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Resents result about Loc (SP) hollow-fuzzy modules

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Abstract

In this work, introduce the concept of Loc (SP) hollow-fuzzy Modules & study the fundamental features of this kind. Also, add several new concepts with their proofs and investigate several conditions under which specific Lc (SP) hollow-fuzzy Modules and Lc-hollow-fuzzy Modules are equivalent.

Subject Classification: *05C38.*

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1. Introduction

Semi prime F- subModules was introduced by [1]. In this work demonstrate (L(SP)-$\hbar$) Moduless to (Loc (SP)- $\hbar$) F -M). Moreover, generalize many features of (Lc (SP)- $\hbar$) FM. In this article write (f -Sub mention to Fuzzy subModules, F-$\leq_{\text{Sub}}$ to mention to Fuzzy small subModules & $\hbar$ - f M to mention that hollow fuzzy Modules & čYč to mention cyclic). This work consists of three section. The first section recall some definition and properties which we need later. The second section includes Lc (SP)- $\hbar$ -F M and basic features about this concept, and Section three: relation between (Lc- hollow, Lc (SP)-)- $\hbar$) F M. Also, we demonstrate the second side is not true by various examples.

Preliminaries:

Definition 1.1: [2], [3] Suppose X is an F-M for R-Modules M, & B F -Sub for the X, hence B be the F- $\leq_{\text{Sub}}$ in X if every F -Sub Ă for the X, s.t B+ Ă =X. implies Ă = f.

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Definition 1.2: [4] A F- M X for the R-Modules M named fuzzy čYč Modules, if $\exists \chi_t \subseteq \chi$ s.t each $y_t \subseteq \chi$ writte as $y_k = r\ell\ x_t$ for some fuzzy singleton $r\ell$ of R where K, ℓ , t $\in$ (0, 1]. Here write $\chi = (\chi_t)$ to denotes a fuzzy čYč Modules generate by X_t.

Definition 1.3: [5] A F -Sub A for the F- M X in the R -Modules M named semi-prime F-Sub if for each F-singleton $r_t \in R$, $x_s \subseteq X$, $(r^2)_t\ x_s \subseteq A$ implies $r_t\ x_s \subseteq A$ such that $A \neq X \in (0,1]$.

Lemma 1.4: [4] If X is a čYč F- M, then X_t is a čYč Modules, $\forall$ k $\in$(0,1].

Proposition 1.5: [6] Let A be the F- Sub in F -M X for the R-Modules M iff A_t be subModules for X_t, t $\in$ (0, 1].

Remark 16: [7] Each prime F-Sub is a semi-prime F-Sub, while the converse is not true.

2. Characterization of Lc (S P) - $\hbar$ -F M:

Hear recall the idea of Lc(S P)-$\hbar$ -F M & study fundamental properties of this kind of Moduless. Also, we added several new concepts with their proofs.

Definition 2.1: The Modules X in R –Modules M named L(S P) - $\hbar$ -F M if X has a unique Semiprime F- Sub contains all F- $\leq_{Sub}$ of X.

Proposition 2.2: [1] If A is F- Sub for Modules X for the R-Modules M so A be semi prime F-Sub of X iff A_t is Semiprime F- Sub of $\forall$ t $\in$(0,1].

Proposition 2.3: Let χ be an Lc(S P)- $\hbar$ -F M of R-Modules M$\Longleftrightarrow$ X_t is Lc(S P)- $\hbar$ -F M $\forall$ t $\in$(0,1].

Proof: Let X be Lc (S P) - $\hbar$ -F M, then every F-Sub in X has a unique Semiprime F- Sub contains all F-$\leq_{Sub}$ for the X. if Ă is the F -Sub of X, then $Ă_t$ is sub in X_t by (1.7), but Ă Semiprime f -Sub contains all F-$\leq_{Sub}$ for the χ. Therefore, is Semiprime f -Sub for Xt by Proposition (2.2).Thus, X_t is Lc (S P) - $\hbar$ - f M $\forall$ t $\in$(0,1] the other side, Let χ_t be a Lc(S P) - $\hbar$ -F M. Suppose that B_t be a Semiprime f -Sub of X_t , so B is f -$\leq_{Sub}$ of X.Implies that X is Lc(S P)- $\hbar$ - f M.

Example 2.4: M= Z_9, A = $\langle 3 \rangle$, & χ: M $\rightarrow$ [0, 1] as $X_{(X)}$ =1 $\forall$ x in M & we defined A: M$\rightarrow$ [0, 1]

$$A(t) = \begin{cases} t \text{ where } x \in \langle 3 \rangle\ , \forall\ t \text{ in the } (0,\ 1] \\ 0{,}5 \text{ otherwises} \end{cases}$$

A unique semi-prime F-sub of M= Z9, containing F-Sub of M= Z9, contains F-Sub of M= $Z_{9,\ \text{contains F-}}$ contain F- $\leq_{Sub}$. That means there exist $t \in R$

for F- singletons $\mathfrak{r}_t$ for the R & $x_s \subseteq X$,, $\mathfrak{r}_t^2\, x_s \subseteq A$ implies $r_t\, x_s \subseteq A$ where $A \neq \chi$ .Hece χ is Lc (SP) - $\hbar$ - f M. Obviously, that is A is f ub of χ and $\breve{A}_t = N$ is sub of X , and hence $\breve{A}_t$ is semi prime sub of X_t=M. implies, every subModules of χ_t is Semiprime ten A is S emiprime $\forall$ t $\in$ (0, 1]. Thus, A is a unique smi-prime F-sub of χ, again by (2.2). Impliesgleton $\mathbf{3}_{1/3}$ of R, and $\mathbf{2}_{1/2} \subseteq X$, $\mathbf{3}^2_{1/3}.\ \mathbf{2}_{1/2} = \mathbf{18}_{1/3} \subseteq$ A, Since A(18) = 1$\geq$ 1/3 implies that $3_{1/3}\ .2_{1/2} = 6_{1/3} \subseteq$ A and since A(6) = 1$\geq$1/3, where A $\neq$ X. Hence X is Loc (SP) - $\hbar$ -F M by proposition (2. 3).But X=Z_6 as Z-Modules is not is Lc (SP) – $\hbar$ -F M. Since Z_6 has two semi prime F-Sub and $\langle 3\rangle \langle\overline{3}\rangle$. Let M= Z_6 , R = Z, defined X: M $\rightarrow$ [0, 1]

$$X_{(x)} = \begin{cases} 1 \text{ where x in M} \\ 0 \text{ otherwise} \end{cases} \quad \& \quad A:\mathcal{M} \rightarrow [0, 1] \text{ as } A_{(x)} = \begin{cases} 1 \text{ when } x \in \langle\overline{2}\rangle \\ 0 \text{ otherwises } \forall\ t \in (0, 1] \end{cases}$$

$$B(x) \ \&\ B :M \rightarrow [0, 1] \text{ as } = \begin{cases} 1 \text{ if } x \in \langle\overline{3}\rangle\ B_{(x)} \\ 0 \text{ otherwis} \end{cases}$$

Direct $A_t = \langle\overline{2}\rangle$ and $B_t = \langle 3\rangle$,$X_t = M = Z_6\ \forall$ t $\in$ (0, 1], $\langle\overline{2}\rangle$ & $\langle\overline{3}\rangle$ are semi-prime F- Sub of Z6, but X=Z6 as Z-Modules is not is Loc (S P) - $\hbar$ -F M since Z6 has two semi-prime F- Sub.It is clear that A_t and B_t are semi-prime subModuless of $X_t\ \forall$ t in (0, 1] & A & B semi-prime subModuless for X But Xt=Z6 is not in Loc (S P) - $\hbar$ -F M, since Z6 has two semi-prime subModuless. Notes ; there exists; F- Singleton $4_{1/3}$ of R, $2_{1/2} \subseteq$ X, then we have $\mathbf{4}^2_{1/3}.\ \mathbf{2_{1/2}} = 32_{1/3} \subseteq$ A and A (32) = 1$\geq$ 1/3 , implies $4_{1/3}\ .4_{1/3}.\ 2_{1/2} = 8_{1/3} \subseteq$ A also and A (8) = 1$\geq$ 1/3 where A $\neq$ X. Similarly, B_t. also B is not F-$\leq_{Sub}$ of X.

Example 2.5: Each Lc (SP) - $\hbar$ -F M is not necessarily a simple F- M. For example, Z- Modules Z25 is Lc (SP) - $\hbar$ -F M but not a simple F- M. Let R=Z, M=$Z_{25.}$ Then A_t =N $\forall$ t $\in$(0, 1], and X_t = M. Let X: M $\rightarrow$ [0, 1], as $X_{(x)} = \{$ 1 where x in M 0 otherwises $\forall$ t in the (0, 1].

Defined A: A: M $\rightarrow$ [0, 1], as, $A_{(x)} = X_{(x)} = \{$ 1 when x $\in$ M 0 otherwises $\forall$ t in the (0, 1] where N = 5Z. so X & A are F -M (F-Sub) A_t = N, X_t = M. So That's mean there exists $r_t \in$ R for fuzzy singletons r_t for the R and $x_s \subseteq$ X, = Z_{25} , $r_t^2\ x_s \subseteq$ A implies $r_t\, x_s$ A3, where A X.Hence X is Lc (SP) -$\hbar$ -F M., And it is trivial that M=$Z_{25.}$ is not simple.Clear that X and A are F -M & A is F-Sub of χ. Therefore, A_t = N Semiprime subModules of X_t = M $\forall$ t $\in$ (0, 1] by [6] get the result. Note that: there exists: f- singleton $3_{1/5}$ of R, $5_{2/5} \subseteq$ X, $3^2_{1/5}.5_{2/5} = 45_{1/5} \subseteq$ A & A (45) = 1 $\geq$ 1/3 implies $3_{1/5}.\ 5_{2/5} = 15_{2/5} \subseteq$ A also and A(15) = 1 $\geq$ 1/3 . Hence, A is a unique semiprime F-sub of X by proportion (2.3). But X_t=Z_{25} is not simple f- M. Therefore, X is not simple f- M.

Proposition 2.6: Let X_1 and X_2 are two F- M & let f: $X_1 \rightarrow X_2$ is a protective cover of X_2. If X_2 is Lc (SP)- $\hbar$ -F M, then X_1 is Lc (SP) - $\hbar$ -F M.

Proof: Let X_2 be Lc (SP) -$\hbar$ -F M and f: $X_1 \rightarrow X_2$ is an epimorphism of X1/kerf, which is an isomorphism to X2, and since it is Lc (SP) - $\hbar$ -F M, and Kerf is an F-subset of. Therefore, by [12, prop], this leads X1 to be Loc (SP)- $\hbar$ -F M**.** The property that every Loc -$\hbar$ -F M is lifting F -M is not true in Lc (SP) - $\hbar$ -F M. Now, we can show that in the next example.

Example 2.7: The Z Modules Q is Lc (SP) - $\hbar$ -F M, but is not lifting F- M. Consider the Z-Modules Q (when Q set for rational number), this F -M is Lc (SP) - $\hbar$ -F M, but is not lifting F -M. Since Q has only Semiprime F- Sub is 0_1, and it is not F-$\leq_{Sub}$. To spur our example, as we say in section one, every $\hbar$ -F M is Semiprime – $\hbar$ -F M, but the converse is not true. if X: Q$\rightarrow$ [0, 1], define as $X_{(x)}$ ={ 1 for x $\in$ Z 0 otherwise, so X_t=Z, X_0= Q, hence 0_1 is only prime F- Sub in Q by Example (2.4) , that is, Semiprime F- Sub is F-$\leq_{Sub}$ of Q $\forall$ t $\in$ (0, 1]. because each prime F- Sub implies to semi-prime F- Sub by [8] whilst Q doesn't F-$\leq_{Sub}$. Therefore, Q is not lifting. If X is finitely generating F -M. Then Lc (SP) -$\hbar$ -F M could be lifting F -M, as follows.

Proposition 2.8: Every finitely generating Lc (SP) - $\hbar$ -F M is lifting F -M.

Proof: Suppose that X is Lc (SP)- $\hbar$ -F M. There exisit unique f-Sub Ă that contains all F-$\leq_{Sub}$ of X. So X = X $\oplus$ $\{0_1\}$ where $\{0_1\}$ is f-Sub of A, also A $\cap$ X = A is f -$\leq_{Sub}$ [9]. But χ Lc(SP)- $\hbar$ -F M. Also A$\cap$ X = A be the F -$\leq_{Sub}$ for the χ.Hence, X lifting f-M [8].

Remark 2.9: The opposite of prop (9.2) is not always correct, as well as in the next example. The Z-Modules M= $Z_3 \oplus Z_4$, R = Z. ifX: M $\rightarrow$ [0, 1], define as X (a, b) = 1,$\forall$ (a, b)$\in$ = Z3 $\oplus$ Z4. Let A= Z3, B = Z4, and X where $\bar{}$1. Let A: M $\rightarrow$ [0, 1], Let B: M $\rightarrow$ [0, 1], define as

A (a, b) = { 1 if (a,b) $\in <(\bar{1}, \bar{2})>$, B (a, b)= { 1 if (a,b) $\in <(\bar{1}, \bar{2})> 0$ otherwise.

If A_t F- Sub of B_t, then A_t F- Sub of X_t $\forall$ t $\in$ [0, 1] by [6]. But X_t =Z= $Z_3 \oplus Z_4$, so X_t =M= $Z_3 \oplus Z_4$, hence A_t subModules of X_t and A is an F-sub of X. so A is an F-sub for X & $X_t \cap A_t = A_t$ unique Semiprime F- Sub say A, Since X is lifting Modules by example (2.4) and X is lifting F- M, while let C: M$\rightarrow$ [0, 1], define as follows: C (a, b) = { 1/2 if a $\in$3Z 0 otherwise. Note that: There exists the F- singleton $3_{1/3}$ of R, and $6_{1/3}(3.0) \subseteq X$, then $3^2{}_{1/5}$. $3^2{}_{1/5}$. $6(3,0)_{1/5}$ = $(162,0)_{1/5} \subseteq$ C because $(162,0)_{1/5}$ = 1/2 $\geq$ 1/35.

And $3_{1/5}$.$6_{1/5}$ $(3,0)_{1/5}$ = $(54,0)_{1/5} \subseteq$ C, because C (54,0) = 1/2 $\geq$ 1/5, Obviously, C is F- Sub of X and C_t= 3Z $\oplus$ (0) is a Semiprime subModules of X which not contain each F-$\leq_{Sub}$. Hence, C be semiprime F- Sub for the X not containing all F-$\leq_{Sub}$ of X.

3 . Relation between (Lc –hollow and Lc (SP)-hollow) F- Moduless

If f čYč f- M then χ is Lc –$\hbar$ -f M and Lc (SP)-$\hbar$ - f M are equivalent.

Remark 3.1: Every Loc – $\hbar$ -F M is Lc (SP) - $\hbar$ -F M, but the converse is not correct in general. This appears as follows:

Examples 3.2: 1-Each Lc- $\hbar$ -F M is Lc (SP) - $\hbar$ -F M, but an opposite not always true. The Z-Modules Q is Lc (SP) -$\hbar$ -F M, if $X_{(X)} = \{ 1$ where $X \in Z\ 0$ o.w direct that $X_t=Z$, $X_0= Q$, hence 0_1 is only Semiprime F- Sub in Q, that is F- $\leq_{Sub}$ of Q $\forall$ t $\in$ (0, 1]. Whilst Q is not Lc - $\hbar$ -F M, where Q does not have a unique maximal F-Sub, containing all F- Sub .2-Each Lc - $\hbar$ -F M is Lc (SP) - $\hbar$ -F M, but the opposite is not always correct.

Proposition 3.3: Suppose that X be the f- M for the R-Modules M, X is Lc -$\hbar$ - F M$\Leftrightarrow$ X is Lc (SP) - $\hbar$ - F M & has a unique Semiprime F- Sub.

Proof: If X is Lc– $\hbar$ - F M, therefore χ is Lc(SP)– $\hbar$ - f M immediately by remark (3.1). Hence, X has a unique Semiprime F- Sub; also by definition (2. 1), χ has a unique Semiprime F- Sub.

Conversely 3.4: [10] Suppose that X is Lc(SP) - hollow and has a unique Semiprime F- Sub say A, that is enough to prove that X is čYč F- M, let v $\in$X with v $\notin$ A where $X_t=X$, $A_t=A$, $\forall$ t $\in$ (0, 1], v_t is F- Sub of χ. Implies that Rv is F - Sub for the X. So X be the Loc (SP) -$\hbar$ - F M, thus A is unique Semiprime F - Sub of X , hence X = (v_t) $\forall$ t $\in$ (0, 1]. Consequently, χ is čYč f - M. So through (3.2) X is Lc - $\hbar$ - f M.

Conclusion

To sum up, we discussed and examined the concept Resents result about Loc (SP) hollow - Fuzzy Modules and (Lc –hollow) Fuzzy Modules. Where A module X of R –module M is said L(S P) - $\hbar$ -F M if X has unique Semiprime F- Sub contains all F- Sub of X. Also moreover, we generalized numerous properties about Loc (SP) hollow - Fuzzy Modules, and got several results and fundamental properties about Loc (SP) hollow -Fuzzy Modules, which are argued. Includes the relationship between properties about Loc (SP) hollow - Fuzzy Modules and (Lc –hollow)-Fuzzy Modules.

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