

Pure rationally extending modules

Zahra Moaed Abd Al-Majeed *

Department of Chemical Engineering and Oil Pollution

College of Chemical Engineering

University of Technology

Baghdad

Iraq

Zainab Raad Shaker [†]

Department of Registration and Student Affairs

University of Technology

Baghdad

Iraq

Mahdi Saleh Nayef [§]

Department of Mathematics

College of Education

Mustansiriyah University

Baghdad

Iraq

Abstract

The main purpose for this work, we introduce and study a concept for pure rationally extending modules on the class of rationally extending modules by using pure submodules and direct summands. This paper presents the concept of pure rationally extending modules as a more general concept than rationally extending modules. Throughout this study, many examples and counterexamples are given. In addition, many properties, propositions, and relationships between pure rationally extending modules and some other concepts are discussed.

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* E-mail: Zahra.m.abdalmajeed@uotechnology.edu.iq (Corresponding Author)

[†] E-mail: zainab.r.shaker@uotechnology.edu.iq

[§] E-mail: mahdisaleh773@uomustansiriyah.edu.iq

1. Introduction

All modules in this work will be unitary left R -modules, where the ring R is associative with unity.

A submodule P of W is named essential (briefly $P \leq_e W$), if $\forall C \leq W$ with $P \cap C = 0$ then must be $C=0$ [1]. W is CS - module If, $\forall P \leq W$, then $P \leq_e D \leq \oplus$, see [2]. Let W as R -module, $P \leq W$ is rational (shortly, $P \leq_r W$) if $\forall y \in W, x \in W \setminus \{0\}, \exists r \in R$ s.t $rx \neq 0$ & $ry \in P$ (If P be the rational extensions for W , then W named rational extensions for the P). Clearly, every rational submodule is essential [1]. W is rationally extending if $\forall N \leq W, \exists D \leq W$ s.t $D \leq_d W$ and $N \leq_r D$ [1]. If $P \leq W$ is called pure if $IP = P \cap IW, \forall$ ideal I of R [3].

W is flat module if $0 \rightarrow N_1 \rightarrow N_2$ is exact for R -modules N_1 and N_2 then $0 \rightarrow W \otimes N_1 \rightarrow W \otimes N_2$ is also exact. W is named Pure extending if $\forall P \leq_p W, \exists D \leq W$ s.t $D \leq_d W$ and $P \leq_e D$ [4]. A rings R be the regular ($\forall a \in R$ be the regular). Each of finitely generate left ideals be the direct - summand in R [3].

We will symbolize rationally extending (R.E) and pure extending (P.E). This paper aims to generalize the theory of rationally extending modules by introducing and investigating a broader class, termed pure rationally extending modules. These are modules in which each pure sub-module is a direct summand. This generalization, based on the concept of purity, yields a class that properly contains the class of rationally extending modules, thereby expanding the scope of the original theory.

Following the introduction of pure rationally extending modules, we present several examples and counterexamples to clarify the distinction between this class and other related module classes. We demonstrate that pure rationally extending modules form a proper generalization of R.E by constructing counterexamples of pure rationally extending modules that are not R.E. Additionally, we establish a sufficient condition for which a pure rationally extending module becomes an R.E module. It is further shown that both direct summands and pure submodules of a pure rationally extending module inherit the pure rationally extending property. Moreover, we prove that direct sum of the pure rationally extend module be itself pure rationally extend. [see prop(2.7)].

Numerous researchers have contributed to the study of rationally extending modules, injectivity, and their various generalizations. Some have introduced concepts stronger than the class of rationally extending modules, while others have explored notions weaker than it. For further details, see [2], [5-12].

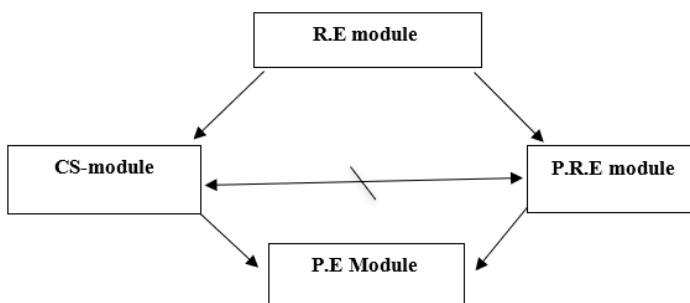
2. Pure Rationally Extending Modules

Definition 2.1: The W as the R -modules is named pure rationally extending module (shortly, P.R.E-modules) where each pure sub - modules for the W be the rational in direct summand for W .

Remark and Example 2.2:

1. Each R.E modules be the P.R.E module, such as Z as a Z -module.
2. All P.R.E module is P.E module. but not conversely for example where $W = Z \oplus Z$ as Z -modules be the P.E by [13] but not P.R.E since $\langle 0, 4 \rangle$ is a pure submodule but not rational in any direct summand.
3. Every rationally extending module is a pure extension. The convers is not true in general by example $W = Z \oplus Z$ as Z - module is pure extending but not rationally extending.
4. Each semi-simple (so simple) and monoform module is a pure rationally extending module since all of them are R.E. modules.
5. Z as Z - module is P.R.E since only 0 and Z are pure submodules.
6. $W = Z/pZ$ is P.R.E but $W = \prod_{p \in P} Z/\langle p \rangle$ is not P.R.E module since it is not P.E module [11], [13].

The following Figure 1, shows the relations between the Pure rationally extending-module and some related concepts.

**Figure 1**

Pure rationally extending-module and some related concepts

Proposition 2.3: If W be the pure rationally extending module and P be the pure sub - modules, so P be the pure rationally extending modules.

Proof: If W is P.R.E and $P \leq_p W$. Consider $K \leq_p P$. Then by [7, prop(7.2)], $K \leq_p W$, and W is P.R.E. Then there exists $N \leq_d W$ s.t K be the rational sub - modules for the N , so K be the rational sub - modules for the $N \cap P$. we know $N \leq_d W$ then $W = N \oplus N/\ , N/\leq W$. Now, $W \cap P = (N \oplus N/\) \cap P$. Implies $P = (N \cap P) \oplus (N/\cap P)$. Hence P is P. R. E.

Proposition 2.4: If W is a pure rationally extending module and $P \leq_d W$, then P is a pure rationally extending module.

Proof: If P the direct summand sub - modules for the W , and we know every direct summand is a pure submodule, so by proposition (2.3) we get P is P.R.E.

Proposition 2.5: If R is a regular ring, W is pure rationally extending iff W is rationally extending.

Proof: $\Rightarrow$ Since R is regular, each module is flat by [8] and we know that when R is regular, and W is flat, we get that every submodule of a flat module is pure.

Then, for every submodule of W , W is pure; by hypothesis, W is pure, so rationally extending this implies each sub – modules be the rational in direct - summand.

So W is R.E module.

$\Leftarrow$ By remake (2.2(1)).

Proposition 2.6: If W is pure rationally extending, $N \leq W$, then W/N is pure rationally extending.

Proof: If W be the P.R.E & N be the sub - module for the W . To prove $\square$ $N \leq W/N$ be the rational indirect summand. Let $\pi : W \rightarrow W/N, \pi(w) = w + N \forall w \in W$ be a natural projection. Suppose that $K \leq_p W/N$, so $P = \pi^{-1}(K) = \{w \in W : \pi(w) \in K\} \leq_p W$. Since W is P.R.E, then there exists $D \leq W$ s.t $P \leq D$ and P is rational in D , thus $P = P \cap D$, $\pi(D) \leq W/N$. Since $D \leq_d W$, then $\pi(D) \leq_d W/N$. Moreover, $K = \pi(P) \leq \pi(D)$, T.P K is rational in $\pi(D)$. $K = \pi(P) = \pi(P \cap D) = \pi(P) \cap \pi(D)$.

Hence, $K = K \cap \pi(D)$. Implies that K is rational in $\pi(D)$ (direct summand).

Now we shall discuss the case of the direct sum in relation to the property of a P.R.E module.

Proposition 2.7: Let $W = \bigoplus_{i \in \lambda} W_i$ where λ arbitrary index. W is P.R.E iff $\forall i \in \lambda, W_i$ is P.R.E module.

Proof: $\Rightarrow$ If W is P.R.E so by prop (2.4), W_i is P.R.E.

$\Leftarrow$ Assume that each W_i is a pure rationally extending module, T.P W is P.R.E. Let $P \leq_p W, P = \bigoplus_{i \in \lambda} P_i$. By assuming each P_i is rational in direct summand $D_i \leq W_i$ such that $D = \bigoplus_{i \in \lambda} D_i$. Since each P_i is rational in D_i . Then the direct sum P is rational in D . This implies that P is a pure ‘submodules for the direct summand. Hence, W is P.R.E.

A ring R is a local ring iff R_R (or ${}_R R$) is a local modules (where W have largest submodules, i.e. the proper submodules that includes every other proper sub modules, then W is called a local).

A pure simple module be the nonzero modules over ring s.t the only pure submodules of it are itself.

Proposition 2.8: Assume R be the local rings, so R_R be the P.R.E. modules.

Proof: Because every local ring be the pure simple by [[11]th(4)], so R_R and 0 are only pure submodules. Hence, R_R is the P.R.E. module.

A ring R is called left PDS if pure submodules of a left R -module are direct summand and the ring R named PDS where its both left & right PDS [13].

Proposition 2.9: Suppose R be the PDS rings, each R -modules be the pure rationally extending.

Proof: If W is module, and P is a pure submodule. By hypothesis, R is a PDS ring (every pure be direct - summand). So, $\exists$ sub-module L s.t $W=P \oplus L$, and P already direct - summand for the W , $P \leq P \oplus L$, and since P is pure. So, P is rational in $P \oplus L$. Then W is the P.R.E. module. We can now proceed to introduce the next theorem.

Theorem 2.10: Let W is nonsingular of a regular ring R , the following statements are equivalent.

- 1) W is R.E module
- 2) W is P.R.E module.
- 3) W is P.E module.
- 4) W is CS – module.

Proof: $1 \Rightarrow 2$ by [remark (2.2(1))], $2 \Rightarrow 3$ by [Remark (2.2(2))]

$3 \Rightarrow 4$ [since R is regular [13]], $4 \Rightarrow 1$ [since W is nonsingular]

3. Discussion and Conclusion

Through this work, it was concluded that all rationally extending modules are pure rationally extending modules, and the opposite is generally only achieved under a certain condition, which is that the ring R is regular. It was also concluded that there is no relationship between the concept of extending and pure rational extending, and examples of that were given. We have reached a relationship between our concept and other concepts, such as pure extending. A comprehensive theorem was also given to verify the equivalence of the following concepts: rationally extending, extending, pure extending, and pure rationally extending under the conditions of a nonsingular module over a regular ring.

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