

Rational closed-duo-modules

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Abstract

The first aim of this study is to generalize the concept of a Duo-module, introducing the Rational Closed Duo Module (RC-DU-module). The second aim is to discuss its relationship with other concepts and obtain many results. During this time, some examples are given to explain these new concepts.

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1. Introduction

Let R be a ring, and $\mathcal{M}$ be a left module over R for (abbreviated $\mathcal{M}$ is R -mod). A sub module S is claimed to be rational (written as $\hookrightarrow^r \mathcal{M}$), if $\forall m \in \mathcal{M}$ and $0 \neq m \in \mathcal{M}$, $\exists r \in R$ with $rm \in S$ and $rm \neq 0$ [1]. A module $\mathcal{M}$ is claimed to be monoform if each nonzero sub module S of $\mathcal{M}$ is rational [1]. A sub module claimed to be a rationally-closed sub module (written as $S \hookrightarrow^{rc} \mathcal{M}$) if S does not have a proper rational extension in $\mathcal{M}$ [2]. A sub module is claimed to be fully invariant (abbreviated FI-submod written as $S \hookrightarrow^{FI} \mathcal{M}$) if $\mathcal{F}(S) \subseteq S, \forall f \in \text{End}(\mathcal{M})$ [3]. A module $\mathcal{M}$ is claimed to be a duo-module (abbreviated DU-mod) if $\forall S \hookrightarrow \mathcal{M}, S$ is FI-sub module [4]. A module $\mathcal{M}$ is claimed to be a weak-duo (abbreviated Weak-DU-mod) if $\forall S \hookrightarrow^{\oplus} \mathcal{M}$ then $S \hookrightarrow^{FI} \mathcal{M}$ [4]. A module $\mathcal{M}$ is claimed to be a CL-duo (abbreviated CL-DU-mod) (if $\forall S \hookrightarrow^c \mathcal{M}$ then $S \hookrightarrow^{FI} \mathcal{M}$ [5].

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2. Rational closed-duo-modules

Now, we introduce a new concept, a generalization of the DU-mod, which is claimed to be the RC-DU-mod.

Definition 2.1: A module $\mathcal{M}$ is claimed to be a rational closed-duo-module (abbreviated RC-DU-mod), if every rational-closed sub-mod of $\mathcal{M}$ is the FI-sub mod of $\mathcal{M}$.

Remark and Examples 2.2:

- 1- Each DU-mod is RC-DU, but generally, the opposite is incorrect. Such as Q as Z - module is RC-DU-mod, actually since Q have exactly two rational-closed sub-mod, 0 and Q itself, both of which are FI-sub mod, but Q is not a DU-mod since the sub-mod Z is not FI-sub mod of Q actually $\exists \mathcal{F}: Q \rightarrow Q$ is a homomorphism such that $\mathcal{F}(X) = \frac{1}{8}X$ and obviously $\mathcal{F}(Z) \not\subseteq Z$.
- 2- Every commutative ring R It is an RC-duo ring. Since, by [4], each commutative ring is a DU-ring, then by (1) we obtain the results.
- 3- Every monoform is an RC-DU-mod, actually, since the monoform module $\mathcal{M}$ has exactly rational-closed sub module are (0) and $\mathcal{M}$, both of which are FI-sub mod.
- 4- Obvious that each RC-DU-mod is CL-DU-mod, but the opposite is incorrect.

Proposition 2.3: Consider $\mathcal{M}$ as a semi-simple module. Then $\mathcal{M}$ is DU-mod, iff $\mathcal{M}$ is RC-DU-mod.

Proof: $\Rightarrow$ by remark (2.2 (1)).

$\Leftarrow$ Suppose $S \hookrightarrow \mathcal{M}$. By assume that $S \hookrightarrow^{\oplus} \mathcal{M}$ then by [6], $S \hookrightarrow^{rc} \mathcal{M}$ And since $\mathcal{M}$ is an RC-DU-mod, S is FI-sub mod. Thus, $\mathcal{M}$ is a DU-mod.

Corollary 2.4: Consider $\mathcal{M}$ as a simple-mod. Then $\mathcal{M}$ is DU-mod iff $\mathcal{M}$ is RC-DU-mod.

Proposition 2.5: If $\mathcal{M}$ is RC-DU-mod, then each direct summand is an RC-DU-mod.

Proof: Suppose that $S \hookrightarrow^{\oplus} \mathcal{M}$, since $\mathcal{M}$ RC-DU-mod, let $A \hookrightarrow^{rc} S$ take $\rho: \mathcal{M} \rightarrow S$ and $j: S \rightarrow \mathcal{M}$ (where ρ is projection and j is injection) assume that $\mathcal{F}: S \rightarrow S$ thus $h \equiv j\mathcal{F}\rho \in \text{End}(\mathcal{M})$, .Since $S \hookrightarrow^{\oplus} \mathcal{M}$ then $S \hookrightarrow^{rc} \mathcal{M}$ by [6], then $A \hookrightarrow^{rc} \mathcal{M}$. But $\mathcal{M}$ is RC-DU-mod so $h(A) \subseteq A$, i.e $(j\mathcal{F}\rho)(A) = \mathcal{F}(A) \subseteq A$, S is RC-DU-mod.

In this example, we will explain the sub-mod of a RC-DU-mod is not RC-DU-mod.

Example 2.6: Suppose $R = Q \oplus Q^2$. (Q denoted as the set of rational numbers on R) as following $(x, y) \cdot (h, k) = (xh, xk + hy)$ for each $x, h \in Q$ and $y, k \in Q^2$ where $Q^2 \cong Q \times Q$. R is commutative, so R is RC-DU-ring, hence it is RC-DU-mod. But the sub module $S = Q \times Q$ is not RC-DU-mod, since if it RC-DU-mod then it is CL-DU-mod, but by [5] S is not CL-DU-mod, (contradiction).

3. Some characterizations of the RC-DU-modules

In this final part we will study some characterizations of RC-DU-mod and also discuss the relationship of our concept with various types of modules. Moreover, we discuss the direct sum of RC-DU-mod.

Proposition 3.1: The R -module $\mathcal{M}$ is RC-DU-mod iff $\forall \mathcal{F} \in \text{End}(\mathcal{M})$ and $\forall \langle m \rangle = Rm \hookrightarrow^{rc} \mathcal{M}$, $\exists t \in R$ s.t $\mathcal{F}(m) = tm$.

Proof: ($\Rightarrow$) Suppose $\mathcal{F} \in \text{End}(\mathcal{M})$. Let $\langle m \rangle \hookrightarrow^{rc} \mathcal{M}$. Because $\mathcal{M}$ is an RC-DU-mod, so $\langle m \rangle$ is a FI-sub mod that is $\mathcal{F}(\text{module of } \langle m \rangle) \subseteq \langle m \rangle$. Thus, $\exists t \in R$ s.t $\mathcal{F}(m) = tm$.

($\Leftarrow$) Assume that $S \hookrightarrow^{rc} \mathcal{M}$, and let $\mathcal{F} \in \text{End}(\mathcal{M})$. $\forall s \in S, \mathcal{F}(s) \in \mathcal{M}$. By assume that, $\exists t \in R$ s.t $\mathcal{F}(m) = ts \in S$. Hence $\mathcal{F}(S) \subseteq (S)$ i.e, $\mathcal{M}$ is RC-DU-mod.

In Example (2.6), we show that the RC-DU property is not hereditary. But it is possible to get under certain conditions. Before that, we need the following.

Lemma 3.2: Let $S \hookrightarrow A \hookrightarrow \mathcal{M}$. Then $S \hookrightarrow^{rc} \mathcal{M} \Rightarrow S \hookrightarrow^{rc} A$.

Proof: If S is a complement to D in $\mathcal{M}$, it follows easily from the definition that S is complement to $D \cap A$ in $\mathcal{M}$. Therefore $S \hookrightarrow^{rc} A$.

$\mathcal{M}$ is claimed to be countably generated (shortly cont. gen.). Now, if a mod $\mathcal{M}$ gen. by count, set then we say that $\mathcal{M}$ is cont. gen. [5].

Proposition 3.3: If each cont. gen. sub-mod of $\mathcal{M}$ is RC-DU-mod, then $\mathcal{M}$ is RC-DU-mod.

Proof: Suppose $\langle m \rangle = Rm \hookrightarrow^{rc} \mathcal{M}$, and let $\mathcal{F} \in \text{End}(\mathcal{M})$. Let the sum of sub modules of $\mathcal{M}$: $Rm + R(\mathcal{F}(m)) + R(\mathcal{F}^2(m)) + \dots$ is a cont. gen. sub-mod of $\mathcal{M}$. If we represented the above sum by S and restrict $\mathcal{F}$ to S , thus we obtain $\mathcal{F}_S \in \text{End}(S)$. By lemma (3.2) $\langle m \rangle \hookrightarrow^{rc} S$, and since S is RC-DU-mod, so by prop. (3.1) $\exists t \in R$ such that $\mathcal{F}(m) = tm$. Also, by using prop.(3.1), therefore, $\mathcal{M}$ It is an RC-DU mod.

Theorem 3.4: If $\mathcal{M}$ is a semi-simple R -mod. Below, they give the same result.

1) $\mathcal{M}$ is RC-DU-mod 2) $\mathcal{M}$ is CL-DU-mod 3) $\mathcal{M}$ is DU-mod 4) $\mathcal{M}$ is Weak-DU-mod 5) $\mathcal{M}$ It is a fully stable mod.

Proof: (1) $\Rightarrow$ (2) clear by remark (2.2(4))

(2) $\Leftarrow$ (1) Let $\hookrightarrow^{rc} \mathcal{M}$. And $\mathcal{M}$ is semi-simple implies $S \hookrightarrow^{\oplus} \mathcal{M}$ and by [3], then $S \hookrightarrow^c \mathcal{M}$. While $\mathcal{M}$ is CL-DU-mod then $S \hookrightarrow^{FI} \mathcal{M}$, therefore $\mathcal{M}$ is RC-DU-mod. Clearly (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) by [5].

The following theorem gives the conditions under which a Weak-DU-mod becomes an RC-DU-mod.

Theorem 3.5: If $\mathcal{M}$ is a Rational Extending Module (RCS-mod). Below, they give the same result : 1) $\mathcal{M}$ is RC-DU-mod 2) $\mathcal{M}$ is Weak-DU-mod.

Proof: $\Rightarrow$) Let $S \hookrightarrow^{\oplus} \mathcal{M}$ by [6] then $S \hookrightarrow^{rc} \mathcal{M}$. Because $\mathcal{M}$ is RC-DU-mod then $\hookrightarrow^{FI} \mathcal{M}$, therefore, $\mathcal{M}$ is a Weak-DU-mod.

$\Leftarrow$) Let $\hookrightarrow^{rc} \mathcal{M}$, since $\mathcal{M}$ is a RCS-mod, then $S \hookrightarrow^{\oplus} \mathcal{M}$. But $\mathcal{M}$ is a Weak-DU-mod that is $S \hookrightarrow^{FI} \mathcal{M}$. So, $\mathcal{M}$ is RC-DU-mod.

In the next example, we prove that the direct sum of an RCS-mod is not necessarily an RCS-mod in general, where, $\mathcal{M} = Z_p \oplus Z$ is not an RCS-mod by [6]. Before adding the required condition, we first present the following.

Lemma 3.6: [7] Let $\mathcal{M} = \bigoplus_{i \in I} \mathcal{M}_i$. If $S \hookrightarrow^{FI} \mathcal{M}$ then $S = \bigoplus_{i \in I} (S \cap \mathcal{M}_i)$.

Theorem 3.7: Let $\mathcal{M}$ be RC-DU-mod, such that $\mathcal{M} = \bigoplus_{i \in I} \mathcal{M}_i$, where $\mathcal{M}_i$ be a sub mod of $\mathcal{M}$, $\forall i \in I$. Then $\mathcal{M}$ is RCS-mod iff, $\mathcal{M}_i$ is an RCS-mod.

Proof: $\Rightarrow$) Clear by [6]

$\Leftarrow$) Let $S \hookrightarrow^{rc} \mathcal{M}$ because $\mathcal{M}$ is RC-DU-mod, then $S \hookrightarrow^{FI} \mathcal{M}$. So by lemma (3.6), $S = \bigoplus (S \cap \mathcal{M}_i)$. Now, since $S \cap \mathcal{M}_i \hookrightarrow^{\oplus} S$, then $S \cap \mathcal{M}_i \hookrightarrow^{rc} S$. But $S \hookrightarrow^{rc} \mathcal{M}$, thus $S \cap \mathcal{M}_i \hookrightarrow^{rc} \mathcal{M}$ by [6], but $S \cap \mathcal{M}_i \hookrightarrow \mathcal{M}_i$ then easy to show that $S \cap \mathcal{M}_i \hookrightarrow^{rc} \mathcal{M}_i$ ($\forall i \in I$) and $\mathcal{M}_i$ is a RCS-mod ($\forall i \in I$) hence $S \cap \mathcal{M}_i \hookrightarrow^{\oplus} \mathcal{M}_i$, so $(S \cap \mathcal{M}_i) = S \hookrightarrow^{\oplus} \mathcal{M}$, therefore, $\mathcal{M}$ is an RCS-mod.

In the following example, we will show that the direct sum of the RC-DU-mod does not hold in general [8], [9].

Example 3.8: The Z -module $\mathcal{M} = Z \oplus Z$. Clear that Z is RC-DU-mod. But $\mathcal{M}$ is not RC-DU-mod, since if it RC-DU-mod then it is CL-DU-mod, but $\mathcal{M}$ is not CL-DU-mod by [5], (contradiction).

Proposition 3.9: Let $\mathcal{M}$ is a RC-DU-mod. If $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$, then $\text{Hom}(\mathcal{M}_1, \mathcal{M}_2) = 0$

Proof: Because $\mathcal{M}_1 \hookrightarrow^{\oplus} \mathcal{M}$, then $\mathcal{M}_1 \hookrightarrow^{rc} \mathcal{M}$, but $\mathcal{M}$ is a RC-DU-mod, so $\mathcal{M}_1 \hookrightarrow^{FI} \mathcal{M}$ and by [4] then $\text{Hom}(\mathcal{M}_1, \mathcal{M}_2) = 0$.

Proposition 3.10: Let $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ ($\mathcal{M}_1$ and $\mathcal{M}_2$, sub modules of $\mathcal{M}$) s.t $\text{ann } \mathcal{M}_1 + \text{ann } \mathcal{M}_2 = R$. Then $\mathcal{M}$ RC-DU-mod iff $\mathcal{M}_1, \mathcal{M}_2$ are RC-DU-mod with $\text{Hom}(\mathcal{M}_i, \mathcal{M}_j) = 0$, with $i \neq j$

Proof: $\Rightarrow$), by Prop.(2.5) and (3.9).

$\Leftarrow$) Suppose that $\hookrightarrow^{rc} \mathcal{M}$. Since $\text{ann } \mathcal{M}_1 + \text{ann } \mathcal{M}_2 = R$ so by [10] $S = A_1 \oplus A_2$ for some $A_1 \hookrightarrow \mathcal{M}_1$ and $A_2 \hookrightarrow \mathcal{M}_2$ it is clear that $A_1 \hookrightarrow^{rc} \mathcal{M}_1$ and $A_2 \hookrightarrow^{rc} \mathcal{M}_2$. Now, let $\mathcal{F} \in \text{End}(\mathcal{M})$, then $p_j \mathcal{F} i_j \in \text{End}(\mathcal{M}_j)$, where $j = 1, 2$ and (where p_j is projection and i is injection). Because $\mathcal{M}_j$ a RC-DU-mod $\forall j = 1, 2$, then $p_j \mathcal{F} i_j (A_j) \subseteq A_j$. Thus, we get $\mathcal{F}(A_1) + \mathcal{F}(A_2) \subseteq p_1 \mathcal{F} i_1 (A_1) + p_2 \mathcal{F} i_2 (A_2) = \sum_{j=1}^2 p_j \mathcal{F} i_j (A_j)$. But $\mathcal{F}(A_1) + \mathcal{F}(A_2) = \mathcal{F}(A_1 \oplus A_2) = \mathcal{F}(S)$, therefore $\mathcal{F}(S) \subseteq \sum_{j=1}^2 p_j \mathcal{F} i_j (A_j) \subseteq \sum_{j=1}^2 (A_j) = S$ thus $\mathcal{F}(S) \subseteq S$ therefore, RC-DU-mod [11].

4. Conclusion

From this work, we get that: DU-mod $\Rightarrow$ RC-DU-mod $\Rightarrow$ CL-DU-mod $\Rightarrow$ Weak-DU-mod. But the convers is not true. The converse holds when $\mathcal{M}$ is semi-simple. Moreover, the direct sum of RC-DU-mod does not hold in general.

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