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On e-primary ideals of commutative rings

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Abstract

This work studies a novel generalization of primary ideals called ȩ-primary ideals in commutative rings. We aim to investigate properties of primary ideals that extend naturally to ȩ-primary ideals. If π is the proper ideals for the commutative rings with unity H, and ȩ $\in H$ an idempotent element such that ȩ $\notin \pi$. The ideals π is named ȩ-primary ideals in the rings H if for $h, k \in H$ with $hk \in \pi$, we have that ȩ$h \in \pi$ or ȩ$k \in \mathrm{rad}(\pi)$. We settle and generalize multiple results from primary ideals to ȩ-primary ideals. Furthermore, we study

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certain characterizations of ȩ-primary ideals in the idealizations for the module $H(+)M$ and amalgamations duplications for ring $H \bowtie^{\Psi} z$.

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1. Introduction

We assume, H be the commutative rings with a nonzero unity element. Moreover, ȩ will be an idempotent element of H. The radical of an ideal $\text{п} \subseteq H$ will be denoted by $rad(\text{п})$. Remember that an ideal п for the H be termed the primary ideals for H if for all $h, k \in H$ with $hk \in \text{п}$, then $h \in \text{п}$ or $k \in rad(\text{п})$ [1]. Primary ideals have numerous applications in commutative algebra, which has drawn many researchers to investigate several notions that extend primary ideals. This includes various types of ideals such as weakly primary ideals [2], 2-absorbing primary ideal [3], weakly quasi-primary ideal, and quasi-primary ideal [4]. Recently, a new class of ideals termed ȩ-Prime ideals were introduced and investigated in [5] as an extended prime ideal concept.

This article, extended ȩ-prime ideal concept to ȩ-primary ideal notion as the novel classes for ideal in a field for generalizing primary ideals. Let п be an ideal of a commutative ring H. Let ȩ $\in H$ an idempotent element where ȩ $\notin$ п. The ideal п is ȩ-primary in the ring H where $h, k \in H$ with $hk \in \text{п}$, we have that ȩ$h \in \text{п}$ or ȩ$k \in rad(\text{п})$. It will become clear that the class of ȩ-primary ideals is an interesting notion that builds on the notion of primary ideals. At one hand, ȩ-primary ideals enjoy many properties similar to those of primary ideals. But, they nontrivially generalize a class for primary ideal (see Examples 2.3 and 2.13). The paper is in three sections. Among the notable results of Section 2, we proved an interesting identification of ȩ-primary ideals using primary ideals (see Proposition 2.4). Moreover, we investigated ȩ-primary ideals in the context of ideal multiplication, ring extensions, direct images and preimages of ring homomorphisms (see Theorem 2.7). Also, we gave useful characterizations of ȩ-primary ideals in the localization and direct product of commutative rings (see Theorems 2.11 and 2.12). Then, in Section 3, we investigated the properties for ȩ-primary ideal.

2. *e*-Primary ideals and their properties

Definition 2.1: For п is the ideal in H, & ȩ $\in H$ is the idempotent element where ȩ $\notin$ п. The ideal п named ȩ-primary in H if for $h, k \in H$ with $hk \in \text{п}$, we have that ȩ$h \in \text{п}$ or ȩ$k \in rad(\text{п})$.

Remark 2.2: Let H be a connected ring (a ring with only 0 and 1 as idempotent elements). Then in H, an primary ideal and ȩ-primary ideal coincide. In fact, ȩ $= 1$ is the only idempotent that does not belong to any proper ideal in H and so the two notions coincide in this case.

Example 2.3: We explore an example of ҽ-primary ideal which not primary ideal. For H is ring & п be the primary ideal in H. The ideal $п \times п$ is evidently not a primary ideal of $H \times H$ but it is ҽ-primary ideal with respect to $ҽ = (0,1) \notin п \times п$. Let $(h, k), (h', k') \in H \times H$ such that $(h, k)(h', k') = (hh', kk') \in п \times п$. Then, $k \in п$ or $k' \in rad(п)$ because п is a primary ideal of H. Thus, $(0,1)\ (h, k) = (0, k) \in п \times п$ or $(0,1)\ (h', k') = (0, k') \in rad(п \times п) = rad(п) \times rad(п)$ so $п \times п$ is ҽ-primary in $H \times H$ but not primary of $H \times H$.

A next results provides a criterion for ҽ-primary ideals in term primary ideal.

Proposition 2.4: Suppose п is the ideal in H & $ҽ \in H$ is idempotent element where $ҽ \notin п$. Then, п is ҽ-primary iff $(п :_H ҽ)$ is primary.

Proof: ($\Rightarrow$) if $h, k \in H$ where $hk \in (п :_H ҽ)$. Then, $ҽhk \in п$. Hence, $ҽҽh = ҽ^2 h = ҽh \in п$ or $ҽk \in rad(п)$ since п is ҽ-primary ideal of H. Therefore, $h \in (п :_H ҽ)$ or $(ҽk)^n = ҽ^n k^n = ҽk^n \in п$ where integer $n \geq 1$. So $h \in (п :_H ҽ)$ or $k^n \in (п :_H ҽ)$ hence $h \in (п :_H ҽ)$ or $k \in rad((п :_H ҽ))$, thus $(п :_H ҽ)$ is a primary ideal of H. ($\Leftarrow$) Let $h, k \in H$ such that $hk \in п \subseteq (п :_H ҽ)$. Then, $h \in (п :_H ҽ)$ or $k \in rad((п :_H ҽ))$ since $(п :_H ҽ)$ is a primary ideal of H. It follows that $ҽh \in п$ or $k^n \in (п :_H ҽ)$ for some integer $n \geq 1$. Consequently, $ҽh \in п$ or $(ҽk)^n = ҽ^n k^n \in п$. thus $ҽh \in п$ or $ҽk \in rad(п)$, and so п is ҽ-primary ideal of H.

Corollary 2.5: *Let* п *be an ideal of a ring* H, *and let* $ҽ \in H$ *be an idempotent element where* $ҽ \notin п$. *If* $\bar{ҽ} = ҽ + п \in H/п$ *is a regular element in* $H/п$, *then* п *is ҽ-primary iff it is primary.*

The next result shows that rad(п) is an ҽ-prime ideal whenever п is ҽ-primary ideal, extending a well-known property of primary ideals.

Proposition 2.6: Let us assume п is ideal in H, & $ҽ \in H$ is the idempotent element where $ҽ \notin п$. Then, п is ҽ-primary ideal of H implies that $rad(п)$ is an ҽ-prime ideal of H.

Proof: Let $h, k \in H$ with $hk \in rad(п)$. Then, $h^n k^n = (hk)^n \in п$ for some integer $n \geq 1$. Since п is ҽ-primary ideal of H, we obtain that $ҽh^n \in п$ or $ҽk^n \in rad(п)$. Then, it becomes clear that $(ҽh)^n = ҽ^{n-1}ҽh^n \in п$ or $(ҽk)^n = ҽ^{n-1}ҽk^n \in rad(п)$, thus $ҽh \in rad(п)$ or $ҽk \in rad(rad(п)) = rad(п)$, so $rad(п)$ is an ҽ-prime ideal of H as stated.

The following result extends some of the properties of primary ideals to ҽ-primary ideals.

Theorem 2.7: *Let* ҽ, $ҽ_1$, *and* $ҽ_2$ *be idempotent elements in* H. *Then:*

1. *If* п *an* $ҽ_1$*-primary ideal in* H, *so* $пz$ *an* $ҽ_1ҽ_2$*-primary ideals in* H *for every ideal* z *of* H *where* $ҽ_2 \in z$ *but* $ҽ_1ҽ_2 \notin пz$.

2. *Let $H \subseteq L$ be an extension of rings. Then,* п *is* $\mathfrak{e}$*-primary ideal of* L *implies that* $\text{п} \cap H$ *is* $\mathfrak{e}$*-primary ideal of* H.
3. *Let* $\Psi: H \to L$ *be a ring homomorphism, and* п *an* $\Psi(\mathfrak{e})$*-primary ideal of* L. *Then* $\Psi^{-1}(\text{п})$ *is* $\mathfrak{e}$*-primary ideal of* H.
4. *Let* $\Psi: H \to L$ *is onto ring homomorphism, &* п $\mathfrak{e}$*-primary ideals of* H *s.t* $ker(\Psi) \subseteq \text{п}$. *Then* $\Psi(\text{п})$ *is an* $\Psi(\mathfrak{e})$*-primary ideals in L.*

Corollary 2.8: *If* H *is the ring,* $z \subseteq \text{п}$ *be proper ideals of* H, *and* $\mathfrak{e} \in H$ *an idempotent element. Then,* п *is* $\mathfrak{e}$*-primary in* H *iff* $\text{п}/z$ *is an* $\bar{\mathfrak{e}}$*-primary in* H/z, *for* $\bar{\mathfrak{e}} = \mathfrak{e} + z \in H/z$.

The ideal-wise definition of $\mathfrak{e}$-primary ideals can be obtained via the following result.

Theorem 2.9: *If* п *is ideal in* H, *&* $\mathfrak{e} \in H$ *is idempotent element where* $\mathfrak{e} \notin \text{п}$. *Then,* п *is* $\mathfrak{e}$*-primary ideal iff* P, Q *ideals in*H *with* $PQ \subseteq \text{п}$, *we have that* $\mathfrak{e}P \subseteq \text{п}$ *or* $\mathfrak{e}Q \subseteq rad(\text{п})$.

An induction argument gives the following more general characterization of $\mathfrak{e}$-primary ideals.

Corollary 2.10: *If* п *is ideal in* H, *and let* $\mathfrak{e} \in H$ *be an idempotent element where* $\mathfrak{e} \notin \text{п}$. *Then,* п *is* $\mathfrak{e}$*-primary iff* $P_1, \dots, P_n$ *are ideals of* H *with* $P_1 \cdots P_n \subseteq \text{п}$, *we have that* $\mathfrak{e}P_i \subseteq \text{п}$ *or* $\mathfrak{e}P_j \subseteq rad(\text{п})$ *for some* $i, j \in [1{:}\,n]$.

The next result gives an interesting equivalence between $\mathfrak{e}$-primary ideals and their localization with respect to multiplicative subsets.

Theorem 2.11: *Let* H *be a ring, and let* S *be a multiplicative subset of* H. *Let* п *be an ideal of* H *such that* $\text{п} \cap S = \emptyset$ *and* $\mathfrak{e} \in H$ *be an idempotent element where* $\frac{\mathfrak{e}}{1} \notin S^{-1}\text{п}$. *Then,* п *is* $\mathfrak{e}$*-primary iff* $S^{-1}\text{п}$ *an* $\frac{\mathfrak{e}}{1}$*-primary in* $S^{-1}H$.

Proof: ($\Rightarrow$) We easily see that $\frac{\mathfrak{e}}{1}$ is an idempotent element of $S^{-1}H$. Let $\frac{h}{s_1}, \frac{k}{s_2} \in S^{-1}H$ such that $\frac{h}{s_1} \cdot \frac{k}{s_2} = \frac{hk}{s_1 s_2} \in S^{-1}\text{п}$. Then, there is some $t \in S$ such that $thk \in \text{п}$. Note that $t\mathfrak{e} \notin rad(\text{п})$ for all $t \in S$ since otherwise $\frac{\mathfrak{e}}{1} \in S^{-1}\text{п}$ which is absurd. Since п is $\mathfrak{e}$-primary ideal of H and $t\mathfrak{e} \notin rad(\text{п})$, we conclude that $\mathfrak{e}hk \in \text{п}$, and thus $\mathfrak{e}h = \mathfrak{e}^2 h = \mathfrak{e}\mathfrak{e}h \in \text{п}$ or $\mathfrak{e}k \in rad(\text{п})$. Consequently, $\frac{\mathfrak{e}}{1} \cdot \frac{h}{s_1} = \frac{\mathfrak{e}h}{s_1} \in S^{-1}\text{п}$ or $\frac{\mathfrak{e}}{1} \cdot \frac{k}{s_2} = \frac{\mathfrak{e}k}{s_2} \in S^{-1}rad(\text{п}) = rad(S^{-1}\text{п})$, hence $S^{-1}\text{п}$ is an $\frac{\mathfrak{e}}{1}$-primary ideal of $S^{-1}H$. ($\Leftarrow$) This follows from Theorem 2.7(3) using the map $\Psi: H \to S^{-1}H$ defined by $\Psi(h) = \frac{h}{1}$.

We have the following result about $\mathfrak{e}$-primary ideals in direct products of rings.

Theorem 2.12: *Let H_i be a ring, and π_i be an ideal of H_i. Let $e_i \in H_i$ be an idempotent element for $i = 1, 2$. Then, the ideal $\pi_1 \times \pi_2$ is (e_1, e_2)-primary ideal of $H_1 \times H_2$ if and only if $e_1 \in \pi_1$ and π_2 is an e_2-primary ideal of H_2 or $e_2 \in \pi_2$ and π_1 is an e_1-primary ideal of H_1.*

Proof: $(\Rightarrow)$ Evidently, (e_1, e_2) is an idempotent element of $H_1 \times H_2$. Moreover, $(1,0)(0,1) \in \pi_1 \times \pi_2$ and so $(e_1, e_2)(1,0) \in \pi_1 \times \pi_2$ or $(e_1, e_2)(0,1) \in rad(\pi_1 \times \pi_2)$ since $\pi_1 \times \pi_2$ is an (e_1, e_2)-primary ideal of $H_1 \times H_2$. Therefore, $e_1 \in \pi_1$ or $e_2 \in \pi_2$ (since $e_2^n = e_2$ for all integers $n \geq 1$). Without loss of generality, suppose that $e_1 \in \pi_1$. Since $(e_1, e_2) \notin \pi_1 \times \pi_2$ by assumption, we conclude that $e_2 \notin \pi_2$. Let $h, k \in H_2$ such that $hk \in \pi_2$. It follows that $(0, h)(0, k) \in \pi_1 \times \pi_2$ and hence we have $(e_1, e_2)(0, h) = (0, e_2 h) \in \pi_1 \times \pi_2$ or $(e_1, e_2)(0, k) = (0, e_2 k) \in rad(\pi_1 \times \pi_2) = rad(\pi_1) \times rad(\pi_2)$. It follows that $e_2 h \in \pi_2$ or $e_2 k \in rad(\pi_2)$ and so π_2 is an e_2-primary ideal of H_2. $(\Leftarrow)$ Suppose that $e_1 \in \pi_1$ and π_2 is an e_2-primary ideal of H_2. In particular, $e_2 \notin \pi_2$ and so $(e_1, e_2) \notin \pi_1 \times \pi_2$. Let $(h, k), (h', k') \in H_1 \times H_2$ such that $(h, k)(h', k') = (hh', kk') \in \pi_1 \times \pi_2$. Then, $kk' \in \pi_2$ and so $e_2 k \in \pi_2$ or $e_2 k' \in rad(\pi_2)$ since π_2 is an e_2-primary ideal of H_2. so $(e_1, e_2)(h, k) = (e_1 h, e_2 k) \in \pi_1 \times \pi_2$ or $(e_1, e_2)(h', k') = (e_1 h', e_2 k') \in \pi_1 \times rad(\pi_2) \subseteq rad(\pi_1) \times rad(\pi_2) = rad(\pi_1 \times \pi_2)$, and so $\pi_1 \times \pi_2$ is an (e_1, e_2)-primary ideal of $H_1 \times H_2$. The other possibility can be handled using the exact same type of argument.

Example 2.13: We can use Theorem 2.12 to construct more examples of e-primary but not primary. For a ring $\mathbb{Z}_{30}$, the element $\overline{15}$ is an idempotent and $\overline{15} \notin (\bar{2})$ which is a primary ideal. Therefore, $(\bar{2})$ is a $\overline{15}$-primary ideal. By Theorem 2.12, we see that the ideal $(\overline{15}) \times (\bar{2})$ is an $(\overline{15}, \overline{15})$-primary ideal of the ring $\mathbb{Z}_{30} \times \mathbb{Z}_{30}$ but it is evidently not a primary ideal of $\mathbb{Z}_{30} \times \mathbb{Z}_{30}$.

3. Results on e-primary ideals in rings of the types $H(+)M$ and $H \bowtie^f \Theta$.

Proposition 3.1: If π is ideal in H, & $e \in H$ is idempotent element. Then π is e-primary in H iff $\pi(+)M$ is $(e, 0)$-primary ideal of $H(+)M$.

Proof: $(\Rightarrow)$ Firstly, $(e, 0) \notin \pi(+)M$ since we are assuming that $e \notin \pi$. Let $(h, m), (k, n) \in H(+)M$ such that $(h, m)(k, n) = (hk, hn + km) \in \pi(+)M$. Then, $hk \in \pi$ and thus $eh \in \pi$ or $ek \in rad(\pi)$ since π is e-primary ideal of H. It follows that $(e, 0)(h, m) = (eh, em) \in \pi(+)M$ or $(e, 0)(k, n) = (ek, en) \in rad(\pi)(+)M = rad(\pi(+)M)$, and thus $\pi(+)M$ is an $(e, 0)$-primary ideal of $H(+)M$. $(\Leftarrow)$ We see that $e \notin \pi$ for $(e, 0) \notin \pi(+)M$ by assumption. Let $h, k \in H$ such that $hk \in \pi$. Consequently, $(h, 0)(k, 0) = (hk, 0) \in \pi(+)M$ and hence$(e, 0)(h, 0) = (eh, 0) \in \pi(+)M$ or $(e, 0)(k, 0) = (ek, 0) \in rad(\pi)(+)M$ $= rad(\pi(+)M)$ for $\pi(+)M$ is an $(e, 0)$-primary ideal of $H(+)M$. This shows that $eh \in \pi$ or $ek \in rad(\pi)$ and so π is e-primary ideal of H.

Let H and L be a pair of rings with unity, z an ideal of L and $\Psi: H \to L$ is the homomorphism. Let

$$H \bowtie^{\Psi} z = \{(a, \Psi(a) + j) \mid a \in H \text{ and } j \in z\}$$

It was shown in [6], [7] that $H \bowtie^{\Psi} z$ is a subring of $H \times L$ with respect to the usual component-wise operations and they were studied extensively in [6], [7]. These rings are termed amalgamations duplications in$H$ with L along z relative to Ψ. Note that

$$п \bowtie^{\Psi} z = \{(i, \Psi(i) + j) \mid i \in п\}$$

and

$$V^{\Psi} = \{(a, \Psi(a) + j) \mid a \in H, j \in z, \Psi(a) + j \in V\}$$

is ideal for $H \bowtie^{\Psi} z$ whenever п is an ideal of H and V is an ideal of $\Psi(H) + z$. Moreover, $(ę, \Psi(ę))$ is an idempotent element of $H \bowtie^{\Psi} z$ whenever ę is an idempotent element of H [8].

Theorem 3.2: *Let $H \bowtie^{\Psi} z$ be amalgamations duplications in H with L along z on Ψ. if* п *is ideal in H with* ę $\in H$ *is idempotent element. Then,* п *is ę-primary in H iff* $п \bowtie^{\Psi} z$ *an* $(ę, \Psi(ę))$*-primary in* $H \bowtie^{\Psi} z$.

Proof: ($\Rightarrow$) Note that $(ę, \Psi(ę)) \notin п \bowtie^{\Psi} z$ since we are assuming that $ę \notin п$. Let $(h, \Psi(h) + j), (k, \Psi(k) + j') \in H \bowtie^{\Psi} z$ such that $(h, \Psi(h) + j)\,(k, \Psi(k) + j') = (hk, (\Psi(h) + j)(\Psi(k) + j')) \in п \bowtie^{\Psi} z$. Consequently, we have that $hk \in п$ and thus $ęh \in п$ or $ęk \in rad(п)$ since п is ę-primary ideal of H. Therefore, we have that $(ę, \Psi(ę))(h, \Psi(h) + j) = (ęh, \Psi(ęh) + \Psi(ę)j) \in п \bowtie^{\Psi} z$ or $(ę, \Psi(ę))(k, \Psi(k) + j') = (ęk, \Psi(ęk) + \Psi(ę)j') \in rad(п) \bowtie^{\Psi} z = rad(п \bowtie^{\Psi} z)$, see [8, Lemma 8]) and therefore $п \bowtie^{\Psi} z$ is an $(ę, \Psi(ę))$-primary ideal of $H \bowtie^{\Psi} z$. ($\Leftarrow$) Firstly, $ę \notin п$ since $(ę, \Psi(ę)) \notin п \bowtie^{\Psi} z$ by assumption. Let $h, k \in H$ such that $hk \in п$. Then, $(h, \Psi(h))(k, \Psi(k)) = (hk, \Psi(hk)) \in п \bowtie^{\Psi} z$ hence $(ę, \Psi(ę))(h, \Psi(h)) = (ęh, \Psi(ęh)) \in п \bowtie^{\Psi} z$ or $(ę, \Psi(ę))(k, \Psi(k)) = (ęk, \Psi(ęk)) \in rad(п) \bowtie^{\Psi} z = rad(п \bowtie^{\Psi} z)$ since $п \bowtie^{\Psi} z$ is an $(ę, \Psi(ę))$-primary ideal of $H \bowtie^{\Psi} z$. Consequently, we see that $ęh \in п$ or $ęk \in rad(п)$ so п is ę-primary ideal in H.

Theorem 3.3: *Let $H \bowtie^{\Psi} z$ be amalgamations duplication for H & L along z on Ψ. Let V is ideal in $\Psi(H) + z$ & ę $\in H$ is idempotent element. Then V is an $\Psi(ę)$-primary in $\Psi(H) + z$ iff V^{Ψ} be $(ę, \Psi(ę))$-primary in $H \bowtie^{\Psi} z$.*

Proof: ($\Rightarrow$) We have that $(ę, \Psi(ę)) \notin V^{\Psi}$ because we're assuming $\Psi(ę) \notin V$. Let $(h, \Psi(h) + j), (k, \Psi(k) + j') \in H \bowtie^{\Psi} z$ such that $(h, \Psi(h) + j)\,(k, \Psi(k) + j') = (hk, (\Psi(h) + j)(\Psi(k) + j')) \in V^{\Psi}$. So $(\Psi(h) + j)(\Psi(k) + j') \in V$ and so $\Psi(ę)(\Psi(h) + j) = \Psi(ęh) + \Psi(ę)j \in V$ or $\Psi(ę)(\Psi(k) + j') = \Psi(ęk) + \Psi(ę)j' \in rad(V)$ since V is an $\Psi(ę)$-primary ideal of $\Psi(H) + z$. It follows that $(ę, \Psi(ę))(h, \Psi(h) + j) = (ęh, \Psi(ęh) + \Psi(ę)j) \in V^{\Psi}$ or $(ę, \Psi(ę))(k, \Psi(k) +$

$j') = (ȩk, \Psi(ȩk) + \Psi(ȩ)j') \in rad(V)^{\Psi} = rad(V^{\Psi})$, see [8, Lemma 9]. This shows that V^{Ψ} is an $(ȩ, \Psi(ȩ))$-primary ideal of $H \bowtie^{\Psi} z$. ($\Leftarrow$) Note that $\Psi(ȩ) \notin V$ for $\left(ȩ, \Psi(ȩ)\right) \notin V^{\Psi}$ by assumption. Let $\Psi(h) + j, \Psi(k) + j' \in \Psi(H) + z$ such that $(\Psi(h) + j)(\Psi(k) + j') \in V$. Therefore, we see that $(h, \Psi(h) + j)(k, \Psi(k) + j') \in V^{\Psi}$. It follows that $\left(ȩ, \Psi(ȩ)\right)(h, \Psi(h) + j) = (ȩh, \Psi(ȩh) + \Psi(ȩ)j) \in V^{\Psi}$ or $\left(ȩ, \Psi(ȩ)\right)(k, \Psi(k) + j') = (ȩk, \Psi(ȩk) + \Psi(ȩ)j') \in rad(V^{\Psi}) = rad(V)^{\Psi}$ since V^{Ψ} is an $(ȩ, \Psi(ȩ))$-primary ideal of $H \bowtie^{\Psi} z$. Hence, we have that $\Psi(ȩ)(h, \Psi(h) + j) \in V$ or $\Psi(ȩ)(k, \Psi(k) + j') \in rad(V)$ so V is an $\Psi(ȩ)$-primary ideal of $\Psi(H) + z$.

Conclusion

This study introduced ȩ-primary ideals as a generalization of primary ideals in commutative rings and extended several classical properties to this new setting. The work also provided characterizations of ȩ-primary ideals in idealizations and amalgamated duplication structures, contributing to the development of generalized ideal theory in commutative algebra.

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