

P-essential coessential submodules and P-essential coclosed submodules

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Abstract

This work studies the concept for P-essential coessential submodules and P-essentials coclosed sub-module that is over commutating ring with identity. Consider Q as S -module with $V \leq W \leq Q$, then V is known a P-essential coessential submodule in Q of W if $W/V \ll_{pe} Q/V$. On the other hand, a submodule W of Q is called a P-essential coclosed, if W has no proper P-essential coclosed submodule in Q . The main objective is to investigate the properties and characterizations of these submodules.

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1. Introduction

Let S be commutative rings with identities, Q be the S -modules (for short S -Mod). The proper's submodules (sub-mod) W for the Mod Q is called small ($W \ll Q$) if $W + E = Q$, for any sub-mod E of Q implies $E = Q$, [1]. A sub-mod W of an S -Mod Q is essential ($W \leq_e Q$), where $W \cap V \neq (0)$, for each non-zero sub-mod V of Q , [2], [3]. The sub-mod W of an S -Mod Q is called pure if $W \cap IQ = IW$ for each ideal of a ring S , [4]. Al-Thani in [5] defined P-essentials submodule, that generalization of essentials sub-mod, the submodule W of an S -Mod Q is called P-essential ($W \leq_{pe} Q$), if $W \cap E = (0)$, for each pure sub-mod E of Q implies $E = (0)$. Let $V \leq W \leq Q$, V named co-essential submodule W for Q ($V \leq_{ce} W$) where $\frac{W}{V} \ll \frac{Q}{V}$, & W named co-

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closed for Q and write $(W \leq_{cc} Q)$, where $\frac{W}{V} \ll \frac{Q}{V}$ implies that $V = W$. Equivalently, if W has no proper essential coessential sub-mod in Q , [6], [7]. In [8] we defined P-essential small (Pe-small), which is a generalization of small sub-mod, where the *submodule* W for the S -Mod Q named Pe-small, so write $(W \ll_{pe} Q)$, if $W + L = Q$, for each pure sub-mod E of Q implies $E = Q$. In the section 2 we introduce the concept a P-essential coessential sub-mod, such as a sub-mod V is known as P-essentials coessentials *submodule* for W of Q shortly (Pe-coessentials) write $(V \leq_{pe-ce} W)$, if $\frac{W}{V} \ll_{pe} \frac{Q}{V}$. Finally, the third section studies the concept a P-essential coclosed, as a beodening of coclosed sub-mod, if *submodule* W for the S -Mod Q named P-essential coclosed (shortly Pe-coclosed) and denoted by $(W \leq_{pe-cc} Q)$, if $\frac{W}{V} \ll_{pe} \frac{Q}{V}$ inferred $W = V$, where $V \leq_{pe-ce} W$.

2. P-Essential Coessential Submodules

Definition 2.1: Let Q be an S -Mod, with $V \leq W \leq Q$. Then V is named P-essentials coessentials *submodule* for the W of Q shortly (Pe-coessentials) and write $\leq_{pe-ce} W$, if $\frac{W}{V} \ll_{pe} \frac{Q}{V}$.

Remark & Example 2.2:

1. Evidently, all coessential *submodule* be the Pe-coessential, because all small sub-mod is Pe-small, as in [8], but the opposite in general is untrue. As an instance: In Z -module Z_6 , $\frac{\{\bar{0}, \bar{3}\}}{\{\bar{0}\}} \ll_{pe} \frac{Z_6}{\{\bar{0}\}}$ and $\{\bar{0}\} \leq_{pe-ce} \{\bar{0}, \bar{3}\}$, but $\{\bar{0}, \bar{3}\}$ is not small in Z_6 , since $\{\bar{0}, \bar{3}\} + \{\bar{0}, \bar{2}, \bar{4}\} = Z_6$ but $\{\bar{0}, \bar{2}, \bar{4}\} \neq Z_6$, hence $\{\bar{0}\} \not\leq_{ce} \{\bar{0}, \bar{3}\}$ in Z_6 .
2. In the Z -module Z , the sub-mod $4Z$ is Pe-coessential of $2Z$, since $\frac{2Z}{4Z} \cong \{\bar{0}, \bar{2}\} \ll_{pe} Z_4 \cong \frac{Z}{4Z}$, see [8].
3. Let Q be an S -Mod and $W \leq Q$. Then $W \ll_{pe} Q$ iff $\{\bar{0}\} \leq_{pe-ce} W$ in Q .
4. If Q is semi simple module, then $\{\bar{0}\}$ is Pe-coessential of any sub-mod in Q [9].

Proposition 2.3: Allow W , V and E are sub-mods of an S -Mod Q with $W \leq V \leq E \leq Q$. Thus $V \leq_{pe-ce} E$ iff $\frac{V}{W} \leq_{pe-ce} \frac{E}{W}$ in $\frac{Q}{W}$.

Proof: $\Rightarrow$) To prove $\frac{V}{W}$ is Pe-coessential of $\frac{E}{W}$ in $\frac{Q}{W}$. Since $V \leq_{pe-ce} E$, then $\frac{E}{V} \ll_{pe} \frac{Q}{V}$. Next by the third isomorphism theorem $\frac{E}{V} \cong \frac{\frac{E}{W}}{\frac{V}{W}}$ and $\frac{Q}{V} \cong \frac{\frac{Q}{W}}{\frac{V}{W}}$, hence $\frac{\frac{E}{W}}{\frac{V}{W}} \ll_{pe} \frac{\frac{Q}{W}}{\frac{V}{W}}$. Thus $\frac{V}{W} \leq_{pe-ce} \frac{E}{W}$ in $\frac{Q}{W}$.

$\Leftarrow$) Assume $\frac{V}{W} \leq_{Pe-ce} \frac{E}{W}$ in $\frac{Q}{W}$, so from third isomorphisms theorems $\frac{E}{V} \cong \frac{\frac{E}{W}}{\frac{V}{W}}$ and $\frac{Q}{V} \cong \frac{\frac{Q}{W}}{\frac{V}{W}}$, hence $\frac{\frac{E}{W}}{\frac{V}{W}} \ll_{Pe} \frac{\frac{Q}{W}}{\frac{V}{W}}$. hence $\frac{E}{V} \ll_{Pe} \frac{Q}{V}$, thus $V \leq_{Pe-ce} E$ for the Q .

Corollary 2.4: Let W, V, E are sub-mods of an S -Mod Q such that $W \leq V \leq E \leq Q$. If $V \leq_{Pe-ce} E$, thus $\frac{V}{W \cap V} \leq_{Pe-ce} \frac{E}{W \cap V}$.

Proof: Since $W \cap V \leq V \leq E$ in Q and $V \leq_{Pe-ce} E$, then by (prop 2.3) $\frac{V}{W \cap V} \leq_{Pe-ce} \frac{E}{W \cap V}$.

Corollary 2.5: Let W, V and E are sub-mods of an S -Mod Q with $W \leq V \leq E \leq Q$. If $W + V \leq_{Pe-ce} E$, then $\frac{W+V}{W} \leq_{Pe-ce} \frac{E}{W}$.

Proof: Since $W \leq W + V \leq E$ and $+V \leq_{Pe-ce} E$, so from (2.3) $\frac{W+V}{W} \leq_{Pe-ce} \frac{E}{W}$.

Proposition 2.6: Each nonzero isomorphic images for the Pe-coessential is Pe-coessential.

Proof: Is $f: Q \rightarrow Q'$ the isomorphism W, V the sub-mods of an S -Mod Q and $W \leq V \leq Q$. If $W \leq_{Pe-ce} V$ in Q , implies that $\frac{V}{W} \ll_{Pe} \frac{Q}{W}$, but f is an isomorphism, then by [8] $f\left(\frac{V}{W}\right) \ll_{Pe} f\left(\frac{Q}{W}\right)$, hence $\frac{f(V)}{f(W)} \ll_{Pe} \frac{f(Q)}{f(W)}$, thus $\frac{f(V)}{f(W)} \ll_{Pe} \frac{Q'}{f(W)}$. Therefore $f(W) \leq_{Pe-ce} f(V)$ in $(Q) = Q'$.

Proposition 2.7: Let W, V, E be sub-mods of an S -Mod Q and $W \leq V \leq Q$ with $W \leq_P Q$. Suppose $V = W + E$ furthermore $E \ll_{Pe} Q$, then $W \leq_{Pe-ce} V$.

Proof: We have to prove that $\frac{V}{W} \ll_{Pe} \frac{Q}{W}$. Let $\frac{V}{W} + \frac{U}{W} = \frac{Q}{W}$ such that $\frac{U}{W} \leq_{Pe} \frac{Q}{W}$, since $W \leq_P Q$, then by [8] $U \leq_{Pe} Q$. Now, we have $= U + V = U + W + E$, then $Q = U + E$ but $E \ll_{Pe} Q$ and $U \leq_{Pe} Q$ implies that $Q = U$. Thus $\frac{V}{W} \ll_{Pe} \frac{Q}{W}$. Therefore $W \leq_{Pe-ce} V$.

Proposition 2.8: Let W, V and E are sub-mods of an S -Mod Q such that $W \leq V \leq E \leq Q$. If $\leq_{Pe-ce} E$, then $W \leq_{Pe-ce} V$ in Q .

Proof: Suppose that $W \leq_{Pe-ce} E$, then $\frac{E}{W} \ll_{Pe} \frac{Q}{W}$, since $\frac{V}{W} \leq \frac{E}{W} \leq \frac{Q}{W}$, then by [8] $\frac{V}{W} \ll_{Pe} \frac{Q}{W}$, hence $W \leq_{Pe-ce} V$ in Q .

Proposition 2.9: Consider Q be an S -Mod, and W, V, E be sub-mods of Q such that $W \leq V \leq E \leq Q$, with $V \leq_P Q$ and $E \ll_{Pe} Q$. If $\leq_{Pe-ce} E$, then $V \leq_{Pe-ce} E$ in Q .

Proof: We must to prove $\frac{E}{V} \ll_{Pe} \frac{Q}{V}$. Let $\frac{E}{V} + \frac{U}{V} = \frac{Q}{V}$ such that $\frac{U}{V} \leq_{Pe} \frac{Q}{V}$, $E + U = Q$. Since $\frac{U}{V} \leq_{Pe} \frac{Q}{V}$ and $V \leq_P Q$ then by [8] $U \leq_{Pe} Q$, but $E \ll_{Pe} Q$ hence $U = Q$, thus $\frac{U}{V} = \frac{Q}{V}$. Therefore $\frac{E}{V} \ll_{Pe} \frac{Q}{V}$, thus $V \leq_{Pe-ce} E$.

Proposition 2.10: Consider W, V, E and D be sub-mods of an S -Mod Q . Then, the following claims are interchangeable:

1. If $W \leq_{Pe-ce} W + V$ in Q , then $W \cap V \leq_{Pe-ce} V$ in Q .
2. If $W \leq_{Pe-ce} V$ in Q , then $W \cap E \leq_{Pe-ce} V \cap E$ in Q .
3. If $W \leq_{Pe-ce} V$ and $\leq_{Pe-ce} D$, then $W \cap E \leq_{Pe-ce} V \cap D$.

Proof: $1 \Rightarrow 2$) Let $\leq_{Pe-ce} V$. Since $W \leq V$ and $\cap E \leq V$, then $W + (V \cap E) \leq V$. Since $W \leq W + (V \cap E)$ in Q , then by proposition (2.7) $W \leq_{Pe-ce} W + (V \cap E)$ and by (1) $W \cap (V \cap E) \leq_{Pe-ce} V \cap E$ but $W \cap V \leq W$, hence $W \cap E \leq_{Pe-ce} V \cap E$.

$2 \Rightarrow 3$) Suppose that $W \leq_{Pe-ce} V$ and $E \leq_{Pe-ce} D$. Since $E \leq Q$, then by (2) $W \cap E \leq_{Pe-ce} V \cap E$. Also $E \leq_{Pe-ce} D$ and $\leq Q$, then by (2) $V \cap E \leq_{Pe-ce} V \cap D$. Then by (prop2.8) $W \cap E \leq_{Pe-ce} V \cap D$ in Q .

$3 \Rightarrow 1$) Suppose that $\leq_{Pe-ce} W + V$. Since $V \leq_{Pe-ce} V$ in Q , then by (3) $W \cap V \leq_{Pe-ce} (W + V) \cap V$. Thus $W \cap V \leq_{Pe-ce} V$.

P-Essential Coclosed Submodules.

Definition 3.1: Let Q be an S -Mod. A sub-mod W of Q is named P-essential coclosed sub-mod of Q for short (Pe-coclosed), if $\frac{W}{V} \ll_{Pe} \frac{Q}{V}$, ($V \leq_{Pe-ce} W$) implies that $W = V$, for each $V \leq W$ and denoted by $(W \leq_{Pe-cc} Q)$.

Equivalently, a sub-mod W of an S -Mod Q is named P-essential coclosed sub-mod of Q , if W has no proper P-essential coessential sub-mod in Q .

Remark & Example 3.2

1. Since every small submodule be Pe-small, so each Pe-coclosed is coclosed.

Proof: Let $\frac{W}{V} \ll \frac{Q}{V}$, then by [8] $\frac{W}{V} \ll_{Pe} \frac{Q}{V}$. But W is Pe-coclosed, hence $W = V$. Thus W is coclosed.

2. The opposite of 1 in generally, is untrue. As instance: the sub-mod $\{\bar{0}, \bar{2}, \bar{4}\}$ in Z_6 as Z - module is coclosed, but not Pe-coclosed, since $\{\bar{0}\} \leq_{Pe-ce} \{\bar{0}, \bar{2}, \bar{4}\}$ and $\{\bar{0}\} \neq \{\bar{0}, \bar{2}, \bar{4}\}$.
3. The semi simple module is not Pe-coclosed, see (2).

4. The sub-mod $\{\bar{0}, \bar{2}\}$ in Z_4 as Z -module is not Pe-coclosed of Z_4 , since $\{\bar{0}\}$ is Pe-coessential of $\{\bar{0}, \bar{2}\}$ in Z_4 , so $\frac{\{\bar{0}, \bar{2}\}}{\{\bar{0}\}} \ll_{Pe} \frac{Z_4}{\{\bar{0}\}}$ but $\{\bar{0}\} \neq \{\bar{0}, \bar{2}\}$.
5. Each simple modules not Pe-co - closed.
6. for Z -modules Z , the sub-mod $2Z$ is not Pe-coclosed sub-mod of Z , because $4Z \leq_{Pe-cc} 2Z$, but $4Z \neq 2Z$, see Remarks and Examples (2.2, 2).

Proposition 3.3: Allow W is nonzero submodule for the S -Mod Q , where $W \leq_{Pe-cc} Q$, so W not Pe-small for the Q .

Proof: Suppose $0 \neq W \ll_{Pe} Q$ furthermore $W \leq_{Pe-cc} Q$. Since $\{\bar{0}\} \leq W$, so $W \cong \frac{W}{\{\bar{0}\}} \ll_{Pe} \frac{Q}{\{\bar{0}\}} \cong M$, but $W \leq_{Pe-cc} Q$ implies that $W = \{\bar{0}\}$ which is contradiction. Therefore W is not Pe-small in Q .

Proposition 3.4: Allow W, V is the nonzero submodule for S -Mod Q , with $W \leq V \leq Q$. If $\leq_{Pe-cc} Q$, therefore $\frac{V}{W} \leq_{Pe-cc} \frac{Q}{W}$.

Proof: Assume that $V \leq_{Pe-cc} Q$, and let $\frac{E}{W} \leq \frac{Q}{W}$ such that $\frac{\frac{V}{E}}{\frac{W}{E}} \ll_{Pe} \frac{\frac{Q}{E}}{\frac{W}{E}}$, then by

[1] $\frac{V}{E} \cong \frac{\frac{V}{E}}{\frac{W}{E}}$ and $\frac{Q}{E} \cong \frac{\frac{Q}{E}}{\frac{W}{E}}$, hence $\frac{V}{E} \ll_{Pe} \frac{Q}{E}$. but $V \leq_{Pe-cc} Q$, then $V = E$ and $\frac{V}{W} = \frac{E}{W}$. Therefore $\frac{V}{W} \leq_{Pe-cc} \frac{Q}{W}$.

Proposition 3.5: Allow Q be an S -Mod, and W, V be sub-mods of Q such that $W \leq V \leq Q$. Suppose $\leq_{Pe-cc} Q$, then $W \leq_{Pe-cc} V$.

Proof: Let $E \leq W$ such that $\frac{W}{E} \ll_{Pe} \frac{V}{E} \leq \frac{Q}{E}$, then by [8] $\frac{W}{E} \ll_{Pe} \frac{Q}{E}$. Since $\leq_{Pe-cc} Q$, hence $W = E$. Therefore $W \leq_{Pe-cc} V$.

Proposition 3.6: Allow $Q = Q_1 \oplus Q_2$ be an S -Mod, and $\leq_{Pe-cc} Q_1$, then $W \leq_{Pe-cc} Q$.

Proof: Let $E \leq W$ such that $\frac{W}{E} \ll_{Pe} \frac{Q}{E} = \frac{Q_1 \oplus Q_2}{E}$. Then $\frac{W}{E} \ll_{Pe} \frac{Q_1}{E} \oplus \frac{Q_2 + E}{E}$, then by [8] $\frac{W}{E} \ll_{Pe} \frac{Q_1}{E}$. Since $W \leq_{Pe-cc} Q_1$ implies $E = W$. Therefore $W \leq_{Pe-cc} Q$.

Proposition 3.7: Allow Q be an S -Mod and $W \leq_{Pe-cc} Q$. If $V \ll_{Pe} Q$, therefore $V \ll_{Pe} W$.

Proof: Let $V + E = W$ such that $E \leq_{Pe} W$. To prove $E = W$. Claim: $\frac{W}{E} \ll_{Pe} \frac{Q}{E}$. To prove that, assume $\frac{W}{E} + \frac{D}{E} = \frac{Q}{E}$ such that $\frac{D}{E} \leq_{Pe} \frac{Q}{E}$. Then $W + D = Q$ and $D \leq_{Pe} Q$ by [8]. But $W = V + E$, then $V + E + D = Q$, hence $V + D = Q$,

since $V \ll_{Pe} Q$ then $D = Q$ and then $\frac{D}{E} = \frac{Q}{E}$. Therefore $\frac{W}{E} \ll_{Pe} \frac{Q}{E}$, but $W \leq_{Pe-cc} Q$ implies that $W = E$. Thus $V \ll_{Pe} W$.

Proposition 3.8: Each of Pe-coclosed submodule for Pe-hollow modules is Pe-hollow.

Proof: If Q be the Pe-hollow S -Mod, & W is Pe-coclosed for the Q . Consider E is proper sub-mod in W with D is a P-essential sub-mod in W such that $W = E + D$. Since Q is Pe-hollow module, then by [8] $\frac{Q}{D}$ is Pe-hollow. Now, if $\frac{W}{D}$ is a proper sub-mod of $\frac{Q}{D}$, then $\frac{W}{D} \ll_{Pe} \frac{Q}{D}$ since W is Pe-coclosed in Q , thus $W = D$ and E is Pe-small sub-mod of W . Therefore W is Pe-hollow.

Proposition 3.9: Allow Q be an S -Mod and let $W \leq V \leq Q$. Suppose $W \ll V$ and $\frac{V}{W} \leq_{Pe-cc} \frac{Q}{W}$, thus $V \leq_{Pe-cc} Q$.

Proof: Let $E \leq V$ such that $E \leq_{Pe-ce} V$ (i.e. $\frac{V}{E} \ll_{Pe} \frac{Q}{E}$). Let $\pi: Q \rightarrow \frac{Q}{V}$ be the natural epimorphism, since $E \leq_{Pe-ce} V$, then $\pi(E) \leq_{Pe-ce} \pi(V)$, so by [8] $\frac{E+W}{W} \leq_{Pe-ce} \frac{V}{W}$, implies that $\frac{\frac{V}{W}}{\frac{E+W}{W}} \ll_{Pe} \frac{\frac{Q}{W}}{\frac{E+W}{W}}$, since $\frac{V}{W} \leq_{Pe-cc} \frac{Q}{W}$ implies $\frac{V}{W} = \frac{E+W}{W}$, then $V = E + W$, but $W \ll V$ we get $E = V$. Therefore $V \leq_{Pe-cc} Q$.

Conclusion

The concepts P-essentials coessentials and P-essentials coclosed submodule, are generalizations for the coessentials submodule. In this paper we give many properties and results.

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