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On commutativity of prime Γ-rings

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Abstract

In the context of automorphism σ defined on a prime Γ-ring Q, we explored various results related to commutativity within Q by examining a generalized reverse σ-derivations Ħ and Ǵ of Q that satisfy any one of the properties:

(i) If Ħ(Q) ⊂ Z(Q).
(ii) If $[Ħ(r), Ǵ(s)]_\alpha = 0$ and Ħ is commuting.
(iii) If Ǵ is commuting s.t. (Ħ(r)αs + Ħ(s)αr) + (rαǴ(s) + sαǴ(r)) = 0.
(iv) If $[Ħ(r), s]_\alpha = [Ħ(s), r]_\alpha$ and Ħ is commuting. For each r, s belongs to Q and α in Γ, hence Q is commutative.

This study aims to analyze the commutativity of the primitive gamma ring (Γ-ring) by introducing the concept of generalized reverse σ-derivatives associated with the automorphism σ. Also explore the properties of two of these derivatives, Ħ and Ǵ, and investigate the conditions that guarantee Γ-ring commutativity. The main results include an analysis of the relationship between these derivatives and their influence on the Γ-ring structure, while generalizing previous results in this field. The study provides a comprehensive theoretical framework for a deeper understanding of the behavior of these Γ-rings under the influence of generalized reverse σ-derivatives, opening new research horizons in the theory of non-commutative rings.

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1. Introduction

In 1964, the notion of a Γ-ring was proposed as a more general structure than a ring [1]. Extensive studies on Γ-rings have led to several results that extend classical concepts from ring theory (see [1]–[6] and [7]). The commutativity of prime rings with generalized reverse derivations was also investigated in 2019 [3]. Furthermore, reverse derivations have been studied in various algebraic contexts on Γ-ring M (Γ-$ring_M$) was characterized as additive mapping in 2013 $d: M \to M$ in order that $d(\mathrm{x}\alpha y) = \alpha \mathrm{x} d(y) + y\alpha d(\mathrm{x})$, for every $\mathrm{x}, y \in M$ and $\alpha \in \Gamma$ [8]. In 2015, the notion of reverse derivations on a prime Γ-near ring M was used to investigate the commutativity conditions of a nonzero invariant subset U of M [1]. In 2014, the definition of generalized reverse derivations on a Γ-ring M was introduced, as a supplementary mapping $f: M \to M$ with a reverse derivation $d: M \to M$ s.t $(\mathrm{x}\alpha y) = f(y)\alpha \mathrm{x} + y\alpha d(\mathrm{x})$, $\forall\, \mathrm{x}, y \in M$, $\alpha \in \Gamma$ [6]. Then, the concepts of skew derivation and generalized skew derivation associated with an automorphism were introduced θ of Γ-$ring_M$ by an supplementary map $d, F: M \to M$ and s.t. $d(x\alpha y) = d(x)\alpha y + \theta(x)\alpha d(y)$, and $F(x\alpha y) = F(x)\alpha y + \theta(x)\alpha d(y)$, for all $x, y \in M$ and $\alpha \in \Gamma$, additionally, they demonstrated some commutativity results for prime Γ-$ring_M$ that accepts a generalized skew derivation F that satisfies certain requirements [9], the commutativity of a prime Γ-$ring_Q$ by introduce some properties of two generalized reverse σ-derivations Ħ and Ġ of Q with an automorphism σ of Q. Addition we studying the relationship between Ħ and Ġ. The study consists 3-sections. begins with an introduction that explains the theoretical background of Γ−rings and the importance of studying them. This is followed by the basics section, which presents the basic definitions and concepts used in the study. This is followed by the main results section. The research concludes with the conclusions section, which summarizes the most important findings reached [10].

2. Preliminaries

Definition 2.1: Take Q be Γ-ring and g be a supplementary mapping from Q into Q itself. g is namely the reverse σ-derivation on Q if ∀x,y belong Q, g holds $\mathrm{g}(x\alpha y) = \mathrm{g(y)}\alpha x + \sigma(y)\alpha \mathrm{g(x)}$, for every $x, y \in Q$ and $\alpha \in \Gamma$, a σ automorphism on Q used here.

Definition 2.2: Take Q to be a Γ-ring with a reverse σ-derivation g, and σ is an automorphism on Q. An additive mapping Ġ: Q → Q, namely generalized reverse σ-derivation of Q with g, if $\dot{G}(r\alpha z) = \dot{G}(z)\alpha r + \sigma(z)\alpha \mathrm{g(r)}$, for each, $\in Q$, $\alpha \in \Gamma$.

We will assume in our results (*) $r\alpha z\beta s = r\beta z\alpha s$ ∀r, z, s ∈ Q and α, β in Γ, and make advantage of the identities $[r\beta y, s]_\alpha = r\beta[z, s]_\alpha + [r, s]_\alpha \beta z$ and $[r, z\beta s]_\alpha = z\beta[r, s]_\alpha + [r, z]_\alpha \beta s$.

3. The Results

Lemma 3.1: *If G is a nonzero generalized reverse σ-derivation of a prime Γ-ring (P-Γ-ring) Q with a nonzero reverse σ-derivations g, and let t∈ Q. If t ΓG(Q)=(0) or G(Q)Γt=(0), hence t = 0.*

Proof: Suppose that tΓG(Q)= 0, for t∈Q.

Take r in Q and α in Γ, then we have tαG(r) = 0 (1)

Replacing r by rβs in (1), we get

tαG(rβs)= tαG(r)βs + tασ(r)βg(s)= 0 (2), for all r,s∈Q and α,β in Γ.

Through the use of (1) in (2), we obtain tασ(r)βg(s)= 0, and since σ is an automorphism, we obtain tα r βg(s)= 0, ∀r,s∈Q and α,β∈Γ. Since Q is prime and g≠ 0, obtain t=0. If G(Q)Γt= 0, some reasoning applies.

Theorem 3.2: *For a nonzero generalized reverse σ-derivation H of a P-Γ-ring Q with a nonzero reverse σ-derivations h, If H(Q)⊂ Z(Q), then Q is commutative.*

Proof: If H(r)∈Z(Q), r ∈ Q, for z ∈ Z and $\alpha \in \Gamma$, we have H(r) αz = z αH(r) (3) Replacing r by rβs in (3), we have H(rβs) αz = z αH(rβs), hence (H(s)βr + σ(s)βh(r)) αz = zα (H(s)βr + σ(s)βh(r)), this lead H(s)βrαz - zαH(s)βr +σ(s)βh(r)αz - zασ(s)βh(r)) = H(s)β[r,z]$_\alpha$ + σ(s) β[h(r), z]$_\alpha$ = 0, ∀r, s, z ∈Q, $\alpha, \beta \in \Gamma$ (4)

Replacing h(r) by z in (4), we have H(s) β[r, z]$_\alpha$ = 0, By Lemma (3.1), we obtain [r, z]$_\alpha$ = 0, for each r, z ∈ Q, $\alpha \in \Gamma$ Consequently, Q is commutative.

Theorem 3.3: *Let H and G be two nonzero generalized reverse σ-derivations of a P-Γ-ring Q with a nonzero reverse σ-derivations h and g respectively. If [H(r), G(s)]$_\alpha$ = 0, for each r, s ∈ Q, α ∈ Γ, and H is commuting, then Q is commutative.*

Proof: Let [H (r), G(s)]$_\alpha$ = 0, for each r, s ∈Q, $\alpha \in \Gamma$ (5).

Replacing s by rβs in (5), we have [H(r), G(rβs)]$_\alpha$ = [H(r), G(s)βr]$_\alpha$ + [H(r), σ(s)βg(r)]$_\alpha$ = 0, for each r, s ∈ Q and $\alpha, \beta \in \Gamma$ (6).

By using (5) in (6), we get

G(s)β [H(r), r]$_\alpha$ +σ(s)β [H(r), g(r)]$_\alpha$ + [H(r), σ(s)]$_\alpha$ βg(r)= 0 (7)

By hypotheses H is commuting, (7) become

σ(s)β[H(r), g(r)]$_\alpha$ + [H(r), σ(s)]$_\alpha$ βg(r)= 0, for each r, s ∈ Q and $\alpha, \beta \in \Gamma$ (8)

Replacing g(r) by G(s) in (8) and applied (5), we get [H(r), σ(s)]$_\alpha$ β G(s) = 0,

By lemma (3.1), we have [H(r), σ(s)]$_\alpha$ = 0, for each r, s ∈ Q and $\alpha \in \Gamma$ (9)

Replacing H(r) by H(r)βt in (9), and using (9) we have

H(r)β [t, σ(s)]$_\alpha$ = 0, By lemma (3.1), and since σ is automorphism, we have [t, s]$_\alpha$ = 0, for each s, t ∈ Q and $\alpha \in \Gamma$, therefore Q is commutative.

Theorem 3.4: *If Q is a prime Γ-ring , and H, G are two nonzero generalized reverse σ-derivations of Q with a nonzero reverse σ-derivations h and g respectively. If G is commuting on Q such that $(H(r)\alpha s + H(s)\alpha r) + (r\alpha G(s) + s\alpha G(r)) = 0$, for each $r, s \in Q$ and $\alpha \in \Gamma$, implies Q is commutative.*

Proof: Let $(H(r)\alpha s + H(s)\alpha r) + (r\alpha G(s) + s\alpha G(r)) = 0$, $\forall r,s \in Q$ and $\alpha \in \Gamma$ (10) Replacing r by sβr in (10), we have $(H(s\beta r)\alpha s + H(s)\alpha s\beta r) + (s\beta r\alpha G(s) + s\alpha G(s\beta r)) = (H(r)\beta s\alpha s + \sigma(r)\beta h(s)\alpha s + H(s)\alpha s\beta r) + (s\beta r\alpha G(s) + s\alpha G(r)\beta s + s\alpha\sigma(r)\beta g(s)) = 0$, for each r, s ∈Q and $\alpha, \beta \in \Gamma$ (11)

On the other hand by multiplying (1) with βs from right, we have

$$(H(r)\alpha s\beta s + H(s)\alpha r\beta s) + (r\alpha G(s)\beta s + s\alpha G(r)\beta s) = 0 \quad (12)$$

By combining (11) and (12), this implies

$$\sigma(r)\beta h(s)\alpha s + H(s)\alpha s\beta r - H(s)\alpha r\beta s + s\beta r\alpha G(s) - r\alpha G(s)\beta s + s\alpha\sigma(r)\beta g(s) = 0 \quad (13)$$

If we replacing r by s in (13), and sine G is commuting, this implies

$$\sigma(s)\beta h(s)\alpha s + s\alpha\sigma(s)\beta g(s) = 0, \text{ for each } s \in Q \text{ and } \alpha, \beta \in \Gamma \quad (14)$$

Replacing σ(s) by rδσ(s) in (14), we have,

$$r\delta\sigma(s)\beta h(s)\alpha s + s\alpha r\delta\sigma(s)\beta g(s) = 0 \quad (15)$$

Multiplying (14) with rδ from left, we have

$$r\delta\sigma(s)\beta h(s)\alpha s + r\delta s\alpha\sigma(s)\beta g(s) = 0 \quad (16)$$

Combining (15) and (16), we have $[s, r]_\alpha \delta\sigma(s)\beta g(s) = 0$, for each s, r∈ Q and $\alpha, \beta, \delta \in \Gamma$. But, σ is an automorphism, and by primeness of Q, we get $[s, r]_\alpha = 0$, for each s, r ∈ Q and $\alpha \in \Gamma$, therefore, Q is commutative.

Corollary 3.5: *Suppose that Q is a prime Γ-ring, H and G be two nonzero generalized reverse σ-derivations of Q with a nonzero reverse σ-derivations h and g respectively such that $[H(r), s]_\alpha = [H(s), r]_\alpha$, for each $r, s \in Q$ and $\alpha \in \Gamma$Hence, Q is commutative when H commutes.*

Proof: Same proof of theorem (3.4) with substitute G by H.

Theorem 3.6: *If Q be a P-Γ-ring, H and G be two nonzero generalized reverse σ-derivations of Q with reverse σ-derivations h and g respectively. If $H(s)\alpha G(r) = r\alpha\sigma(s)$, for each $r, s \in Q$ and $\alpha \in \Gamma$Hence, H commutes with g, or $g = 0$.*

Proof: Let $H(s)\alpha G(r) = r\alpha\sigma(s)$, for each r, s ∈Q and $\alpha \in \Gamma$ (17)

Replacing r by sβr in (17), we have

$H(s)\alpha G(s\beta r) = s\beta r\alpha\sigma(s)$, for each r, s ∈Q and $\alpha, \beta \in \Gamma$, this implies

$$H(s)\alpha G(r)\beta s + H(s)\alpha\,\sigma(r)\beta g(s) - s\beta r\alpha\sigma(s) = 0 \quad (18)$$

By substituting (17) in (18), we get $[r, s]_\beta \alpha\sigma(s) + H(s)\alpha\sigma(r)\beta g(s) = 0$ (19)

Replacing r by sδr in (19), we have $[s\delta r, s]_\beta \alpha\sigma(s) + H(s)\alpha\sigma(s\delta r)\beta g(s) = s\delta[r,s]_\beta \alpha\sigma(s) + H(s)\alpha\sigma(s)\delta\sigma(r)\beta g(s) = 0$, $\forall r, s \in Q$ and $\alpha, \beta, \delta \in \Gamma$ (20)

Substitute (19) in (20), we get, - sδ Ħ(s)ασ(r)βg(s)+ Ħ(s)ασ(s)δσ(r)βg(s) = 0, since σ is an automorphism, we have $[Ħ(s), s]_α$ δσ(r)βg(s)=0, for each s, r ∈ Q and $\alpha, \beta, \delta \in \Gamma$, by primness of Q, $[Ħ(s), s]_α$ =0, that's Ħ is commuting, ∀ s∈ Q and $\alpha \in \Gamma$, or g(s)= 0, ∀ s ∈Q, that's g=0.

Theorem 3.7: *Let Ħ and Ġ be two nonzero generalized reverse σ-derivations of a P-Γ-ring Q with a nonzero reverse σ-derivations h and g respectively. If Ġ is commuting on Q such that* $[Ħ(r), σ(s)]_α ± [Ġ(r), s]_α = 0$, *for each r,s* $\in Q$ *and* $\alpha \in \Gamma$, *then Ħ is commuting.*

Proof: Suppose that $[Ħ(r),σ(s)]_α$-$[Ġ(r),s]_α$=0, for all r, s∈Q and $\alpha \in \Gamma$ (21) Replacing s by rβs in (21), we have ,$[Ħ(r), σ(rβs)]_α$ - $[Ġ(r), rβs]_α$ =σ(r)β $[Ħ(r), σ(s)]_α$+ $[Ħ(r), σ(r)]_α$ βσ(s)- rβ$[Ġ(r), s]_α$- $[Ġ(r), r]_α$βs (22) Left multiplying (21) with rβ, we get, rβ $[Ħ(r), σ(s)]_α$- rβ$[Ġ(r), s]_α$= 0 (23) Subtract (22) from (23), and σ is automorphism, we have, $[Ħ(r),σ(r)]_α$ βσ(s) - $[Ġ(r), r]_α$ βs= 0 (24), Since G is commuting, we get , $[Ħ(r), σ(r)]_α$ βσ(s)= 0 Replacing σ(s) by tδσ(s), we have, $[Ħ(r), σ(r)]_α$ β t δ σ(s)= 0, since Q is a prime and σ(s) ≠ 0 , get $[Ħ(r), σ(r)]_α$ = 0, for each r∈ Q and $\alpha \in \Gamma$, and σ is automorphism, we have Ħ is commuting.

A slight modification in the proof of the theorem (3.7) yields the following.

Theorem 3.8: *Let Ħ and Ġ be two nonzero generalized reverse σ-derivations of P-Γ-ring Q with a nonzero reverse σ-derivations h and g respectively. If Ħ is commuting on Q such that* $[Ħ(s), r]_α ± [Ġ(s), σ(r)]_α = 0$, *for each r, s* $\in Q$ *and* $\alpha \in \Gamma$, *hence Ġ is commuting.*

Theorem 3.9: *Let Q be a P-Γ-ring, Ħ and Ġ be two nonzero generalized reverse σ-derivations of Q with reverse σ-derivations h and g respectively. If Ħ(σ(Q)) ± σ(Ġ(Q)) = 0, consequently, h ± g = 0.*

Proof: Suppose that Ħ(σ(Q))-σ(Ġ(Q))= 0.

For r ∈ Q, let Ħ(σ(r))-σ(Ġ(r))= 0 (25)

Replacing r by sαr in (25), hence this implies, Ħ(σ(sαr)) - σ(Ġ(sαr)) = Ħ(σ(s)α σ(r)) - σ(Ġ(r)αs + σ(r)αg(s)) = Ħ(σ(r))α σ(s))+ σ(σ(r)) α h(σ(s)) - σ(Ġ(r)) α σ(s) -σ(σ(r))α σ(g(s))= 0 (26)

Substitute (25) in (26), get, σ(σ(r))α (h(σ(s)) - σ(g(s)) = 0 (27)

σ is automorphism, we have σ(r)α(h(s)- g(s)) = 0 (28)

Replacing σ(r) by σ(s)βt in (28), we have

σ(s)β t α(h(s) - g(s)) = 0. By primness of Q,

we get h(s) - g(s) = 0, for eachh s ∈ Q. then h = g.

4. Conclusions

In this study, we examined the commutativity of a prime Γ-ring Q by present the concept of generalized reverse σ-derivations of Q, An additive mapping Ġ:Q→Q namely generalized reverse σ-derivation of Q with a reverse σ-derivation g, and an automorphism σ on Q, if $Ġ(xαy) = Ġ(y)αx +$

σ-derivation g, and an automorphism σ on Q, if $G(x\alpha y) = G(y)\alpha x + \sigma(y)\alpha g(x)$, for each $x, y \in Q$ and $\alpha \in \Gamma$. We studied some properties of Q that make it commutative via the relationship between two generalized reverse σ-derivations, denoted Ħ and G, of Q.

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