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On the diameter of INZD-graph

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Abstract

Let R be a ring and let J be a proper ideal of R. In this work, we focus on the ideal-based non-zero divisor graph $\emptyset_J(R)$, whose vertices consist of the elements lying outside J, while adjacency is determined relative to this fixed ideal. main interest here is to understand how distances behave in this setting, especially when the graph is assumed to be connected. show that once $\emptyset_J(R)$ is connected and contains at least five vertices, the distance between any two vertices does not exceed two, apart from a few exceptional situations that are treated separately. In this way, the classical diameter bounds known for zero-divisor and non-zero divisor graphs naturally extend to the ideal-based context, while at the same time revealing how the presence of an ideal can subtly reshape the overall structure of the graph.

Subject Classification: *05C25, 05C12, 05C05, 05C40.*

Keywords: *ZD-graph, Diameter, Trees, Connectivity.*

1. Introduction

Graph-theoretic techniques provide a natural combinatorial perspective for studying algebraic structures. One of the first graph constructions connecting ring theory with graph theory is the zero-divisor graph (ZD-graph). In this graph, nodes represent the non-regular elements of R, and an arc is established among separate nodes whenever their multiplication vanishes [1]. The origin of this idea can be traced back to Beck's work on the coloring of commutative rings [2], and it was subsequently given a rigorous and systematic treatment by authors [3]. After the zero-divisor graph was formally introduced, i found that researchers began exploring it from a variety of structural perspectives. Over

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time, substantial attention was given to its fundamental properties, such as connectivity, diameter, clique number, and the structure of cut-sets, as these features often reflect essential characteristics of the underlying ring [4-7] what becomes clear from this body of work is how closely certain graph invariants are tied to intrinsic algebraic properties. At the same time, further investigations examined different graph invariants and topological indices associated with zero-divisor graphs and their extensions, which helped clarify the ongoing dialogue between algebra and graph theory [8-10]. More recently, interest has shifted toward quotient-based constructions and toward spectral or energy-related parameters, uncovering additional layers of algebraic information encoded in these graphical representations [11]. In particular, domination in the zero-divisor graph of Z_n has been studied in detail, where it becomes apparent that the arithmetic structure of n plays a decisive role in shaping minimal dominating sets and their configurations [12]. Seeking a more refined notion of adjacency that incorporates the influence of ideals, Redmond [7] introduced a variation of the zero-divisor graph relative to a fixed ideal I of a commutative ring. In this setting, adjacency depends explicitly on the chosen ideal, which opened the door to several further developments and adaptations. From a different perspective, the non-zero divisor graph (NZD-graph) was later proposed to focus on elements that are not zero divisors, thereby highlighting structural features that do not appear in the classical construction [13-15]. Within this setting, the constructed graph establishes a meaningful bridge between the algebraic behavior of the ring and the structural influence of its ideals. Building upon these advances, this study concentrates on the ideal-based non-zero divisor graph (INZD-graph) and explores its global distance-related characteristics. In particular, we study conditions ensuring connectivity and establish sharp bounds for the diameter of the graph. Our main result shows that if the $\boldsymbol{INZD}$-graph is connected and has at least five vertices, then its diameter is at most two, apart from a small number of exceptional cases. These findings contribute to a clearer understanding of how ideals influence the overall structure of the associated graphs and provide a foundation for further research in this area. The paper is organized as follows. In Section 2, we recall the basic definitions and establish preliminary observations. Section 3 is devoted to the main diameter result under connectivity assumptions. Several illustrative examples are provided to clarify the behavior of the graph for different ideals. Finally, we conclude with a summary of the main findings and possible directions for further investigation.

2. Preliminaries and Notation

Throughout this paper, R denotes a ring and I a proper ideal of R. The vertex set of the ideal-based non-zero divisor graph $\emptyset_I(R)$ consists of elements in $R \setminus \{I, 1, -1\}$. Two distinct vertices x and y are said to be adjacent whenever they are not $I - orthogonal$. The concept of ideal-based graphs was introduced in the context of zero-divisor graphs by authors [7]. The present construction adapts this idea to the non-zero divisor setting.

Consider R as a ring that contains the proper ideal J, then $x, y \in R$ are $J-orthogonal$ if xy and yx lies in J. By utilizing this concept, we formulate of $INZD$ −graph definition:

Definition 2.1: The $INZD$ −graph is a simple graph recognized by $\emptyset_J(R)$ possesses a node set $V\left(\emptyset_J(R)\right)$ that is taken to be $\{x \in R \,|x \notin \{J, \mp 1\}$. Moreover, vertices $x \neq y$ produce a line in $E\left(\emptyset_I(R)\right)$ when they are not I-orthogonal.

The transition from the classical non-zero divisor graph to the ideal-based setting is not merely a formal modification. In the classical construction, adjacency depends only on multiplication within the ring. In contrast, in the ideal-based framework, adjacency is determined relative to a fixed ideal, which may significantly alter the local and global structure of the graph. In particular, different choices of the ideal may produce non-isomorphic graphs even when the underlying ring is fixed. This dependence introduces new structural behaviors that do not arise in the classical setting.

Example 2.2: Let $R = <2> \times <4> = \{a = (0,0), b = (0,1), c = (0,2), d = (0,3), e = (1,0), f = (1,1), g = (1,2), h = (1,3)\}$. we consider the following ideals of R: $I_0 = \langle (0,0) \rangle$, $I_1 = \langle (0,2) \rangle$, $I_2 = \langle (1,0) \rangle$, $I_3 = \langle (0,1) \rangle$ and $I_4 = \langle (1,2) \rangle$. For each ideal I_i , the corresponding vertex set and adjacency relations of *INZD*-graph vary accordingly. Figure 1 illustrates the resulting graphs and shows how changing the ideal alters the overall structure of the graph, As shown in Figure 1.

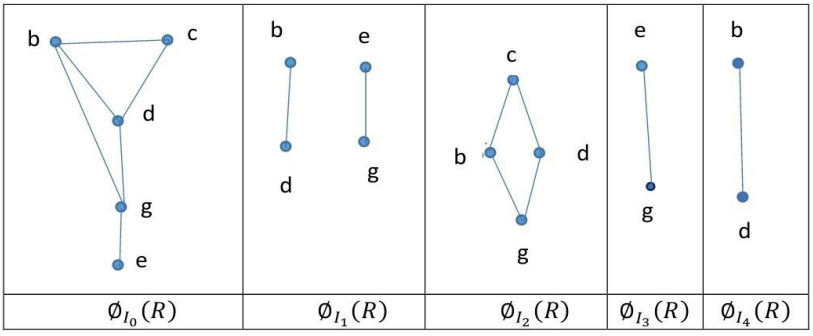


Figure 1
Graphical representation of $\emptyset_{I_i}(R)$, $i = 0, 1, 2, 3, 4$.

In the case of $\emptyset_{I_0}(R)$, the maximum distance between two vertices is three, and hence its diameter equals three, which explains why this configuration is excluded in the main theorem. For the connected graphs corresponding to I_2, I_3, and I_4, every pair of non-adjacent vertices has a common neighbor, and therefore the diameter in each case is at most two.

3. Diameter Bounds of $INZD$ −graph

This section examines the specifications for the diameter of $INZD$ −graph.

Theorem 3.1: [14] *Assume that* $\ncong < 2 > \times < 4 >$ *, then the connectedness of* NZD *−graph ensures that its maximum eccentricity is less than 3.*

Theorem 3.1 is a particular case of the following theorem, which we will demonstrate for each ideal $\subseteq R$.

Theorem 3.2: *Let* $\ncong < 2 > \times < 4 >$ *, then* $INZD$ *−graph of J, under the criteria of being connected with an order of at least 5, has maximum eccentricity less than* 3.

Proof: Consider two vertices u and v in $\emptyset_J(R)$, with $uv \in J \; and \; vu \in J$. by using the $INZD$ −graph attributes (connectivity and vertex set size more than 4), there exist $a, b \in V\left(\emptyset_J(R)\right)$ with $d(u,a) = d(v,b) = 1$. In the case $a = b$, the conclusion follows immediately. Otherwise, assume $a \neq b$, if $(b,u) = 1 \; or \; d(v,a) = 1$, then in either case $Diam\left(\emptyset_J(R)\right) = 2$. If $d(u,v) > 2$, this means that $\{ub, bu, av, va\} \subseteq J$, we will choose the element $a \pm b$. Thus, we have the following subcases:

1- If $a \pm b \neq 1$, we will prove that $a \pm b \in V\left(\emptyset_J(R)\right)$: assume not, if $a \pm b \in J$, then $u(a \pm b) = ua \pm ub \in J$, but $ub \in J$, this gives $ua \in J$, which is contradiction , by the same way we can prove that $au \in J$, which also contradiction. Next, we show that there exists a path in the graph connecting the vertices $a \pm b$, u and v. Otherwise $\{u(a \pm b), (a \pm b)u$, $v(a \pm b)$, $(a \pm b)v\} \subseteq J$, we show that one of the cases lead to a contradiction with the hypotheses, while the remaining cases can be treated analogously, since $v(a \pm b) \in J$, then $va \pm vb \in J$, but $va \in J$, then $\pm vb \in J$, contradiction. Thus, we notice that $Diam\left(\emptyset_J(R)\right) = 2$.

2- If $a + b = 1 \; and \; a - b = 1$, we get a set of relationships that will help us later in the proof, $2a = 2, 2b = 0, au + bu = u, av - bv = y$. This observation gives rives to the following subcases:

   i. $\{u^2, v^2\} \subseteq J$, take $+1$, we show that $v + 1 \in V\left(\emptyset_J(R)\right)$ and that $v + 1$ provides a connection between u and v as following: If $u(v + 1) = uv + u \in J$, but $uv \in J$, then $u \in J$, which is contradiction. If $(v + 1)v \in J$, then it follows that $v \in J$, this contradicts our initial assumption. When $+1 = u$, we have $uv + u = u^2 \in J$ and $uv \in J$, then it follows that $u \in J$, again this contrary to our assumption. If $v + 1 = -1$, then $v = -2$, and since $av \in J$, then $v \in J$, contradiction. We can conclude from all of the above partial situation that $m(\emptyset_J(R)) = 2$.

ii. If $u^2 \in J$ and $v^2 \notin J$ If, then $u \mp 1, v \mp a$ connect u with v and the above conclusion fails for the ring $Z_2 \times Z_4$. Indeed, in this we have $Diam\left(\emptyset_J(R)\right) = 3$ as showed in Example 2.2. Else, we may take $z \in V\left(\emptyset_J(R)\right)$, such that $\notin \{u, a, v, b\}$, and if $d(u,z) = d(z,v) = 1$, we done. If $d(u,z) = 1$ and $d(v,z) = 0$, take $z + b$ to connect u with v as following: If $+z = 1$, then $z = 1 - b = a$, this contradicts our initial assumption. Now when $+b = -1$, we have $z = a$, again we get a contradiction. If $u(z+b) = uz + ub \in J$, since $\in J$, we obtain $uz \in J$. And if $v(z+b) = vz + vb \in J$, since $vz \in I$, we obtain $vb \in J$, in both cases we have reached a contradiction. Therefor $Diam(\emptyset_I(R)) = 2$. Also, if $d(z,u) = 0$ and $d(v,z) = 1$,then $z + a$ to connect u with v and $Diam(\emptyset_I(R)) = 2$. Finally take $z + 1$ to connect x and y in other cases.

iii. If $u^2 \notin I and\ v^2 \in J$, take $v + 1$, we can prove that $v + 1$ belong to the vertices of graph $\emptyset_I(R)$ and $v + 1$ connected u with v as following:

Case 1: If $ChR \neq 2$, take $u + b \neq -1$ to connect u with v as following:

If $+b = 1$, then $u = 1 - b = a$, contradiction. If $+b \in I$, then $vb \in I$, contradiction. If $+b = a$, then u= $a - b = 1$, contradiction. If $+b = b$, then $u = 0$, contradiction If $u + b = y$, then $u^2 = uv - ub \in I$, contradiction. And if $+b = -1$, take $v + a \neq -1$ to connect u with y as following: If $v + a = 1$, then $v = b$, contradiction. If $+a \in I$, then $xa \in I$, contradiction. If $+a = a$, then $v = 0$, contradiction. If $+a = b$, then $v = -1$, contradiction. If $+a = b$, then $v = -1$, contradiction. If $+a = x$, then $v^2 = va - vu \in I$, contradiction. And if $+a = y$, then $a = 0$, contradiction. Now if $u + b = -1$ and $+a = -1$, then $v - u = -1$, and since $v + u = 1$, then $2v = 0$, but $ChR \neq 2$, thus $v = 0$, contradiction.

Case 2: If $hR = 2$, let $w \notin \{u, v, a, b\}$, To show that $Diam(\emptyset_I(R)) = 2$, we must examine four distinct situations. Whenever $(w,u) = d(w,v) = 1$, we are done. If $d(w,u) = 1$ and $(w,v) = 0$, then take $w + v$ to connect u with v. If $d(u,w) = 0$ and $d(v,w) = 1$, then take $w + u$ to connect u with v.And if $(u,w) = 0$ and $d(w,v) = 0$, then take $w + 1$ to connect u with v as following:

If $w + 1 = 1$, then $w = 2 = 0$, contradiction. if $+1 = u$, then $w = u$, contradiction. And if $+1 \in J$, then $uw + x \in J$, thus $u \in J$, contradiction. Therefor $Diam(\emptyset_I(R)) = 2$.

3- If $a + b = -1\ and\ a - b = -1$, then $au = -u - bu, 2b = 0, 2a = -2$ and $bv = av + v$. Thus, we may apply similar argument as in 1 to show that $Diam(\emptyset_I(R)) = 2$

4- If $a + b = -1$ and $a - b = 1$, we get $au = -u - bu, 2a = 0, 2b = -2$ and $bv = av - v$. Also, we consider the following subcases:

i. If $u^2 \in J$ and $v^2 \in J$, take $u+1$ to connect u with v. Therefor $Diam\left(\emptyset_I(R)\right) = 2$.

ii. If $u^2 \in J$ and $v^2 \notin J$, then $u \mp 1\ belong\ to\ V\left(\emptyset_J(R)\right)$ and provides a connection between u and v.

iii. If $u^2 \notin J$ and $v^2 \in J$, then $\{u \mp b, v \mp 1\} \subseteq V\left(\emptyset_J(R)\right)$ and these vertices connect u with v.this conclusion not holed when R isomorphic to $Z_2 \times Z_4$,which has diameter 3 when J is zero ideal. Otherwise, take $z \in V\left(\emptyset_J(R)\right)$ such that $z \notin \{u, a, -a, v, b\}$, such that, if $d(z,u) = d(z,v) = 1$, we done. If $d(z,u) = 1$ and $d(z,v) = 0$, take $z+b \in V\left(\emptyset_J(R)\right)$ and connect u with v.And if $d(z,u) = 1$ and $d(z,u) = 0$, take $z+b \in V\left(\emptyset_J(R)\right)$ and connect u with v. Finally take $z+1 \in V\left(\emptyset_J(R)\right)$ and connect u with v in other cases.

5- If $a+b=1\ and\ a-b=-1$, by the same way of (4). □

Corollary 3.3: *Let* $\ncong < 2 > \times < 4 >$ *, then* $INZD-graph\ of\ J$*, under the restriction of being connected with a minimum order of five, possesses the following features:*

i. Has a shared neighbor for all non-adjacent vertices.

ii. Its neither {path, cycle} graph nor tree.

Conclusion:

In this work, we investigated the global distance properties of the ideal-based non-zero divisor graph. Under the assumption of connectivity and a minimum order of five vertices, we established that the diameter is bounded above by two, with only limited exceptional cases. This result demonstrates that the introduction of an ideal does not destroy the strong connectivity behavior observed in related graph constructions, while simultaneously enriching the structural diversity of the model. These findings contribute to a deeper understanding of the interaction between algebraic properties of ideals and the geometric structure of the associated graphs.

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