r. to I for any $I < R, I \neq 0$. As a special case, every submodule of Z-module Q, Z-module Z_{p^∞} is sp.s.r. to I, $\forall I < Z$.

Every semiprime submodule is sp.s.r. to I, $\forall$ ideal $I < R$, but not conversely.

Example: Consider Z_{24} as Z-module, $I = 6Z$, $(\overline{0})$, is not SP.s.r. to I, since $3.\overline{8} = \overline{0} = (\overline{0})$, but $\overline{8} \notin \overline{(0)} + IZ_{24} = (\overline{6})$ and $3 \notin (0:_Z Z_{24}) + 6Z = 6$. On the other hand ($(\overline{0})$ is an SP.s.r. to I, since $2^2.\overline{6} = \overline{0}$and $2.\overline{6} \in IZ_{24} = (\overline{6})$,

$6^2.\overline{2} = \overline{0}$ and $6.\overline{2} \in IZ_{24} = (\overline{6})$, $3^2.\overline{8} = \overline{0}$ and $3.\overline{8} \in IZ_{24} = (\overline{6})$,$12^2.\overline{3} = \overline{0}$ and $12.\overline{3} \in IZ_{24} = (\overline{6})$,

Let $P < W, I < R, I \leq (P: W)$. Then P is an sp.s.r. of W identical with P is an SP.s.r. to I.

Theorem 2.3: *Let $P < W, I < R$. Then P is an sp.s.r. to I if and only if $J^2K \leq P$, (where $K \leq W$) implies $JK \leq P + IW$.*

Proof: $\Longrightarrow$ If $J^2K \leq P$, hen $J^2(x) \leq P \forall x \in K$ and so $j^2x \in P$ for each $j \in J$, since P is an sp.s.r. to I, $jx \in P + IW$. Hence $JK \leq P + IW$.

$\Longleftarrow$ Let $a^2x \in P$. Then $< a^2 >< x > \leq P$ and hence by hypothesis $< a >< x > \leq P + IW$; that implies $ax \in P + IW$.

Theorem 2.4: *Suppose W is an R module and $P < W$, $I < R$. Focus on these sentences*

1. P is an sp.s.r.to I.
2. $P:_R x)$ is an sp. S.r. to I, $\forall$ x $\notin$ P.
3. $(P:_R K)$ is an sp.s.r. to I, $\forall$ K > P. Then $(2) \Rightarrow (1), (2) \Leftrightarrow (3)$ and $(1) \nRightarrow (2)$.

Proof: (2)$\Rightarrow$ (1) Let $r \in R$ with $r^2x \in P$. If $x \in P$, then no thing to prove since $rx \in P \leq P + IW$.If $x \notin P$, then $r^2 \in (P: x)$. As $\in (P: x)$ is an sp.s.r. to I, $r \in (P: x) + I$ and so $r = a + i$ for some $a \in (P: x)$ and $i \in I$. Hence $ax \in P$ and $ix \in IW$. Thus $rx = ax - ix \in P + IW$. Thus P is an sp.s.r. to I.

(2) $\Rightarrow$ (3) Let $r^2 \in (P: K), K > P$. Then $\exists x \in K$ and $x \notin PW$ so $r^2x \in P$; that is $r^2 \in (P: x)$. Hence $r \in (P: x) + I$. So that $r \in (P: K) + I$.

(3) $\Rightarrow$ (2) Let $r \in R$ with $r^2 \in (P: x)$ and $x \notin P$. Put $K = P+< x > \gneqq P$. $r^2K = r^2(P: < x >) \leq P$, hence $r^2 \in (P: K)$.As $(P: K)$ is an sp.s.r. to I, $r \in (P: K) + I$; that is $r = a + i$ for some $a \in (P: K), i \in I$. Hence, $aK = a(P: < x >) \leq P$ and ax view submission P, $\in (P: x)$. Thus $r = a + i \in (P: K) + I$. Therefore, $(P: x), \forall x \notin P$ is an sp.s.r. to I.

Example 2.5: Consider Z_8 as Z-module, $I = 4Z$, $(\overline{0})$ is an sp.s.r. to I, since $2^2\overline{2} \in (\overline{0})$and $\overline{4}$=2. $\overline{2} \in (\overline{0}) + IZ_8 = (\overline{4})$, $4^2\overline{2} \in (\overline{0})$ and $4\overline{2} = \overline{0} \in (\overline{0}) \leq (\overline{0}) \leq \overline{0} + IZ_8$. But $(\overline{0}:_Z \overline{2}) = 4Z$ is not sp.s,r. to $I = 4Z$, since $2^2 \in 4Z$, but $\in 4Z + I = 4Z + 4Z = 4Z$. Note that $\overline{2} \notin (\overline{0})$. Thus, $(\overline{0}:_Z \overline{2})$ is not sp.s.r. to I.

A module W is refered as finitely generated (shortly f.g). W= $Rx_1 + = Rx_2 + \ldots + Rx_n$, for some $x_1, x_2, \ldots, x_n$ in W. Also, W is named a multiplication R-module (briefly m. m) if every submodule H of W, $H = IW$ for some I< R , see [9] and [10].

Theorem 2.6: *Let W be a finitely generated multiplication R-module, $P < W$, P is an sp.s.r. to I if and only if $(P: W)$ is an sp.s.r. to I.*

Proof: $\Rightarrow$ Let $a \in R$ with $a^2 \in (P: W)$. Then $a^2W \leq P$ and so $aW \leq P + IW$ by Theorem 2.3. Hence $aW \leq (P: W)W + IW$, since W is a multiplication R-module. Then, and hence by [8, Theorem 3.1] $(a) \leq (P: W) + I$, implies $a \in (P: W) + I$.

$\Longleftarrow$ Let $a^2x \in P$, $a \in R, x \in P$. Since W is a multiplication R-module, then $< x >= JW$ for some $J \leq R$ and $P = (P:W)W$. Thus $a^2JW \leq (P: W)W$, hence $a^2J \leq (P: W) + annW$, because W is a finitely generated [4]. It follows that

$a^2J \leq (P: W)$. But $(P: W)$ is an sp.s.r. to I, then $aJ \leq (P: W) + I$ which implies $aJW \leq (P: W)W + IW$. Therefore, $a < x >= P + IW$, then $ax \in P + IW$.

Remark 2.7: The condition W is a multiplication module is necessary in Theorem 2.6. Consider the following example:

Take the Z-module $W = Z_{P^\infty} \oplus Z$, let $P = Z_{P^\infty} \oplus 4Z$, $I = 4Z$. Then $(P:_Z W) = 4Z$. It is clear that 4Z is not A SP.s.r. to I. But P is a sp.s.r to I, see Remarks and Examples 2.2(2).

Corollary 2.8: *Let Wbe f.g. faithful multiplication R-module,* $P < R$ *,* $I < R$*. Then the following statements are equivalent:*

P sp.s.r. to I .

$(p: W)$ is an sp.s.r. to I.

$P = JM$ for some I, sp.s.r. to I.

Proposition 2.9: Let $f: W_1 \to W_2$ be an R-epimorphism, $I < R$, $H < W$with Ker $f \leq H$. If His an sp.s.r. to I, then f(H) is an sp.s.r. to I.

Proof: Let $r \in R, u \in W_2$ with $r^2u \in f(H)$. Since fis an epimorphism, $u = f(x)$ for some $x \in W$. Thus $f(r^2u) = f(h)$ for some $h \in H$ and so $r^2x - h \in \text{Ker } f \leq H$. Hence $r^2x \in H$. As H is an sp.s.r. to I, $ru \in H + IW$ which implies $ru \in f(H) + IW$. Therefore, f(H) is an sp.s.r. to I.

Proposition 2.10: Let $W = W_1 \oplus W_2$ be an R-module, $I < R, H < W_1$. If H is an sp.sr.to I, then $H + W_2$ is an sp.s.r.to I and conversely.

Proof: Let $a \in R$ and $(h, y) \in W_1 \oplus W_2$, with $a^2(h, y) \in H \oplus W_2$. Then $a^2h \in H$ and $a^2y \in W_2$. Since $a^2h \in H$ and H is an sp.s.r. to I, then $ah \in H + IW_2$ and so $a(h, y) \in (H + IW_1) \oplus W_2$=$(H + IW_1) \oplus (W_2 + IW_2)$.It follows that: $a(h, y) \in H \oplus W_2) + I(W_2 + W_2)$.

Remember that a submodule A of an R-module W is called fully invariant if whenever $f \in \text{End}(W), f(A) \leq A$, see [10].

Proposition 2.11: Let $W = W_1 \oplus W_2$ be an R-module with $\text{ann}W_1 \oplus \text{ann}W_2 = R$ or H is a fully invariant submodule and $H < W$. If H is an sp.s.r. to I. Then

$H = H_1 \oplus H_2$ where H_1 and H_2 are sp.s.r.to I.

$H = H_1 \oplus W_2$ where H_1 is an sp.s.r.to I.

$H = W_1 \oplus H_2$ where H_2 is an sp.s.r.to I.

Proof: If $\text{ann}W_1 \oplus \text{ann}W_2 = R$, then by the proof of Proposition 4.2 in [9]

Submodule $H = H_1 \oplus H_2$. Also, if H is fully invariant, then $H = (H \cap M_1) \oplus (H \cap M_2)$ by [8]. Thus in each case $H = H_1 \oplus H_2$ for some $H_1 \leq W_1$ and $H_2 \leq W_2$. As $H < W$, then there are three cases:

$H_1 < W_1$ and $H_2 < W_2$

$H = H_1 \oplus W_2$ where $H_1 < W_1$

$H = W_1 \oplus H_2$ where $H_2 < W_2$

Case 1: To prove $H_1 \& H_2$ are sp.s.r. to I in W_1 and H_2 respectively. Assume $a^2x \in H_1$ where $a \in R, x \in W_1$. Then $a^2(x,a) \in H = H_1 \oplus H_2$ and so $a(x,0) \in H_1 \oplus H_2$. Thus $ax \in H_1$, H_1 is an sp.s.r.to I. Similarly H_2 is an sp.s.r. to I.

Case 2: It follows by Proposition 2.10.

Case 3: Similary to case 2.

Proposition 2.12: Assume W is an R-module, S be a multiplicative closed subset of R, $H < W$ and $I < R$. If H is an sp.s.r. to I, then $S^{-1}H$ is an sp.s.r. to H_1 $S^{-1}H$.

Proof: Let $\frac{a}{s} \in S^{-1}R$ and $\frac{x}{t} \in S^{-1}W$ with $(\frac{x}{t})^2 \frac{x}{t} \in S^{-1}H$.Then $\frac{a^2x}{st} = \frac{h}{t_1}$ for some $h \in H$ and $t_1 \in S$ and hence $\exists t_1 \in S$ with $t_2a^2x = t_2sth \in H$, i.e $(at_2)^2x \in H$. Since His an sp.s.r. to I, it follows that $at_2x \in H + IW$. So that $\frac{at_2x}{stt_2} \in S^{-1}(H + IW) = S^{-1}H + S^{-1}I + S^{-1}W$. Thus $\frac{a}{t}.\frac{x}{t} \in S^{-1}H + S^{-1}I.S^{-1}W$.

3. Semiprime modules relative to an ideal

Definition 3.1: For an R-module Wand $I < R$. W is named a semiprime module relative to I (shortly sp.m.r. to I) if the zero submodule of W, is a sp.s.r. to I. A ring R is a semiprime ring relative to I (sp.r.r. to I) if the zero ideal of R is an sp.i.r. to I.

Remarks and Examples 3.2: Every semiprime module is a sp.m.r to I, for any $I < R$. The reverse direction may not be valid, for instance

Consider Z_{24} as Z-module $(\overline{0})$ is a sp.s.r. to $I = 6Z$; Z_{24} is an a sp.m.r. to I. But $\overline{0} = 2^2.\overline{6}$ and $2\overline{6} = \overline{12} \notin (\overline{0})$, that is $(\overline{0})$ is not a semiprime submodule (Z_{24} is not a semiprime module).

Every divisible R-module W is a sp.m.r. to $I, \forall I < R$.

The following proposition follows directly by Theorem 2.4 by considering $P = (\overline{0})$.

Proposition 3.3: For R – module W, $I < R$. Consider these sentences:

1. W is an sp.m.r. to I.
2 $\operatorname{ann}(x)$ is an sp.i.r. to I.
3 $\operatorname{ann}(K)$ is an sp.i.r. to I.

Then $(3) \Leftrightarrow (2) \Rightarrow (1)$.

Proposition 3.4: Assume W is a f.g and m.m R-module, then W, then W is an sp.m.r. to I is equivalent with $\operatorname{ann}_R M$ is an sp.i.r. to I.

Consider the following example:

Example 3.5: Let W be the Z-module $Z\oplus Z_4$, $I = 4Z$. It is clear that W is not a $m.m$ Z-module $2^2(0,\overline{1}) = (0,\overline{0})$, but $2(0,\overline{1})$=$(0,\overline{2}) \notin (0,\overline{0}) + IW = 4Z + (\overline{0})$; so that W is not an sp.m.r. to I. But $ann_Z W = (0)$ is prime ideal in Z, and so it is a sp.i.r. to I. Thus condition W is an important in Proposition 3.4.

Proposition 3.6: Let W be a m. m, $J \leq R, I \leq R$ with $J \nleq ann_R W$ with $(ann_R W:_R J)$. Then W is an sp.m.r. to I.

Proof: Let $a \in R, m \in W$ with $a^2 m = 0, m \neq 0$. Since W is a multiplication R-module $\exists\, J \leq R$ such that $< m >= JW$, hence $J \nleq annW$ and $a^2 JW = (0)$, that is $a^2 \in (ann_R W:_R J)$. As $(ann_R W:_R J)$ is an sp.i.r. to I, then $a \in (ann_R W:_R J) + I$ and $a = r + j$ for some $r \in (ann_R W:_R J), i \in I$. Hence $aJw = rJW + iW = iW$ that $a < m >= IW$ and so $am \in (0) + IW$. Therefore, M is an sp.m.r. to I.

Proposition 3.7: Let $W = W_1 \oplus W_2$ be an R-module, $I < R$. Then W is an sp.m.r. to I if and only if W_1 and W_2 are sp.m.r. to I.

Proof: $\Rightarrow$ $\boldsymbol{\rho_1}: W_1 \oplus W_2 \to W_1$ be the natural projection since the zero submodule (0,0) is an sp.s.r. to I then $\boldsymbol{\rho_1(0,0) = (0)_{W_1}}$ is an sp.s.r. to , by Proposition 2.9. Thus W_1 is an sp.m.r. to I . Similarly, W_2 is an sp.m.r. to I.

$\Leftarrow$ Let $a^2(x_1, x_2) = (0,0)$, where $(x_1, x_2) \in W$, $a \in R$. Then $a^2 x_1 = 0_{W_1}$ and $a^2 x_2 = 0_{W_2}$. Since W_1 and W_2 are sp.m.r. to I. $ax_1 \in IW_1$ and $ax_2 \in IW_2$ and so $a(x_1, x_2) \in I(W_1 \oplus W_2) = IW$. Thus W is an sp.m.r. to I [11].

Conclusion

In this work, Semiprime submodules relative to an ideal and some related concepts is defined and studied are introduced and studied. Beside these many descriptions, results and examples related with these concepts are presented.

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Received May, 2025