Journal of Discrete Mathematical Sciences & Cryptography
ISSN 0972-0529 (Print), ISSN 2169-0065 (Online)
Vol. 29 (2026), No. 5, pp. 2153–2160
DOI : 10.47974/JDMSC-2718

Purely small compressible modules and purely small retractable modules

Haider M. Ridha *
Nuhad S. Al-Mothafar †
Department of Mathematics
College of Science
University of Baghdad
Baghdad
Iraq

Abstract

Let S be a commutative ring with identity and let U be an (left) unitary S-module (for brief S-mod). This paper presents a detail and study the concepts of purely small compressible (pu-s-comp) and purely small retractable (pu-s-retract) as a generalization of compressible S-mod and retractable S-mod, and give some of their properties, characteriztions, and examples. On the other hand, we study the relation between pu-s-comp and pu-s-retract and provide illustrative example to support the obtained result.

Subject Classification: *Primary 93A30, Secondary 49K15.*

Keywords: *Purely small retractable S-mod, Compressible S-mod, Retractable S-mod, Pu-S sub-mod, Pu-S-comp, Pu-hollow S-mod.*

1. Introduction

A commutative ring $\boldsymbol{S}$ with identity, let $\boldsymbol{U}$ be an (left)unitary S-mod. It is widely known that a sub-mod $\boldsymbol{V}$ of $\boldsymbol{U}$ is termed pure if $\mathbb{I}\boldsymbol{U} \cap \boldsymbol{V} = \mathbb{I}\boldsymbol{V}$ for all ideal $\mathbb{I}$ of $\boldsymbol{S}$ [1]. In [2], We present and study the concept of purely small (pu-s), such that a sub-mod $\boldsymbol{V}$ of an $\boldsymbol{S}$-mod $\boldsymbol{U}$ is called pu-s sub-mod ($\boldsymbol{V} \ll_{\boldsymbol{pu}} \boldsymbol{U}$) if for any pure sub-mod $\mathbf{C}$ of $\boldsymbol{U}$, $\boldsymbol{V} + \mathbf{C} = \boldsymbol{U}$ implies $\mathbf{C} = \boldsymbol{U}$. Also [3] study some properties of a pu-s sub-mod. An S-mod $\boldsymbol{U}$ is called pu-s-comp if there exists a monomorphism (for brief mono) from $\boldsymbol{U}$ in to each of it's non-zero pu-s sub-mod. In addition, an S-mod $\boldsymbol{U}$ is called pu-s-retract if $\boldsymbol{Hom_S}(\boldsymbol{U}, \boldsymbol{V}) \neq \mathbf{0}$ for each non-zero pu-s sub-mod $\boldsymbol{V}$ of $\boldsymbol{U}$.

* *E-mail:* **haidar.reda2503p@sc.uobaghdad.edu.iq (Corresponding Author)**
† *E-mail:* **nuhad.salim@sc.uobaghdad.edu.iq**

In this paper, we introduce and study the concept of pu-s-comp and pu-s-retract as generalization of compressible module and retractable modules. We investigate their properties, characterizations, and interrelations. In addition, we see that under conditions, pu-s-comp, small compressible (s-comp) and compressible (comp) are identical. Moreover, we study the relationship between pu-s-comp, pu-s-retract and several of class of module.

2. Preliminaries

Definition 2.1: Take U be an S-mod and $V \leq U$:

(1) V is termed small sub-mod (s-sub-mod) of U, ($V \ll U$) if $V + \mathrm{C} = U$ implies $\mathrm{C} = U$, for each sub-mod C of U [1].

(2) An S-mod U is termed hollow if for all a proper sub-mod is small in U [4]

(3) A sub-mod V of U is termed pure if $\mathbb{I}U \cap V = \mathbb{I}V$ for every ideal $\mathbb{I}$ of S [1].

(4) A sub-mod V is termed pu-s sub-mod of U, ($V \ll_{pu} U$) if $V + \mathrm{C} = U$, if for any pure sub-mod C of U, implies $C = U$ [2].

(5) An S-mod U is called purely hollow (pu-hollow) if for all proper sub-mod in U is pu-s, [3].

(6) An S-mod U is said to be s-comp if there exists a mono from U in to each of it's non-zero small sub-mod of U, [5], [6].

(7) An S-mod U is termed retractable if $Hom_S(U, V) \neq 0$ for each non- zero sub-mod V in U, [6], [7].

(8) An S-mod U is termed small retractable (s-retract) if $Hom_S(U, V) \neq 0$ for each non- zero small sub-mod V in U, [6], [8].

Remark 2.2: [3] Each small sub-mod is pu-s. But in generally, the opposite is untrue. As instance: $2\mathbb{Z}$ is pu-s sub-mod of the $\mathbb{Z}$-mod $\mathbb{Z}$. In fact, the only pure sub-mod of $\mathbb{Z}$ is (0) and $(0) + 2\mathbb{Z} \neq \mathbb{Z}$. But $2\mathbb{Z}$ is not s-sub-mod with $3\mathbb{Z} + 2\mathbb{Z} = \mathbb{Z}$, while $2\mathbb{Z} \neq \mathbb{Z}$.

3. Purely Small Compressible Modules

Herein, we present the notion of pu-s-comp S-mod in general of comp S-mod; give some of basic proparties, illustrations and characteriztions of this concept.

Definition 3.1: Let U be an S-mod is called pu-s-comp if for each non-zero pu-s sub-mod $V$of U there exists a mono $f: U \longrightarrow V$.

A ring S is called pu-s-comp if the S-mod S is pu-s-comp.

Examples and Remarks 3.2.

(1) Every comp S-mod is pu-s-comp but in generally, the opposite is untrue. As instance: as $\mathbb{Z}$-mod $\mathbb{Z}_6$ is not comp however $\mathbb{Z}_6$ is pu-s-comp since 0 is only pu-s sub-mod of $\mathbb{Z}_6$.

(2) as $\mathbb{Z}$-mod $\mathbb{Z}_4$ is not pu-s-comp, since $((\bar{2})$ is a pu-s-sub-mod of $\mathbb{Z}_4$ however $\mathbb{Z}_4$ cannot be embeded in $(\bar{2}))$.

(3) Per simple module is pu-s-comp converse not holed, since $\mathbb{Z}$ as a $\mathbb{Z}$-mod is pu-s-comp but not simple.

(4) Every pu-s-comp module is s-comp.

(5) A homomorphic image of a pu-s-comp S-mod not be pu-s-comp in general for example $as\ \mathbb{Z}$ -mod $\mathbb{Z}$ is a pu-s-comp and $\frac{\mathbb{Z}}{4\mathbb{Z}} \simeq \mathbb{Z}_4$is not pu-s-comp see point (2) above.

Proposition 3.3: A pu-s sub-mod of pu-s-comp module is also pu-s-comp module.

Proof: Let $0 \neq V \ll_{pu} U$ and U be pu-s-comp module and let $0 \neq C \leq V \ll_{pu} U$, then by [2] $C \ll_{pu} U$. Since U is pu-s-comp, so there exists a mono $f: U \longrightarrow C$ and $i: V \longrightarrow U$ is the inclusion homomorphism (inclu-hom), implies that $f \circ i: V \longrightarrow C$ is a mono. Hence V is a pu-s-comp module.

Proposition 3.4: Let U_1 and U_2 be isomorphic S-mod. Then U_1 is pu-s-comp if and onlyif U_2 is pu-s-comp.

Proof: Assume that $U_2$is pu-s-comp and let $0 \neq V \ll_{pu} U_1$. Let $\Phi: U_1 \longrightarrow U_2$ be an isomorphism. Then $0 \neq \Phi(V) \ll_{pu} U_2$ by [2]. Put $C = \Phi(V) \ll_{pu} U_2$, so $\alpha: U_2 \longrightarrow C$ is a mono (from the assumption), let $g = \Phi^{-1}\big|_C$, then $g: C \longrightarrow U_1$ is a mono. $g(C) = \Phi^{-1}(\Phi(V)) = V$. Hence $g: C \longrightarrow V$, then we have $\Psi = g \circ \alpha \circ \Phi$, hence $\Psi: U_1 \longrightarrow V$ is a mono. As a result U_1 is pu-s-comp.

Remark 3.5: The direct sum of pu-comp module need not be pu-s-comp. Observe the following as instance let $\mathbb{Z}_2 \oplus \mathbb{Z}_2 = \mathbb{Z}_4$ as $\mathbb{Z}$-mod, $\mathbb{Z}_2$ is pu-s-com mod, since $\mathbb{Z}_4$ as $\mathbb{Z}$ −mod is not pu-s-comp module see (3.2) point (2).

Proposition 3.6: Take U be a pu-holow S-mod. It follows that statament are equivelent.

(1) U is comp module.

(2) U is pu-s-comp module.

(3) U is s-comp module.

Proof: (1)⇒ (2) It's essay by (3.2) point (1).

(2) ⇒ (3) It's clear by remarks and examples (3.2) point (4).

(3) ⇒ (1) Let $0 \neq V \leq U$. Since U is pu-hollow S-mod and s-comp module then, there exists a mono $f: U \longrightarrow V$. Therefor U is comp module.

Proposition 3.7: An S-mod U is called pu-s-comp if and only if over there a mono $0 \neq g \in End_S(U)$ such that $g \subseteq V$ for each $V \ll_{pu} U$.

Proof: $\Rightarrow$Assume that U is pu-s-comp. Let $0 \neq V \ll_{pu} U$, $f: U \to V$ be a non-zero mono. So there exists a mono $g = jf \in End_S(U)$ where $j: V \to U$ be the inclu-hom and $Im\ g = jf(U) = f(U) \subseteq V$.

$\Leftarrow$Let $0 \neq V \ll_{pu} U$. By hypothesis, there exists a nonzero mono $g \in End_S(U)$ and $g(U) \subseteq V$. Therefore, $g: U \to V$ is a nonzero mono. Thus is U pu-s-comp.

Proposition 3.8: Let U be pu-s-comp S-mod such that every non-zero pu-s sub-mod of U is simple. Then U is simple.

Proof: Assume that U is pu-s-comp and let C be a non-zero pu-s sub-mod of U. Then U can be embedded in C, so U is isomorphic to any sub-mod of C. Since by hypothesis C is simple and $U \cong C$, implies U is simple.

Corollary 3.9: *Let U be a faithful S-mod such that every non-zero pu-s sub-mod of U is simple. If U is pu-s-comp, then S is pu-s-comp.*

Proof: Assume that U pu-s-comp. Then by above proposition U is simple, so by [8], $End_S(U)$ is a division ring. Since U is a faithful S-mod, $End_S \simeq S$ implies S is a division ring. Thus S is pu-s-comp.

Proposition 3.10: Let U be a regular S-mod, so U is s-comp if and onlyif U is pu-s-comp.

Proof: Assume that U is pu-s-comp and let $0 \neq V \ll_{pu} U$, then by [2], $V \ll U$ and since U is s-comp module, therefor U is pu-s-comp. Conversely, it's essay by Remarks and Examples (3.2) point (4).

Proposition 3.11: A direct summand of a pu-s-comp S-mod is also pu-s-comp.

Proof: Let $U = A \oplus D$ be an pu-s-comp S-mod and let $0 \neq C \ll_{pu} A$. Then $C \oplus 0 \ll_{pu} U$ [3], and hence there exists a mono $g: U \longrightarrow C \oplus 0$, clearly $C \oplus 0 \simeq C$, so $g: U \to C$ is a mono and the composition $f \circ j_A: A \longrightarrow U \longrightarrow C$ is a mono and $j_A: A \longrightarrow U$ is the inclu-hom. There for A is pu-s-comp.

Proposition 3.12: Let U be a faithful-finitely-generated multiplication S-mod. Then U is pu-s-comp S-mod iff S is pu-s-comp ring.

Proof: Let $0 \neq \mathbb{I} \ll_{pu} S$, but U is a faithful-finitely-generated multiplication, then by [2] $0 \neq C = \mathbb{I}U \ll_{pu} U$. But U is pu-s-comp, then there exists a mono $g: U \to C$. Put $g(u) = bu$ for some $0 \neq b \in \mathbb{I}$ and $u \in U$. Define $f: S \to \mathbb{I}$ by $f(s) = b\ \ \forall\ s \in S$. It can be easily checked that f is well-define mono. Implies that S is pu-s-comp ring.

Conversely, assume that S is pu-s-comp ring. Let $0 \neq V \ll_{pu} U$. Then $V = \mathbb{I}U$ for a particular nonzero ideal $\mathbb{I}$ of S (since U multiplication S-mod). But

$V \ll_{pu} U$ and U is a faithful finitely generated multiplication S-mod, then by [2], $\mathbb{I} \ll_{pu} S$. Therefor there exists a mono $g: S \to \mathbb{I}$ (since S is pu-s-comp). Put $g(1) = b$ for some $b \in \mathbb{I}$ and $b \neq 0$. Define $f: U \to V$ by $f(u) = bu$ for all $u \in U$. Can be easily check that f is well-defineed mono. Therefor U is pu-s-comp module.

4. Purely Small Retractable Modules

Herein, we present pu-s-retract S-mod in general of retractable module and present their properties, examples and characterzations.

Definition 4.1: S-mod U is called pu-s-retract if $Hom(U, V) \neq 0$ for each non-zero pu-s sub-mod V of U.

That means, U is pu-s-retract if there exists a non-zero homomorphism $g: U \to V$ whenever $0 \neq V \ll_{pu} U$.

A ring S is called pu-s-retract if the S-mod S is pu-s-retract.

Examples and Remarks 4.2.

(1) Every retractable S-mod is pu-s-retract, however in generally, the opposite is untrue. As instance:
we show that $B = \{\begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} \quad c, d \in S\}$ is not retractable [6], however the only
pu-s sub-mod of B is the zero sub-mod, hence B is pu-s-retract.

(2) Q as a $\mathbb{Z}$-mod is not pu-s-retract since $\mathbb{Z}$is pu-s sub-mod of Q but $Hom_S(Q, \mathbb{Z}) = 0$.

(3) Every semi-simple module is pu-s-retract because it is retract, however in generally, the opposite is untrue. As instance: $\mathbb{Z}$ is pu-s-retract $\mathbb{Z}$ -mod however it is not semi-simple.

(4) Every pu-s-retract module is s-retract.

(5) Every pu-s-comp is a pu-s-retract module, however in generally, the opposite is untrue. As instance: $\mathbb{Z}$-mod, $\mathbb{Z}_{12}$ is pu-s-retract however, not pu-s-comp since $(\bar{6})$ is the only pu-s sub-mod of $\mathbb{Z}_{12}$ and $f: \mathbb{Z}_{12} \to (\bar{6})$, such that $f(\bar{x}) = 6\bar{x}, \forall\ \bar{x} \in \mathbb{Z}_{12}$ is ahomomorphism which is not mono.

Proposition 4.3: A pu-s sub-mod of pu-s-retract module is also pu-s-retract module.

Proof: Let $0 \neq C \ll_{pu} U$ and U be pu-s-retract module and let $0 \neq W \leq C \ll_{pu} U$, then $W \ll_{pu} U$ [3]. Since U is pu-s-retract, then there exists a homomorphism $g: U \longrightarrow W$ and $i: C \longrightarrow U$ is the inclu-hom, then $g \circ i: C \to U \longrightarrow W$ be a homomorphism. Therefore C is a pu-s-retract module.

Proposition 4.4: Let U_1 and U_2 be isomorphic S- mod. Then U_1 is pu-s-retract if and only if U_2 is pu-s-retract.

Proof: Let $0 \neq C \ll_{pu} U_1$ and suppose that U_2 is pu-s-retract. Let Ψ: $U_1 \longrightarrow U_2$ be an isomorphism. Then by [2] $0 \neq \Psi(C) \ll_{pu} U_2$. Put $W = \Psi(C) \ll_{pu} U_2$, we get $g: U_2 \longrightarrow W$ is a non-zero homomorphism (from the assumption), let $f = \Psi^{-1}|_W$, then $f: W \longrightarrow U_1$ is a homomorphism. $f(W) = \Psi^{-1}(\Psi(C)) = C$. Hence $f: W \longrightarrow C$, then we have a composition $\phi = f \circ g \circ \Psi$. Hence, $\phi: U_1 \longrightarrow C$ is a homomorphism if $\phi = 0$, then $0 = f\left(g\left(\Psi(U_1)\right)\right) = f(g(U_2))$ implies that $g(U_2) \subseteq \ker f \subseteq \ker \Psi = 0$, then $g(U_2) = 0$ this is contrition, then $\phi \neq 0$. Therefore $U_1$is pu-s-retract module.

Proposition 4.5: Let U be pu-hollow S-mod, the following statements are equivalent.

(1) U is retractable S-mod.

(2) U is pu-s-retract S-mod.

Proof: (1) $\Longrightarrow$ (2) clearly by (4.2) point (1).

(2) $\Longrightarrow$ (1) Let $0 \neq W \leq U$ and U be pu-hollow module.

Thus $W \ll_{pu} U$, but U is pu-s-retract, then there exists a homomorphism $f: U \longrightarrow W$. Therefore U is retractable module.

Proposition 4.6: An S-mod U is called pu-s-retract if and only if over there $0 \neq g \in End_S(U)$ such that $Im\ g \subseteq V, \forall\ 0 \neq V \ll_{pu} U$.

Proof: $\Rightarrow$ Suppose that U is pu-s-retract. Let $0 \neq V \ll_{pu} U$. Then $Hom_S(U, V) \neq 0$. Let $f: U \to V$ be a nonzero hom and $g = jf$ where $j: V \to U$ be the inclu-hom, then $g \in End_S(U)$ and $Im\ g = jf(U) = f(U) \subseteq V$.

$\Leftarrow$ Let $0 \neq V \ll_{pu} U$. From the hypothesis, over there a nonzero endomorphism $g \in End_S(U)$ and $Im\ g \subseteq V$. Then $g: U \to V$ is a nonzero hom this U is pu-s-retract.

Proposition 4.7: An S-mod U is pu-s-retract. If for all nonzero endomorphism of U is mono, then for each nonzero element of $Hom_s(U, C)$ is mono from a nonzero pu-s sub-mod C of U.

Proof: Let $0 \neq C \ll_{pu} U$ and $g: U \to C$ be anon-zero homomorphism. Then $jg \in End_S(U)$ and $jg \neq 0$. For if $jg = 0$, then $jg(U) = g(U) = 0$ implies $g = 0$ which is a contradiction. Hence $0 \neq jg \in End_S(U)$ and (by hypot) jg is a mono, which gives that g is mono.

Corollary 4.8: *Let U be a pu-s-retract S-mod such that every non-zero of U is a mono then U is pu-s-comp.*

Proposition 4.9: An S-mod U is pu-s-retract if and only if for all $0 \neq x \in U$ with $Sx \ll_{pu} U$, $Hom_S(U, Sx) \neq 0$.

Proof: $\Rightarrow$By Proposition (4.7).

$\Leftarrow$ Let $0 \neq V \ll_{pu} U$ and let $0 \neq x \in V$, then $Sx \ll_{pu} U$, so by hypothes, $Hom_S(U, Sx) \neq 0$, impliies that $Hom_S(U, V) \neq 0$ and therefor U is pu-s-retract.

Proposition 4.10: Let S be a pu-s-retract and U be faithful-finitely-generated multiplcation S-mod. Then U is pu-s-retract.

Proof: Assume that $0 \neq C \ll_{pu} U$. Then $C = \mathbb{I}U$ for some $0 \neq \mathbb{I}$ ideal of S (since U multiplication S-mod). But $C \ll_{pu} U$ and U is a faithful finitely-generated multiplication S-mod, by [8] U is a cancellation module implies that $\mathbb{I} \ll_{pu} S$ [2] therefor $Hom(S, \mathbb{I}) \neq 0$ (since S is pu-s-retract). Let $g: S \to \mathbb{I}$ be a nonzero homomorphism. Put $g(1) = b$ for some $b \in \mathbb{I}$ and $b \neq 0$. Define $f: U \to C$ by $f(u) = bu$ for all $u \in U$. Can be easily check that f is well-defineed homomorphism, if $f = 0$, then $bu = 0$ for all $u \in U$ and therefor $b \in ann_S(U)$ (since U is fathful) but $b \neq 0$, therefor a contradiction and hence $Hom_S(U, C) \neq 0$. Therefor U is pu-s-retract.

Proposition 4.11: Every faithful cyclic S-mod is also pu-s-retract.

Proof: By [6], every faithful cyclic S-mod is retractable and hence is pu-s-retract.

Proposition 4.12: Let U be a regular S-mod, then U is s-retract iff U is pu-s-retract.

Proof: Suppose that U is s-retract and let $0 \neq V \ll_{pu} U$, then by [2], $V \ll U$ and since U is s-retract module, therefor U is pu-s-retract. Conversely, it's essay by Remarks and Examples (4.2) number (4).

5. Conclusion

In the current research, extended the theory of comp and retract modules by introducing pu-s-comp and pu-s-retract modules. In addition to examining their essential properties, the study identified the sufficient conditions for these new classes to coincide with traditional comp and retract modules.

References

[1] M. M. Ali and D. J. Smith, "Pure submodules of multiplication modules," *Contributions to Algebra and Geometry,* vol. 45, no. 1, pp. 61–74 (2004).

[2] M. A. Ahmed, I. A. Dhari, and Z. A. Atiya, "Purely small and purely hollow modules," *Iraqi Journal of Science*, vol. 63, no. 12, pp. 5487–5495 (2022).

[3] H. Q. Hamdi and N. S. Al-Mothafar, "Study on-purely small submodules,". *Journal of Discrete Mathematical Sciences and Cryptography*, vol. 27, no. 5, pp. 1723–1728 (2024).

[4] P. Fluery, "Hollow modules and local endomorphism rings," *pacific Journal of Mathematics*, vol. 53, pp. 379-385 (1974).

[5] P. C. Diop and M. L. Dia, "On E-small compressible modules," *Journal of Algebra and Related Topics*, vol. 9, no. 2, pp. 69–81 (2021).

[6] L. S. Mahmood and I. H. Muslem. "Small compressible module and small retractable module," *International Journal of Applied Mathematics and Statistical Sinenes*, vol. 5, no. 2, pp. 15-28 (2016).

[7] M. M. Hameed and A. A. Elewi, "Z-small critically compressible modules," *Journal of Discrete Mathematical Sciences & Cryptography*, vol. 28, no. 4-B, pp. 1309–1316 (2025).

[8] W. A. Ali and N. S. Al. Mothafar, "Small prime and quasi-small prime modules," *Journal of Discrete Mathematical Sciences and Cryptography*, vol. 24, no. 7, pp. 1967–1971 (2021).

Received January, 2026