

Rad coquasi-Dedekind modules

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Abstract

In this article, we introduce and investigate a novel notion called Rad coquasi-Dedekind module which is a proper generalization of coquasi-Dedekind module. A left $\mathcal{R}$ -module $\mathcal{M}$ is called Rad coquasi-Dedekind if for any proper submodule $\mathfrak{N}$ of $\mathcal{M}$ and any $\alpha \in \text{Hom}_{\mathcal{R}}(\mathcal{M}, \mathfrak{N})$, we have $\text{Im}\alpha \cap \text{Rad}(\mathcal{M}) = 0$. Some basic features and a characterization of these modules are established. We also prove some important related results.

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1. Introduction

In this article, each ring $\mathcal{R}$ is an associative with 1, and each module $\mathcal{M}$ is a unitary left. We denoted of the class of all left $\mathcal{R}$ -modules by $\mathcal{R}\text{-Mod}$. The notion coquasi-Dedekind modules recalled as the dual notion of quasi-Dedekind modules. “ $\mathcal{M} \in \mathcal{R}\text{-Mod}$ is called coquasi-Dedekind if any submodule $\mathfrak{N} \neq \mathcal{M}$, $\text{Hom}_{\mathcal{R}}(\mathcal{M}, \mathfrak{N}) = 0$ ”. Several generalizations of the concept of quasi-Dedekind modules have been given by a number of authors; see [1-2]. Then, it is natural to consider further generalizations of the idea of coquasi-Dedekind modules. In this work, we present and investigate the concept of Rad coquasi-Dedekind modules which are a proper expansion of coquasi-Dedekind modules. “ $\mathcal{M} \in \mathcal{R}\text{-Mod}$ is called Rad coquasi-Dedekind if for every submodule $\mathfrak{N} \neq \mathcal{M}$, we have that $\text{Im}\alpha \cap \text{Rad}(\mathcal{M}) = 0$ for any $\alpha \in \text{Hom}_{\mathcal{R}}(\mathcal{M}, \mathfrak{N})$ ”. “The Jacobson radical of $\mathcal{M} \in \mathcal{R}\text{-Mod}$ is denoted by $\text{Rad}(\mathcal{M})$ and defined as $\text{Rad}(\mathcal{M}) = \cap \{\mathfrak{N} \leq \mathcal{M}\} \mid \mathfrak{N}$

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is maximal submodules of $\mathcal{M}$, or $\sum\{\mathfrak{K} \leq \mathcal{M} \mid \mathfrak{K} \ll \mathcal{M}\}$, and a proper (nonzero) submodule $\mathfrak{K}$ of $\mathcal{M} \in \mathcal{R}\text{-Mod}$ is called small (resp., essential) in $\mathcal{M}$, as short $\mathfrak{K} \ll \mathcal{M}$ (resp., $\mathfrak{K} \trianglelefteq \mathcal{M}$) if for all $L \leq \mathcal{M}$ and $\mathcal{M} = \mathfrak{K} + L$ (resp., $\mathfrak{K} \cap L = 0$) implies $L = \mathcal{M}$ (resp., $L = 0$), see [3] and [4]. A nonzero $\mathcal{M} \in \mathcal{R}\text{-Mod}$ is said to be hollow if any proper submodule is small in $\mathcal{M}$ [5]. A submodule $\mathfrak{K}$ of $\mathcal{M} \in \mathcal{R}\text{-Mod}$ is said to be fully invariant if, $f(\mathfrak{K}) \subseteq \mathfrak{K}$ for all $f \in \text{End}(\mathcal{M})$ [5], and $\mathcal{M}$ is said to be duo if all its submodules are fully invariant [6]. Some basic properties of Rad coquasi-Dedekind, are discussed. Moreover, we prove that if, $\mathcal{M} \in \mathcal{R}\text{-Mod}$ is projective Rad coquasi-Dedekind, $\mathfrak{K} \ll \mathcal{M}$, then $\mathcal{M}/\mathfrak{K}$ is also Rad coquasi-Dedekind. Theorem 2.12 establishes an equivalent characterization of Rad coquasi-Dedekind, a nonzero $\mathcal{M} \in \mathcal{R}\text{-Mod}$ is Rad coquasi-Dedekind iff for any $f \in \text{End}_{\mathcal{R}}(\mathcal{M})$ with $\text{Im}f \cap \text{Rad}(\mathcal{M}) \neq 0$, is an epimorphism.

2. Main Results

In this section, we mention the definition of Rad coquasi-Dedekind and discuss some of their properties, characterizations and some important result.

Definition 2.1: A nonzero $\mathcal{M} \in \mathcal{R}\text{-Mod}$ is called Rad coquasi-Dedekind if for all submodule $\mathfrak{K} \neq \mathcal{M}$ and for all $\alpha \in \text{Hom}_{\mathcal{R}}(\mathcal{M}, \mathfrak{K})$, $\text{Im}\alpha \cap \text{Rad}(\mathcal{M}) = 0$.

As example; $\mathbb{Q}$, $\mathbb{Z}$ and $\mathbb{Z}_{p^\infty} \in \mathbb{Z}\text{-Mod}$ are Rad coquasi-Dedekind, while we see $\mathbb{Z}_{p^n} \in \mathbb{Z}\text{-Mod}$ where $1 \neq n \in \mathbb{Z}^+$ is not Rad coquasi-Dedekind, where p any prime number. Note that it is possible to prove the Rad coquasi-Dedekind and the coquasi-Dedekind module are coincide under the concept of hollow. Then, $\mathbb{Z} \in \mathbb{Z}\text{-Mod}$ is not coquasi-Dedekind, in fact, $\mathbb{Z} \in \mathbb{Z}\text{-Mod}$ is not hollow.

However, the epimorphic image and hence the factor module of Rad coquasi-Dedekind module are not always Rad coquasi-Dedekind. In order to make the factor module of Rad coquasi-Dedekind is also Rad coquasi-Dedekind, we shall provide the following necessary case.

Proposition 2.2: Suppose $\mathcal{M} \in \mathcal{R}\text{-Mod}$ is projective and $\mathfrak{K} \ll \mathcal{M}$. If $\mathcal{M}$ is Rad coquasi-Dedekind, then $\mathcal{M}/\mathfrak{K}$ is Rad coquasi-Dedekind.

Proof: Let $\mathfrak{K} \leq \mathcal{H} \leq \mathcal{M}$ and $\mathfrak{K} \ll \mathcal{M}$. Assume that $g \in \text{Hom}_{\mathcal{R}}(\mathcal{M}/\mathfrak{K}, \mathcal{H}/\mathfrak{K})$ such that $\mathcal{H}/\mathfrak{K} \subset \mathcal{M}/\mathfrak{K}$. Consider the diagram shown in Figure 1.

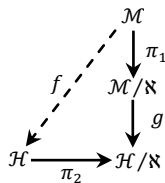


Figure 1

Commutative diagram of the natural projections and induced homomorphisms between $\mathcal{M}$, $\mathcal{H}$ and their quotient modules

Since $\mathcal{M} \in \mathcal{R}\text{-Mod}$ is projective, so there is $f \in \text{Hom}_{\mathcal{R}}(\mathcal{M}, \mathcal{H})$, $g\pi_1 = \pi_2 f$ where π_1, π_2 are natural mappings. It follows that $\text{Im}f \cap \text{Rad}(\mathcal{M}) = 0$. But $\text{Im}g = f(\mathcal{M}) + \mathfrak{K}/\mathfrak{K}$, also from [3, Corollary 9.1.5] $\text{Rad}(\mathcal{M}/\mathfrak{K}) = \text{Rad}(\mathcal{M})/\mathfrak{K}$, then, $\text{Im}g \cap \text{Rad}(\mathcal{M}/\mathfrak{K}) = \mathfrak{K}$ and hence $\mathcal{M}/\mathfrak{K}$ is Rad coquasi-Dedekind.

Corollary 2.3: Suppose $\mathcal{M} \in \mathcal{R}\text{-Mod}$ is free and $\mathfrak{K} \ll \mathcal{M}$. Then $\mathcal{M}/\mathfrak{K}$ is Rad coquasi-Dedekind, whenever $\mathcal{M}$ is Rad coquasi-Dedekind.

We provide the next consequence as a dual result of Proposition 2.2.

Proposition 2.4: Suppose $\mathcal{M} \in \mathcal{R}\text{-Mod}$ is Rad coquasi-Dedekind and $\mathfrak{K} \leq \mathcal{M}$. If every proper submodule of $\mathfrak{K}$ is injective, then $\mathfrak{K}$ is Rad coquasi-Dedekind.

Proof: Let $\mathcal{M} \in \mathcal{R}\text{-Mod}$ be Rad coquasi-Dedekind and $\mathcal{H} \subset \mathfrak{K} \leq \mathcal{M}$. Assume $g \in \text{Hom}_{\mathcal{R}}(\mathfrak{K}, \mathcal{H})$. Consider the diagram shown in Figure 2.

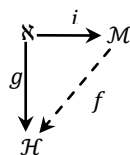


Figure 2

Diagram of the inclusion and homomorphisms between $\mathfrak{K}$, $\mathcal{M}$ and $\mathcal{H}$

where $i: \mathfrak{K} \rightarrow \mathcal{M}$ is the inclusion mapping, under assumption $\mathcal{H}$ is an injective, then there exists $f \in \text{Hom}_{\mathcal{R}}(\mathcal{M}, \mathcal{H})$ and $g = fi$. Because $\mathcal{M}$ is Rad coquasi-Dedekind and $\mathcal{H} \subset \mathcal{M}$ implies $\text{Im}f \cap \text{Rad}(\mathcal{M}) = 0$. We have $\text{Im}g \subseteq \text{Im}f$, then $\text{Im}g \cap \text{Rad}(\mathfrak{K}) \subseteq \text{Im}f \cap \text{Rad}(\mathcal{M}) = 0$ and this is required.

However, we will prove the Rad coquasi-Dedekind property for modules is inherited on by its direct summands as the following consequence.

Proposition 2.5: The class of Rad coquasi-Dedekind modules is closed under direct summands.

Proof: Suppose $\mathcal{M} \in \mathcal{R}\text{-Mod}$ is Rad coquasi-Dedekind and let $\mathfrak{K}$ be any proper direct summand of $\mathcal{M}$ with $L \subset \mathfrak{K}$. Suppose that $g \in \text{Hom}_{\mathcal{R}}(\mathfrak{K}, L)$. Consider $\rho: \mathcal{M} \rightarrow \mathfrak{K}$ is a projection mapping, thus $\text{Im}g \subseteq \text{Im}\rho$ and $\text{Rad}(\mathfrak{K}) \subseteq \text{Rad}(\mathcal{M})$, it follows that $\text{Im}g \cap \text{Rad}(\mathfrak{K}) = 0$ and hence $\mathfrak{K}$ is Rad coquasi-Dedekind.

Remark 2.6: The Rad coquasi-Dedekind property for modules is not closed under the direct sums, see to this; we known $\mathbb{Z}$ and $\mathbb{Q} \in \mathbb{Z}\text{-Mod}$ are Rad coquasi-Dedekind. Consider $\mathcal{M} = \mathbb{Z} \oplus \mathbb{Q}$, $\mathcal{R} = \mathbb{Z}$ and $\mathfrak{K} = 0 \oplus \mathbb{Q} \subset \mathcal{M}$. If $\rho: \mathcal{M}_{\mathcal{R}} \rightarrow \mathfrak{K}_{\mathcal{R}}$ is a projection map, we deduce that $\text{Im}\rho \cap \text{Rad}(\mathcal{M}) = 0 \oplus \mathbb{Q}$, this means that $\mathbb{Z} \oplus \mathbb{Q} \in \mathbb{Z}\text{-Mod}$ is not Rad coquasi-Dedekind.

However, we shall discuss Proposition 2.8, which implies that the finite direct sum of Rad coquasi-Dedekind modules must also be Rad coquasi-Dedekind. First, we must establish the next lemma.

Lemma 2.7: Suppose $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2 \in \mathcal{R}\text{-Mod}$. For $i = 1, 2$, if $\aleph_i \leq \mathcal{M}_i$ and $\aleph_1 \oplus \aleph_2 \subset \mathcal{M}$, then $\aleph_i \subset \mathcal{M}_i$.

Proof: Suppose that $\aleph_1 \oplus \aleph_2 \subset \mathcal{M}$. Consider the projection homomorphism $\rho_i: \mathcal{M} \rightarrow \mathcal{M}_i$, for all $i = 1, 2$, therefore $\rho_i(\aleph_1 \oplus \aleph_2) \subset \mathcal{M}_i$, because if assume $\rho_i(\aleph_1 \oplus \aleph_2) = \mathcal{M}_i$ so $\rho_i(\aleph_1 \oplus \aleph_2) = \begin{cases} \aleph_1 & \text{if } i = 1 \\ \aleph_2 & \text{if } i = 2 \end{cases}$, so $\aleph_1 = \mathcal{M}_1$ and $\aleph_2 = \mathcal{M}_2$, hence $\aleph_1 \oplus \aleph_2 = \mathcal{M}_1 \oplus \mathcal{M}_2$, a contradiction, then $\aleph_1 \subset \mathcal{M}_1$ and $\aleph_2 \subset \mathcal{M}_2$.

Proposition 2.8: Suppose $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2 \in \mathcal{R}\text{-Mod}$ is duo and $\mathcal{M}_1, \mathcal{M}_2 \in \mathcal{R}\text{-Mod}$ are Rad coquasi-Dedekind, then $\mathcal{M}$ is Rad coquasi-Dedekind.

Proof: Let $\mathcal{H} \subset \mathcal{M}_1 \oplus \mathcal{M}_2$. Since $\mathcal{M} \in \mathcal{R}\text{-Mod}$ is duo, then $\mathcal{H} = \aleph_1 \oplus \aleph_2$ where $\aleph_1 \leq \mathcal{M}_1$ and $\aleph_2 \leq \mathcal{M}_2$ by [6, Theorem 2.10]. Let $f \in \text{Hom}_{\mathcal{R}}(\mathcal{M}_1 \oplus \mathcal{M}_2, \aleph_1 \oplus \aleph_2)$, it follows $f = f_1 \oplus f_2$ where $f_1 \in \text{Hom}_{\mathcal{R}}(\mathcal{M}_1, \aleph_1)$ and $f_2 \in \text{Hom}_{\mathcal{R}}(\mathcal{M}_2, \aleph_2)$. Since $\aleph_1 \oplus \aleph_2 \subset \mathcal{M}$, we have that $\aleph_1 \subset \mathcal{M}_1$ and $\aleph_2 \subset \mathcal{M}_2$ from Lemma 2.7, then $\text{Im} f \cap \text{Rad}(\mathcal{M}) = 0$, as $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ is duo and $\mathcal{M}_1, \mathcal{M}_2$ are Rad coquasi-Dedekind modules, hence $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ is Rad coquasi-Dedekind.

Corollary 2.9: Suppose $\mathcal{M} = \bigoplus_{i=1}^n \mathcal{M}_i \in \mathcal{R}\text{-Mod}$ is duo. For any $i \in \{1, 2, \dots, n\}$, if $\mathcal{M}_i$ is Rad coquasi-Dedekind, then $\mathcal{M}$ is Rad coquasi-Dedekind.

Lemma 2.10: Suppose $\mathcal{M} \in \mathcal{R}\text{-Mod}$ and $\bar{\mathcal{R}} = \mathcal{R}/l_{\mathcal{R}}(\mathcal{M})$. Then,

- (1) $\aleph_{\mathcal{R}} \ll \mathcal{M}_{\mathcal{R}}$ iff $\aleph_{\bar{\mathcal{R}}} \ll \mathcal{M}_{\bar{\mathcal{R}}}$.
- (2) $\text{Rad}_{\mathcal{R}}(\mathcal{M}) = \text{Rad}_{\bar{\mathcal{R}}}(\mathcal{M})$.

Proposition 2.11: Suppose $\mathcal{M} \in \mathcal{R}\text{-Mod}$ and $\bar{\mathcal{R}} = \mathcal{R}/l_{\mathcal{R}}(\mathcal{M})$. Then $\mathcal{M} \in \mathcal{R}\text{-Mod}$ is Rad coquasi-Dedekind iff $\mathcal{M} \in \bar{\mathcal{R}}\text{-Mod}$ is Rad coquasi-Dedekind.

Proof: Let $\mathcal{M} \in \mathcal{R}\text{-Mod}$ be Rad coquasi-Dedekind. Assume $\aleph \subset \mathcal{M}_{\bar{\mathcal{R}}}$ where $h \in \text{Hom}_{\bar{\mathcal{R}}}(\mathcal{M}, \aleph)$ then from [3] $h \in \text{Hom}_{\bar{\mathcal{R}}}(\mathcal{M}, \aleph) = \text{Hom}_{\mathcal{R}}(\mathcal{M}, \aleph)$ such that $\aleph \subset \mathcal{M}_{\mathcal{R}}$, so $\text{Im} h \cap \text{Rad}_{\mathcal{R}}(\mathcal{M}) = 0$, Lemma 2.10(2) imply $\text{Im} h \cap \text{Rad}_{\bar{\mathcal{R}}}(\mathcal{M}) = 0$, then $\mathcal{M} \in \mathcal{R}\text{-Mod}$ is a Rad coquasi-Dedekind. The converse is similar.

The next is a characterization of the concept of Rad coquasi-Dedekind modules.

Theorem 2.12: A nonzero $\mathcal{M} \in \mathcal{R}\text{-Mod}$ is Rad coquasi-Dedekind iff for any $f \in \text{End}_{\mathcal{R}}(\mathcal{M})$ and $\text{Im} f \cap \text{Rad}(\mathcal{M}) \neq 0$ is an epimorphism.

Proof: Let $\mathcal{M}$ be nonzero Rad coquasi-Dedekind, and let $f \in \text{End}_{\mathcal{R}}(\mathcal{M})$ with $\text{Im} f \cap \text{Rad}(\mathcal{M}) \neq 0$. If f is not epimorphism, then $\text{Im} f \subset \mathcal{M}$, and then for any

$g \in \text{Hom}_{\mathcal{R}}(\mathcal{M}, \text{Im}f)$, $\text{Im}g \cap \text{Rad}(\mathcal{M}) = 0$. Let $h: \mathcal{M} \rightarrow \text{Im}f$ be a map define by $h(m) = f(m)$ for all $m \in \mathcal{M}$. Thus, $\text{Im}h \cap \text{Rad}(\mathcal{M}) = \text{Im}f \cap \text{Rad}(\mathcal{M}) = 0$, this is a contradiction. Conversely, assume the condition is hold. Let $g \in \text{Hom}_{\mathcal{R}}(\mathcal{M}, \mathfrak{N})$ and $\text{Im}g \cap \text{Rad}(\mathcal{M}) \neq 0$ where $\mathfrak{N} \subset \mathcal{M}$. Let $i: \mathfrak{N} \rightarrow \mathcal{M}$. We have $f = ig \in \text{End}_{\mathcal{R}}(\mathcal{M})$ and $\text{Im}f \cap \text{Rad}(\mathcal{M}) = \text{Im}g \cap \text{Rad}(\mathcal{M}) \neq 0$, then by the assumption $\mathcal{M} = \text{Im}f$ and that implies $\mathcal{M} = \text{Im}g \subseteq \mathfrak{N}$, a contradiction.

Proposition 2.13: Suppose $\mathcal{M} \in \mathcal{R}\text{-Mod}$ is Rad coquasi-Dedekind. For every $s \in \mathcal{R}$, either $s\mathcal{M} \cap \text{Rad}(\mathcal{M}) = 0$ or, $s\mathcal{M} = \mathcal{M}$.

Proof: Let $s \in \mathcal{R}$. We can define an $\mathcal{R}$ -endomorphism $\varphi_s: \mathcal{M} \rightarrow \mathcal{M}$ since $\varphi_s(m) = sm$ for all $m \in \mathcal{M}$. If $s\mathcal{M} \cap \text{Rad}(\mathcal{M}) \neq 0$, i.e., $\text{Im}\varphi_s \cap \text{Rad}(\mathcal{M}) \neq 0$, so φ_s is an $\mathcal{R}$ -epimorphism, by Theorem 2.12, and hence $s\mathcal{M} = \text{Im}\varphi_s = \mathcal{M}$.

Corollary 2.14: Suppose $\mathcal{M} \in \mathcal{R}\text{-Mod}$ is Rad coquasi-Dedekind such that $s\mathcal{M} \subseteq \text{Rad}(\mathcal{M})$ for all $s \in \mathcal{R}$, then $l_{\mathcal{R}}(\mathcal{M})$ is prime.

Proof: Let $a, b \in \mathcal{R}$ and $ab \in l_{\mathcal{R}}(\mathcal{M})$. Assume $a \notin l_{\mathcal{R}}(\mathcal{M})$ and $b \notin l_{\mathcal{R}}(\mathcal{M})$, then $a\mathcal{M} \neq 0$ and $b\mathcal{M} \neq 0$ under the assumption, $a\mathcal{M} \cap \text{Rad}(\mathcal{M}) = a\mathcal{M} \neq 0$ with $b\mathcal{M} \cap \text{Rad}(\mathcal{M}) = b\mathcal{M} \neq 0$. As $\mathcal{M}$ is Rad coquasi-Dedekind, $a\mathcal{M} = \mathcal{M}$ and $b\mathcal{M} = \mathcal{M}$ by Proposition 2.13, then $0 = ab\mathcal{M} = a\mathcal{M} = \mathcal{M}$, a contradiction.

Proposition 2.15: Let $\mathcal{M}_1$ and $\mathcal{M}_2 \in \mathcal{R}\text{-Mod}$ such that $\mathcal{M}_1$ is Rad coquasi-Dedekind and $s\mathcal{M}_1 \subseteq \text{Rad}(\mathcal{M}_1)$ for every $s \in \mathcal{R}$. If $0 \neq \varphi \in \text{Hom}_{\mathcal{R}}(\mathcal{M}_1, \mathcal{M}_2)$, then $l_{\mathcal{R}}(\mathcal{M}_1) = l_{\mathcal{R}}(\text{Im}\varphi)$.

Proof: If $s \in l_{\mathcal{R}}(\mathcal{M}_1)$, $r\mathcal{M}_1 = 0$, and then $\varphi(s\mathcal{M}_1) = s(\text{Im}\varphi) = 0$, therefore $s \in l_{\mathcal{R}}(\text{Im}\varphi)$, so $l_{\mathcal{R}}(\mathcal{M}_1) \subseteq l_{\mathcal{R}}(\text{Im}\varphi)$. Assume $s \in l_{\mathcal{R}}(\text{Im}\varphi)$, then $s\varphi(\mathcal{M}_1) = \varphi(s\mathcal{M}_1) = 0$. Since $\mathcal{M}_1$ is Rad coquasi-Dedekind, then either $s\mathcal{M}_1 = \mathcal{M}_1$, or $s\mathcal{M}_1 \cap \text{Rad}(\mathcal{M}_1) = 0$, Proposition 2.13. If $s\mathcal{M}_1 = \mathcal{M}_1$, so $\text{Im}\varphi = \varphi(s\mathcal{M}_1) = 0$, a contradiction. Hence, $s\mathcal{M}_1 \cap \text{Rad}(\mathcal{M}_1) = 0$, since $s\mathcal{M}_1 \subseteq \text{Rad}(\mathcal{M}_1)$, then $s\mathcal{M}_1 \cap \text{Rad}(\mathcal{M}_1) = s\mathcal{M}_1 = 0$, thus $s \in l_{\mathcal{R}}(\mathcal{M}_1)$, hence $l_{\mathcal{R}}(\text{Im}\varphi) \subseteq l_{\mathcal{R}}(\mathcal{M}_1)$.

Corollary 2.16: Let $\mathcal{M}_1$ and $\mathcal{M}_2 \in \mathcal{R}\text{-Mod}$ such that $\mathcal{M}_1$ is Rad coquasi-Dedekind and $s\mathcal{M}_1 \subseteq \text{Rad}(\mathcal{M}_1)$ for all $s \in \mathcal{R}$. If $\varphi \in \text{Hom}_{\mathcal{R}}(\mathcal{M}_1, \mathcal{M}_2)$ is onto, then $l_{\mathcal{R}}(\mathcal{M}_1) = l_{\mathcal{R}}(\mathcal{M}_2)$.

Lemma 2.17: Suppose $\mathcal{M} \in \mathcal{R}\text{-Mod}$, $m \in \mathcal{M}$ and $l_{\mathcal{R}}(m) = 0$. If J is an ideal of $\mathcal{R}$ such that $Jm = \mathcal{R}m$, then $J = \mathcal{R}$.

Proof: Suppose that $Jm = \mathcal{R}m$. Since $m = 1 \cdot m \in \mathcal{R}m$, then $m \in Jm$, and hence $m = am$ for some $a \in J$. Since $(1 - a)m = m - am = 0$, $1 - a \in l_{\mathcal{R}}(m)$, thus $1 = a + (1 - a) \in J + l_{\mathcal{R}}(m) = J$, therefore $J = \mathcal{R}$.

Theorem 2.18: Suppose $\mathcal{M} \in \mathcal{R}\text{-Mod}$ is Rad coquasi-Dedekind and $l_{\mathcal{R}}(m) = 0$, $m \in \mathcal{M}$. If $f \in \text{Hom}_{\mathcal{R}}(\mathcal{M}, \mathcal{R})$, then either $\text{Im}f \cap \text{Rad}(\mathcal{R}) = 0$ or, $\text{Im}f = \mathcal{R}$.

Proof: Suppose $f \in \text{Hom}_{\mathcal{R}}(\mathcal{M}, \mathcal{R})$ where $l_{\mathcal{R}}(m) = 0$ for some $m \in \mathcal{M}$. Define $f_m: \mathcal{M} \rightarrow \mathcal{R}m$ by $f_m(x) = f(x)m$ for all $x \in \mathcal{M}$, so f_m is an $\mathcal{R}$ -homomorphism (clear). If $i: \mathcal{R}m \rightarrow \mathcal{M}$ is the inclusion map, we put $g = if_m \in \text{End}_{\mathcal{R}}(\mathcal{M})$. As $\mathcal{M}$ is Rad coquasi-Dedekind, then either $\text{Im}g \cap \text{Rad}(\mathcal{M}) = 0$ or $\text{Im}g = \mathcal{M}$, by Theorem 2.12. If assume $\text{Im}g \cap \text{Rad}(\mathcal{M}) = 0$, so $f(\mathcal{M})m \cap \text{Rad}(\mathcal{R})m = 0$, thus $(\text{Im}f \cap \text{Rad}(\mathcal{R}))m = 0$, then $\text{Im}f \cap \text{Rad}(\mathcal{R}) \subseteq l_{\mathcal{R}}(m) = 0$. If $\text{Im}g = \mathcal{M}$, we have $f(\mathcal{M})m = \mathcal{R}m$, but $l_{\mathcal{R}}(m) = 0$, thus $\text{Im}f = \mathcal{R}$, by Lemma 2.17.

Proposition 2.19: Suppose $\mathcal{M} \in \mathcal{R}\text{-Mod}$ is indecomposable. Then $\mathcal{M}$ is Rad coquasi-Dedekind iff for any proper essential submodule $\mathfrak{K}$ of $\mathcal{M}$, and for any $\alpha \in \text{Hom}_{\mathcal{R}}(\mathcal{M}, \mathfrak{K})$, $\text{Im}\alpha \cap \text{Rad}(\mathcal{M}) = 0$.

Proof: The necessity is clear. Conversely, Assume that $0 \neq \mathfrak{K} \subset \mathcal{M}$ and $\mathcal{H}$ a complement of $\mathfrak{K}$, thus $\mathfrak{K} \oplus \mathcal{H} \trianglelefteq \mathcal{M}$ form [4, Proposition 1.3]. Since $\mathcal{M}$ is indecomposable, therefore $\mathfrak{K} \oplus \mathcal{H} \neq \mathcal{M}$. Consider the sequence $\mathcal{M} \xrightarrow{f} \mathfrak{K} \xrightarrow{i} \mathfrak{K} \oplus \mathcal{H}$ where i is the inclusion mapping. It follows $g = if \in \text{Hom}_{\mathcal{R}}(\mathcal{M}, \mathfrak{K} \oplus \mathcal{H})$, then $\text{Im}g \cap \text{Rad}(\mathcal{M}) = 0$, while $\text{Im}g = \text{Im}f$ that implies $\text{Im}f \cap \text{Rad}(\mathcal{M}) = 0$, then $\mathcal{M}$ is Rad coquasi-Dedekind.

Corollary 2.20: Suppose $\mathcal{M} \in \mathcal{R}\text{-Mod}$ is hollow. Then $\mathcal{M}$ is Rad coquasi-Dedekind iff $\text{Im}\alpha \cap \text{Rad}(\mathcal{M}) = 0$ for any proper essential submodule $\mathfrak{K}$ of $\mathcal{M}$ and any $\alpha \in \text{Hom}_{\mathcal{R}}(\mathcal{M}, \mathfrak{K})$.

Proposition 2.21: Suppose $\mathcal{R}$ is a Rad coquasi-Dedekind ring and $J(\mathcal{R}) \trianglelefteq \mathcal{R}$, then $\mathcal{R}$ has no identity.

Proof: Suppose $\mathcal{R}$ has identity, say 1. By assumption, $\mathcal{R}$ is not simple. Let $0 \neq I \subset \mathcal{R}$, then there is $0 \neq a \in I$. Consider $f: \mathcal{R} \rightarrow I$ is an $\mathcal{R}$ -homomorphism define by $f(r) = ra$ for any $r \in \mathcal{R}$. Since $\text{Im}f \cap J(\mathcal{R}) = 0$, thus $\text{Im}f = 0J(\mathcal{R}) \trianglelefteq \mathcal{R}$, then $0 = f(1) = 1 \cdot a = a$, a contradiction, then $\mathcal{R}$ has no identity.

Conclusions

In this article, we defined Rad coquasi-Dedekind modules as a proper generalization of the coquasi-Dedekind modules. Several properties of this concept were proven, and it was noted that new properties not covered in the notion of coquasi-Dedekind modules were also established, as example Propositions 2.2, 2.4, and 2.8. Several of the connections between this kind of modules and other kinds of modules will be studied in future papers.

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