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S-pseudo bounded modules

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Abstract

The M is unitary module and F is commutative rings with identities. The definition of novel classes F-modules that named, S-pseudo bounded modules for short (S-PS.B.). F- The modules has been studied. Our concept is based on the definition of S-pseudo bounded submodules, which was presented in a previous study. Furthermore, we highlight several known types of modules and clarify their relationships, including Dedekind, fully bounded, and quasi-injective modules, as well as other concepts. Finally, new results have been added on some types of modules related to S-PS.B. F-modules.

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1. Introduction

The notion of almost bounded submodules. Some authors introduced and defined the concept of a bounded module by using different ideas. The definition of a bounded start equation script cap F-module presented by ed $\mathcal{F}$-module presented by Faith [5]. Where ($\mathcal{F}$-modules $\mathfrak{M}$ named bounded if $\exists x \epsilon \mathfrak{M}$ s.t $ann_{\mathcal{F}}(\mathfrak{M}) = ann_{\mathcal{F}}(x)$). Moreover, almost bounded submodules notion is studied by Shihab [12]. Later, a generalization of the bounded module was given by authors which called semi-bounded module where $\exists x \epsilon \mathfrak{M}$ s.t $\sqrt{ann_{\mathcal{F}}(\mathfrak{M})} = \sqrt{ann_{\mathcal{F}}(x)}$, so $\mathcal{F}$ -modules $\mathfrak{M}$ named semi-bounded $\mathcal{F}$ -module), [1].

The authors Mahmood and Al-Ani [8], introduced some details regarding the definition of bounded $\mathcal{F}-modules$ that belongs to Faith [5], and studied

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many properties containing the definition of a bounded submodule if $\exists x\epsilon\aleph$ s.t $ann_{\mathcal{F}}(\aleph) = ann_{\mathcal{F}}(x)$, the submodule $\aleph$ of $\mathcal{F}-modules$ named bounded submodules, and then define fully bounded $\mathcal{F}-modules$ when every nonzero submodules for the $\mathfrak{M}$ be bounded [3]. Then, the concept of a finitely annihilated module was presented by Beachy and Blair [4], which is a generalization of a bounded module for Faith [5], where (if $ann_{\mathcal{F}}(\mathfrak{M})$ is the annihilator of a finite subset, then the $\mathcal{F}-modules$ $\mathfrak{M}$ named finitely annihilated) [4]. Recent studies gave an equivalent characterization for bounded submodules when the $\mathcal{F}-submodules$ $\aleph$ named Endo- Restricted Bounded if $\exists \mathcal{k}(x)\epsilon\aleph, \mathcal{k}\epsilon End(\mathfrak{M}), x\epsilon\mathfrak{M}$ implies $ann_{\mathcal{F}}(\aleph) = ann_{\mathcal{F}}\mathcal{k}(x)$, [10].

The aim of our study is to present a new concept (S-pseudo bounded module) with the addition of important and useful results that connect our concept with well-known and established concepts presented by previous researchers in Algebra Theory.

2. Some Preliminary Properties Associated with S-Pseudo Bounded Modules

Definition 2.1: The submodules $\aleph$ for the $\mathcal{F}-modules$ $\mathfrak{M}$ named S-pseudo bounded if $\exists \mathcal{k}(x)\epsilon\aleph, \mathcal{k}\epsilon S = End(\mathfrak{M}), x\epsilon\mathfrak{M}$ implies $\sqrt{ann_{\mathcal{F}}(\aleph)} = \sqrt{ann_{\mathcal{F}}\mathcal{k}(x)}$.

Definition 2.2: An $\mathcal{F}-modules$ $\mathfrak{M}$ named S-PS.B.$\mathcal{F}-modules$ where each proper submodules for the $\mathfrak{M}$ be the S-PS. B $\mathcal{F}-$ submodules for the $\mathfrak{M}$.

Examples 2.3: The $\mathbb{Z}_p$ as the $\mathbb{Z}-modules$ be the S-PS.B.$\mathbb{Z}-modules$, when p be the prime.

Proposition 2.4: Let $\mathfrak{M}$ is the $\mathcal{F}-modules$ & $\mathfrak{M}$ be the S-PS.B.$E-modules$, so we have $\mathfrak{M}$ be the S-PS.B.$\mathcal{F}-modules$, where $E = End(\mathfrak{M})$.

Proof: If $\mathfrak{M}$ be the S-PS.B.$E-modules$ & $\aleph$ be the submodules for the $\mathfrak{M}$. Defined $\mathcal{k}: \mathfrak{M} \longrightarrow \mathfrak{M}$ as $\mathcal{k}(x) = x, \forall x\epsilon\mathfrak{M}$, then $\mathcal{k}(x)\epsilon\aleph$ & $\sqrt{ann_E(\aleph)} = \sqrt{ann_E\mathcal{k}(x)}$. Since $\aleph$ as an $E-submodules$ is a $\mathcal{F}-submodules$ [9], so that $\mathcal{k}(x)\epsilon\aleph$ as $\mathcal{F}-submodules$. Assume that $a\epsilon\sqrt{ann_{\mathcal{F}}\mathcal{k}(x)}$ then $a^n\epsilon ann_{\mathcal{F}}\mathcal{k}(x), n\epsilon\mathbb{Z}^+$ which means $a^n.\mathcal{k}(x) = 0$ hence $a^n.(x) = 0, \forall x\epsilon\aleph$ so $a^n.\aleph = 0$, thus $a\epsilon\sqrt{ann_{\mathcal{F}}(\aleph)}$. Directly, $\sqrt{ann_{\mathcal{F}}(\aleph)} \subseteq \sqrt{ann_{\mathcal{F}}\mathcal{k}(x)}$. Hence $\mathfrak{M}$ be the S- PS.B.$\mathcal{F}-modules$.

Proposition 2.5: Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ two S-PS.B. $\mathcal{F}-moduless$, then $\mathfrak{M}_1\oplus\mathfrak{M}_2$ be the S-PS.B.$\mathcal{F}-modules$.

Proposition 2.6: If $\mathfrak{M}$ is the $\mathcal{F}-modules$ and $I \subseteq ann_{\mathcal{F}}(\mathfrak{M})$ where I be the ideals for the $\mathcal{F}$. so $\mathfrak{M}$ be the S-PS.B. $\mathcal{F}-modules$ iff $\mathfrak{M}$ be the S-PS.B. $\mathcal{F}/I-modules$.

Proof: If $\aleph$ be the submodules for the $\mathfrak{M}$ so $\exists \&\epsilon End(\mathfrak{M})$ s.t $\&: \mathfrak{M} \longrightarrow \mathfrak{M}, x\epsilon\mathfrak{M}, \&(x)\epsilon\aleph$ & $\sqrt{ann_{\mathcal{F}}\&(x)} = \sqrt{ann_{\mathcal{F}}(\aleph)}$. We claim that $\sqrt{ann_{\mathcal{F}/I}\&(x)} = \sqrt{ann_{\mathcal{F}/I}(\aleph)}$. Suppose that $a + I\epsilon\sqrt{ann_{\mathcal{F}/I}\&(x)}$, then $(a + I)^n\epsilon ann_{\mathcal{F}/I}\&(x)$, $n\epsilon\mathbb{Z}^+, (a + I)\epsilon\mathcal{F}/I$ so that $(a + I)^n.\&(x) = 0 + I$ implies that $(a^n + I).\&(x) = 0 + I$ hence $a^n.\&(x) = 0$ and $a\epsilon\sqrt{ann_{\mathcal{F}}\&(x)}$. But $\mathfrak{M}$ be the S-PS.B. $\mathcal{F} - modules$, therefore $a\epsilon\sqrt{ann_{\mathcal{F}}(\aleph)}$ & $(a^n + I).(\aleph) = 0 + I = 0, n\epsilon\mathbb{Z}^+$ implies $(a + I)^n.(\aleph) = 0$, so $a + I\epsilon\sqrt{ann_{\frac{\mathcal{F}}{I}}(\aleph)}, (a + I)\epsilon\mathcal{F}/I$. Thus $\sqrt{ann_{\mathcal{F}/I}\&(x)} \subseteq \sqrt{ann_{\mathcal{F}/I}(\aleph)}, \forall\aleph < \mathfrak{M}$. Clearly, $\sqrt{ann_{\mathcal{F}/I}(\aleph)} \subseteq \sqrt{ann_{\mathcal{F}/I}\&(x)}$, therefore $\mathfrak{M}$ be the S- PS.B. $\mathcal{F}/I - modules$.

Conversely, it is easy. $\mathfrak{M}$ be the S-PS.B. $\mathcal{F} - modules$.

3. Main Results of S-Pseudo Bounded Modules

In this part, several relationships connecting the S-Pseudo bounded module with other modules have been studied, and many important results have been obtained.

Proposition 3.1: For $\mathfrak{M}$ an S-PS.B. $\mathcal{F} - modules$ &each ideals I for the $\mathcal{F}$ be semiprime, we have $\mathfrak{M}$ be the bounded.

Proposition 3.2: If $\mathfrak{M}$ is the bounded uniform $\mathcal{F} - modules$ with $ann_{\mathcal{F}}(x)$ be prime. We have $\mathfrak{M}$ be the S- PS.B. $\mathcal{F} - modules$.

Proposition 3.3: If $\aleph$ be the cyclic submodules for the bounded $\mathcal{F} - modules$ $\mathfrak{M}$, so $\aleph$ be S-PS.B. $submodules$.

Corollary 3.4: *Every proper submodule of a cyclic fully bounded $\mathcal{F} - modules$ $\mathfrak{M}$ be the S- PS.B. $\mathcal{F} - modules$.*

Proposition 3.5: If $\mathfrak{M}$ is the cyclic fully bounded $\mathcal{F} - modules$. Then $\mathfrak{M}$ be S-PS.B. $\mathcal{F} - modules$.

Proof: From the previous corollary and the definition of S- PS.B. $\mathcal{F} - modules$.

Corollary 3.6: *If $\mathfrak{M}$ is cyclic divisible $\mathcal{F} - modules$(when $\mathcal{F}$ be ID). so $\mathfrak{M}$ be the S- PS.B. $\mathcal{F} - modules$.*

Proof: From [7, 13] and previous proposition.

Corollary 3.7: *Let $\mathcal{F}$ is an integral domain. If $\mathfrak{M}$ is cyclic torsion free $\mathcal{F} - modules$, so $\mathfrak{M}$ be the S- PS.B. $\mathcal{F} - modules$.*

Corollary 3.8: *If* $\mathfrak{M}$ *is cyclic faithful* $\mathcal{F}-modules$ *(where* $\mathcal{F}$ *is ID). so* $\mathfrak{M}$ *be the S- PS.B.* $\mathcal{F}-modules$.

Proof: From [3] and using proposition (3.5).

Corollary 3.9: *If* $\mathfrak{M}$ *is a fully stable bounded faithful* $\mathcal{F}-modules$*(where* $\mathcal{F}$ *is ID), then* $\mathfrak{M}$ *be the S- PS.B.* $\mathcal{F}-modules$.

Proof: Using previous corollary and [8].

Proposition 3.10: For $\mathfrak{M}$ be the fully stable quasi-injective S- PS.B. $\mathcal{F}-modules$, where every ideal of $\mathcal{F}$ is semiprime. So $\mathfrak{M}$ be the finitely generated $\mathcal{F}-modules$.

Proof: Using proposition (3.1) and [9, 14].

Corollary 3.11: *If* $\mathfrak{M}$ *be the cyclic dedekind* $\mathcal{F}-modules$. *so* $\mathfrak{M}$ *be the S- PS.B.* $\mathcal{F}-modules$.

Proof: Using proposition (3.5) and [11, 15].

Proposition 3.12: If $\mathfrak{M}$ is the simple monoform $\mathcal{F}-modules$ s.t $ann_{\mathcal{F}}(\mathfrak{M})$ be the prime ideals. so $\mathfrak{M}$be the S- PS.B. $\mathcal{F}-modules$.

Proof: Using proposition (3.5) and from [2, 11].

Proposition 3.13: If $\mathfrak{M}$ is the quasi-Dedekind $\mathcal{F}-modules$ s.t each ideal for the $\mathcal{F}$ be the semiprime. So we have $\mathfrak{M}$ be the S- PS.B. $\mathcal{F}-modules$.

Proof: Let $\hbar \epsilon End(\mathfrak{M})$ define $\hbar: \mathfrak{M} \longrightarrow \mathfrak{M}$ by $\hbar(x) = x, \forall x \epsilon \mathfrak{M}$. If $\aleph$ be the proper's submodules for the $\mathfrak{M}$, let $x \epsilon \aleph$ so that $ann_{\mathcal{F}}(\aleph) \subseteq ann_{\mathcal{F}}(x)$, hence $\sqrt{ann_{\mathcal{F}}(\aleph)} \subseteq \sqrt{ann_{\mathcal{F}}\hbar(x)}$. Let $r\epsilon\sqrt{ann_{\mathcal{F}}\hbar(x)}$, since $\mathfrak{M}$ is quasi-Dedekind then $\aleph$ is a quasi-invertible submodule [6] and if $g: \mathfrak{M}/\aleph \longrightarrow \mathfrak{M}$ is defined $g(x+\aleph) = rx, \forall x \epsilon \mathfrak{M}$ but $g = 0$ which means $r\mathfrak{M} = 0, \forall x \epsilon \mathfrak{M}$ so $r \epsilon ann_{\mathcal{F}}(\mathfrak{M}) \subseteq ann_{\mathcal{F}}(\aleph)$ implies $r\epsilon\sqrt{ann_{\mathcal{F}}(\aleph)}$. Therefore $\sqrt{ann_{\mathcal{F}}\hbar(x)} \subseteq \sqrt{ann_{\mathcal{F}}(\aleph)}$ and since $\aleph$ is an arbitrary submodule of $\mathfrak{M}$, hence $\mathfrak{M}$ be the S- PS.B. $\mathcal{F}-modules$.

Proposition 3.14: If $\mathfrak{M}$ the quasi-injective S- PS.B. $\mathcal{F}-modules$ such that every ideal for the $\mathcal{F}$ be the semiprime. Then $\mathfrak{M}$ be the cyclic over $End(\mathfrak{M})$.

Proof: Using proposition (3.1) and from [5].

Proposition 3.15: If $\mathfrak{M}$ the quasi-injective S- PS.B. $\mathcal{F}-modules$ s.t each ideals for the $\mathcal{F}$ be the semiprime. Then $\mathfrak{M}$ be the finendo.

Proof: From proposition (3.1) and [9, 11].

Corollary 3.16: *If $\mathfrak{M}$ is the semi-simple S- PS.B.* $\mathcal{F} - modules$ *s.t every ideals for the $\mathcal{F}$ is semiprime. Then $\mathfrak{M}$ be the finendo***.**

4. Conclusion

In this work, the class of S-pseudo bounded module has been calculated with some of its properties. A main result for the work can be summarize as new relationships and facts are added to previously studied concepts such as bounded module, finitely generated module, cyclic module by adding specific conditions and obtaining new results.

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