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# Mine.2. absorbing submodules

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**Abstract**

In this work, the Mine.2. Absorbing concept (for short Mi. 2. AB) over a commutative ring with a nonzero identity has been presented. This concept is considered a strong form of 2. Absorbing using the intersection between submodules. In this work, some relationships between different concepts of submodules are also studied. The sufficient conditions for finding relationships between these concepts are explained. Several examples related to the proposed new concept are discussed. The effect of this proposed concept on functions and their inverses is also explained.

***Subject Classification (2020):*** *13D07.*

***Keywords:*** *2. absorbing, Pseudo.2. absorbing, Mine.2. absorbing, Soc of submodule.*

## 1. Introduction

In this article, we assume the rings have a unique identity, and every module will be until the left $\mathrm{R}$. module. The concept 2-Absorbing submodule intr, and w element of script cap X, named 2. Absorb, and w element, script cap X, bing if $zvw \in D$ for $z, v \in R$ and $w \in \mathcal{X}$ whih means $zw \in D$ or $vw \in$ or

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$ẓv \in [D : \mathcal{X}]$ [1]. The notion of 2. absorbing was generalized by a number of authors, as evidenced by the (Nearly.2. Absorbing and Pseudo.2. Absorbing) submodules [2-3] as was done in previous studies to introduce~~d~~ the concepts Restrict Nearly prime sub-modules as the novel generalizations for the prime sub-modules and strong form the (Nearly and Approximately) prime submodules [4-9]. Fundamentals of work:

- $\mathcal{X}$ be multiplication R-modules if each sub-module $D$ for $\mathcal{X}$ have a form $J\mathcal{X}$ for some ideals $J$ for $\Longrightarrow D = [D :_R \mathcal{X}]\mathcal{X}$[9].
- $\mathcal{X}$ is a distributive R-module if $\forall$ D, w and Ɗ of $\mathcal{X}$, $D \cap (w + Ɗ) = (D \cap w) + (D \cap Ɗ)$ or $(D + w) \cap Ɗ = (D \cap Ɗ) + (w \cap Ɗ)$[10].
- Annihilator of $\mathcal{X}$, denoted by $ann_R(\mathcal{X}) = [0 :_R \mathcal{X}]$.
- $\mathcal{X}$ is a semi simple R -module if and only if $soc(\mathcal{X}) = \mathcal{X}$ [11].
- $\mathcal{X}$ is a semi simple R -module $\Longrightarrow D \subseteq soc(\mathcal{X})$, where $D \leq \mathcal{X}$ [12].
- Projective module $\Longrightarrow soc(R)\mathcal{X} = soc(\mathcal{X})$ [9].
- If $D \leq \mathcal{X}$, D be maximal & essential iff $soc(\mathcal{X}) \subseteq D$ [13].
- If $D \leq \mathcal{X} \Longrightarrow soc(D) = D \cap soc(\mathcal{X})$ [14].

The Mine.2. Absorbing submodules is named and proposed to be as a strong form of 2. Absorbing. it studied and explained with its relationship to some other types of partial modules with the influence of some properties and basics on this concept. A new definition with several examples and properties has been strengthened.

## 2. Basic Properties of Mine.2. Absorbing Submodules

The absorption submodule (Mine.2. Absorbing) has been defined. Several of its aspects and its relationship to reinforcement with examples are discussed. Also, some main points are drawn as the conclusions [15].

**Definition 2.1:** If $\leq \mathcal{X}$, D is named to be Mine.2.Absorbing (for short $Mi.2.AB$) sub-module in $\mathcal{X}$ if for any $ẓvw \in D$, for $ẓ, v \in R$, $w \in \mathcal{X}$, so either $ẓw \in D \cap soc(\mathcal{X})$ or $vw \in D \cap soc(\mathcal{X})$ or $ẓv\mathcal{X} \subseteq D \cap soc(\mathcal{X})$.

**Remarks and Examples 2.2**

- If $\mathcal{X} = Z_{60}$, R=Z, $D = <\bar{6}>$ is $Mi.2.AB$ of $\mathcal{X}$. Now, large submodules of $Z_{60}$ are $Z_{60}$ and the $<\bar{2}> \Longrightarrow soc(Z_{60}) = Z_{60} \cap <\bar{2}> = <\bar{2}>$. $\forall\, 3, 2 \in Z$, $\bar{1} \in Z_{60}$, $3.2.\bar{1} = 6 \in <\bar{6}>$, We get $3.\bar{1} = 3 \notin <\bar{6}> \cap <\bar{2}> = <\bar{6}>$ and $2.\bar{1} = 2 \notin <\bar{6}> \cap <\bar{2}> = <\bar{6}>$, but
$$3.2 \in [<\bar{6}> \cap <\bar{2}> = <\bar{6}> : Z_{60}] = 6Z.$$

- If $T \leq \mathsf{D} \leq \mathcal{X}, T$ is $Mi.2.AB$ of $\mathcal{X}$ then, $\mathsf{D}$ not necessarily $Mi.2.AB$ of $\mathcal{X}$ to clarify, $soc(Z_8) = <\bar{4}>$. If $\mathcal{X} = Z_8$ $R = Z$, $\mathsf{D} =< \bar{2} >, T =<\bar{4}>$, $< \bar{4} > \leq < \bar{2} > \leq Z_8$ since for $4,1 \in Z$, $\bar{1} \in Z_8$, $4.1.\bar{1} = 4 \in <\bar{4}>$, we get $4.\bar{1} = 4 \in < \bar{4} > \cap < \bar{4} > = < \bar{4} >$ and that's enough. And $2.1.\bar{1} = 2 \in <\bar{2}>$, We get $2.\bar{1} = 2 \notin < \bar{2} > \cap < \bar{4} > = < \bar{4} >$ and $1.\bar{1} = 1 \notin < \bar{2} > \cap < \bar{4} > = < \bar{4} >$, and $2.1 = 2 \notin [< \bar{2} > \cap < \bar{4} > = < \bar{4} >: Z_{60}] = 4Z$.
- $8Z \leq Z$, $8Z$ is not $Mi.2.AB$ of $Z$ because, $2.2.\bar{2} = 8 \in 8Z$, now $2 \in Z$, but $2.\bar{2} = 4 \notin 8Z \cap soc(Z) = 8Z \cap (0) = (0)$ and $2.\bar{2} = 4 \notin 8Z \cap (0) = (0)$ and $2.2 = 4 \notin [8Z \cap (0): Z] = [(0): Z] = (0)$.
- If $\mathsf{D}$ isnot $Mi.2.AB$ of $\mathcal{X}$, it could $[\mathsf{D} :_R \mathcal{X}]$ isnot $Mi.2.AB$. To clarify, If $\mathcal{X} = Z_{24}$, $R = Z$, $\mathsf{D} =< \bar{8} >$ since, for $2 \in Z$, $2.2.\bar{2} = 8 \in < \bar{8} >$, $soc(Z_{24}) = < \bar{4} >$ we get $2.\bar{2} = 4 \notin < \bar{8} > \cap < \bar{4} > = < \bar{8} >$ and $2.\bar{2} = 4 \notin < \bar{8} > \cap < \bar{4} > = < \bar{8} >$ and $2.2 = 4 \notin [< \bar{8} > \cap < \bar{4} > = < \bar{8} >: Z_{24}] = 8Z$. We note $< \bar{8} >$ of the Z-module $Z_{24}$ is not $Mi.2.AB$ and $[< \bar{8} >: Z_{24}] = 8Z$ is not $Mi.2.AB$ ideal of $Z$ by above part.

**Proposition 2.3:** Every $Mi.2.AB$ of $\mathcal{X}$ is $2.absorbing$ but not conversely.

***Proof:***

$\Rightarrow Mi.2.AB \Rightarrow 2.absorbing$ clear for any submodule.

$\Leftarrow$ If $\mathcal{X} = Z_{24}$, $R = Z$, $\mathsf{D} =< \bar{2} >$ obviously $\mathsf{D}$ is $2.absorbing$. For the opposite, since for $2 \in Z$, $2.1.\bar{1} = 2 \in < \bar{2} >$, $soc(Z_{24}) = < \bar{4} >$ we get $2.\bar{1} = 2 \notin < \bar{2} > \cap < \bar{4} > = < \bar{4} >$ and $1.\bar{1} = 1 \notin < \bar{2} > \cap < \bar{4} > = < \bar{4} >$ and $2.1 = 2 \notin [< \bar{2} > \cap < \bar{4} > = < \bar{4} >: Z_{24}] = 4Z$. We note $< \bar{2} >$ of the Z-module $Z_{24}$ is not $Mi.2.AB$.

**Proposition 2.4:** Every $Mi.2.AB$ of $\mathcal{X}$ is $pseudo.2.absorbing$ but not conversely.

**Proof:**

$\Rightarrow Mi.2.AB \Rightarrow 2.absorbing$ clear for any submodule.

$\Leftarrow$ If $\mathcal{X} = Z_{24}$, $R = Z$, $\mathsf{D} =< \bar{2} >$ obviously $\mathsf{D}$ is $pseudo.2.absorbing$. For the opposite, since for $2 \in Z$, $2.1.\bar{1} = 2 \in < \bar{2} >$, $soc(Z_{24}) = < \bar{4} >$ we get $2.\bar{1} = 2 \notin < \bar{2} > \cap < \bar{4} > = < \bar{4} >$ and $1.\bar{1} = 1 \notin < \bar{2} > \cap < \bar{4} > = < \bar{4} >$ and $2.1 = 2 \notin [< \bar{2} > \cap < \bar{4} > = < \bar{4} >: Z_{24}] = 4Z$. We note $< \bar{2} >$ of the Z-module $Z_{24}$ is not $Mi.2.AB$.

**Proposition 2.5:** If $\emptyset: \mathcal{X} \rightarrow \mathcal{X}'$ be an R-epimorphism and $\mathsf{D}$ is $Mi.2.AB$ of $\mathcal{X}'$ with $ker\ \emptyset \gg \mathcal{X}$. Then $\emptyset^{-1}(\mathsf{D})$ is $Mi.2.AB$ of $\mathcal{X}$.

***Proof:*** If $\mathrm{ƺvw} \in \emptyset^{-1}(\mathsf{D})$, for $\mathrm{ƺ}, \mathrm{v} \in R$, $\mathrm{w} \in \mathcal{X}$, with $\mathrm{w} \notin \emptyset^{-1}(\mathsf{D}) \cap soc(\mathcal{X})$, beyond doubt

$\emptyset(\mathrm{w}) \notin \mathsf{D} \cap \emptyset(soc(\mathcal{X})) \Rightarrow \emptyset(\mathrm{w}) \notin \mathsf{D} \cap soc(\mathcal{X}')$. $\mathrm{ƺvw} \in \emptyset^{-1}(\mathsf{D}) \Rightarrow \mathrm{ƺv}\emptyset(\mathrm{w}) \in \mathsf{D}$, but we know $\mathsf{D}$ is $Mi.2.AB$ of $\mathcal{X}'$ and from above $\emptyset(\mathrm{w}) \notin \mathsf{D}$

$\cap\, soc(\mathcal{X}') \Rightarrow \mathrm{w} \in [\mathrm{D} \cap soc(\mathcal{X}'): \mathcal{X}'] \Rightarrow \mathrm{w}\mathcal{X}' \subseteq \mathrm{D} \cap soc(\mathcal{X}') \Rightarrow \mathrm{w}\emptyset(\mathcal{X}) = \emptyset(\mathrm{w}\mathcal{X}) \subseteq \mathrm{D} \cap soc(\mathcal{X}') \Rightarrow \mathrm{w}\mathcal{X} \subseteq \emptyset^{-1}(\mathrm{D}) \cap soc(\mathcal{X})$. Therefore $\emptyset^{-1}(\mathrm{D})$ is $Mi.2.AB$ of $\mathcal{X}$.

**Proposition 2.6:** If $\mathrm{D}, \mathrm{w} \leq \mathcal{X}$ with $(\mathrm{D} + soc(\mathcal{X})) \cap \mathrm{w}$ is $Mi.2.AB$ of $\mathcal{X}$ and w is maximal essential. Then $\mathrm{D} \cap \mathrm{w}$ is an $psudo\ 2.absorbing$.

***Proof:*** If $\text{ȥ}\mathrm{v}\text{ɇ} \in \mathrm{D} \cap \mathrm{w}$, $\text{ȥ}, \mathrm{v} \in \text{Ɍ}$, $\text{ɇ} \in \mathcal{X}$. Obviously $\mathrm{D} \cap \mathrm{w} \subseteq (\mathrm{D} + soc(\mathcal{X})) \cap \mathrm{w}$. We conclude $\text{ȥ}\mathrm{v}\text{ɇ} \in (\mathrm{D} + soc(\mathcal{X})) \cap \mathrm{w}$. We know $(\mathrm{D} + soc(\mathcal{X})) \cap \mathrm{w}$ is an $Mi.2.AB$, thus $\mathrm{v}\text{ɇ} \in (\mathrm{D} + soc(\mathcal{X})) \cap \mathrm{w}$ or $\text{ȥ}\text{ɇ} \in (\mathrm{D} + soc(\mathcal{X})) \cap \mathrm{w}$ or $\text{ȥ}\mathrm{v}\mathcal{X} \subseteq (\mathrm{D} + soc(\mathcal{X})) \cap \mathrm{w}$. Now, by modular law and w is maximal essential $\Rightarrow soc(\mathcal{X}) \subseteq \mathrm{w}$, we get $(\mathrm{D} + soc(\mathcal{X})) \cap \mathrm{w} = (\mathrm{D} \cap \mathrm{w}) + soc(\mathcal{X})$ then $\mathrm{v}\text{ɇ} \in (\mathrm{D} \cap \mathrm{w}) + soc(\mathcal{X})$ or $\text{ȥ}\text{ɇ} \in (\mathrm{D} \cap \mathrm{w}) + soc(\mathcal{X})$ or $\text{ȥ}\mathrm{v}\mathcal{X} \subseteq (\mathrm{D} \cap \mathrm{w}) + soc(\mathcal{X})$. Therefore, $\mathrm{D} \cap \mathrm{w}$ is an $psudo\ 2.absorbing$ of $\mathcal{X}$.

***Proposition 2.7:*** If $\mathrm{D} \leq \mathcal{X}$ with $\mathcal{X}$ is a semisimple and D $Mi.2.AB$ of $\mathcal{X}$. Then D is an $2.absorbing$.

***Proof:*** If $\text{ȥ}\mathrm{v}\text{ɇ} \in \mathrm{D}$, $\text{ȥ}, \mathrm{v} \in \text{Ɍ}$, $\text{ɇ} \in \mathcal{X}$. We know D is an $Mi.2.AB$, thus $\mathrm{v}\text{ɇ} \in \mathrm{D} \cap soc(\mathcal{X})$ or $\text{ȥ}\text{ɇ} \in \mathrm{D} \cap soc(\mathcal{X})$ or $\text{ȥ}\mathrm{v}\mathcal{X} \subseteq \mathrm{D} \cap soc(\mathcal{X})$.Now, $\mathcal{X}$ is a semisimple $\Rightarrow \mathrm{D} \subseteq soc(\mathcal{X}) \Rightarrow \mathrm{D} \cap soc(\mathcal{X}) = \mathrm{D}$, we get $\mathrm{v}\text{ɇ} \in \mathrm{D}$ or $\text{ȥ}\text{ɇ} \in \mathrm{D}$ or $\text{ȥ}\mathrm{v}\mathcal{X} \subseteq \mathrm{D}$. Therefore, D is an $2.absorbing$ of $\mathcal{X}$. In another way $\mathcal{X}$ is a semisimple $\Rightarrow soc(\mathcal{X}) = \mathcal{X} \Rightarrow \mathrm{v}\text{ɇ} \in \mathrm{D} \cap \mathcal{X}$ or $\text{ȥ}\text{ɇ} \in \mathrm{D} \cap \mathcal{X}$ or $\text{ȥ}\mathrm{v}\mathcal{X} \subseteq \mathrm{D} \cap \mathcal{X} \Rightarrow \mathrm{v}\text{ɇ} \in \mathrm{D}$ or $\text{ȥ}\text{ɇ} \in \mathrm{D}$ or $\text{ȥ}\mathrm{v}\mathcal{X} \subseteq \mathrm{D}$. Therefore, is an $2.absorbing$ of $\mathcal{X}$.

**Proposition 2.8:** If $\mathrm{D} \leq \mathcal{X}$ with $\mathcal{X}$ is a semisimple and every non. zero element is invertible. Then D is an $Mi.2.AB$ of $\mathcal{X}$.

***Proof:*** If $\text{ȥ}\mathrm{v}\text{ɇ} \in \mathrm{D}$, $\text{ȥ}, \mathrm{v} \in \text{Ɍ}$, $\text{ɇ} \in \mathcal{X}$. Now, ȥ is invertible then, $\exists\, \text{ȥ}^{-1} \in \text{Ɍ}$ such that $\text{ȥ}^{-1}\text{ȥ}\mathrm{v}\text{ɇ} \in \text{ȥ}^{-1}\mathrm{D} \subseteq \mathrm{D} \Rightarrow 1\mathrm{v}\text{ɇ} \in \mathrm{D}$. $\mathcal{X}$ is a semisimple $\Rightarrow \mathrm{D} \subseteq soc(\mathcal{X}) \Rightarrow \mathrm{D} = \mathrm{D} \cap soc(\mathcal{X})$, we get $\mathrm{v}\text{ɇ} \in \mathrm{D} \cap soc(\mathcal{X})$. This is enough to prove D is an $Mi.2.AB$ of $\mathcal{X}$.

**Proposition 2.9:** If $\mathrm{D} \leq \mathcal{X}$, with $soc(\mathrm{D})$ is an $2.absorbing$ of $\mathcal{X}$. Then D is an $Mi.2.AB$ of $\mathcal{X}$.

***Proof:*** If $\text{ȥ}\mathrm{v}\text{ɇ} \in soc(\mathrm{D}), \text{ȥ}, \mathrm{v} \in \text{Ɍ}$, $\text{ɇ} \in \mathcal{X}$. Now, $soc(\mathrm{D})$ is $2.absorbing$ thus, $\mathrm{v}\text{ɇ} \in soc(\mathrm{D})$ or $\text{ȥ}\text{ɇ} \in soc(\mathrm{D})$ or $\text{ȥ}\mathrm{v}\mathcal{X} \subseteq soc(\mathrm{D})$ . But $\mathrm{D} \leq \mathcal{X}$ then $soc(\mathrm{D}) = \mathrm{D} \cap soc(\mathcal{X}) \Rightarrow \mathrm{v}\text{ɇ} \in \mathrm{D} \cap soc(\mathcal{X})$ or $\text{ȥ}\text{ɇ} \in \mathrm{D} \cap soc(\mathcal{X})$ or $\text{ȥ}\mathrm{v}\mathcal{X} \subseteq \mathrm{D} \cap soc(\mathcal{X})$. Therefore, D is an $Mi.2.AB$ of $\mathcal{X}$.

**Proposition 2.10:** Let D be an $Mi.2.AB$ submodule of distributive module $\mathcal{X}$ and $H \leq \mathcal{X}$ with $H \subseteq \mathrm{D}$. Then $\frac{\mathrm{D}}{H}$ is an $Mi.2.AB$ submodule of $\frac{\mathcal{X}}{H}$.

***Proof:*** If $\alpha b\ (c\ +\ H\ )\ =\ \alpha be\ +\ H \in \frac{D}{H}$ for $\alpha$ , $b\ \in$ R and $c\ +\ H\ \in \frac{\mathcal{X}}{H}$ , $c \in \mathcal{X}$ , implies that $\alpha bc\ \in D$. But $D$ is $Mi.2.AB$ submodule of $\mathcal{X}$, then $\alpha c \in D \cap soc(\mathcal{X})$ or $bc \in D \cap soc(\mathcal{X})$ or $\alpha b\mathcal{X} \subseteq D \cap soc(\mathcal{X})$. Thus, either $\alpha(\ c+\ H\ ) \in \frac{D\cap soc(\mathcal{X})}{H}$ or $b(\ c+\ H\ ) \in \frac{D\cap soc(\mathcal{X})}{H}$ or $\alpha b\frac{\mathcal{X}}{H} \subseteq \frac{D\cap soc(\mathcal{X})}{H}$ but $\mathcal{X}$ is distributive module then, $\alpha(\ c+\ H\ ) \in \frac{D}{H} \cap \frac{soc(\mathcal{X})}{H} \subseteq \frac{D}{H} \cap soc\left(\frac{\mathcal{X}}{H}\right)$ or $b(\ c+\ U\ ) \in \frac{D}{H} \cap \frac{soc(\mathcal{X})}{H} \subseteq \frac{D}{H} \cap soc\left(\frac{\mathcal{X}}{H}\right)$ or $\alpha b\frac{\mathcal{X}}{H} \subseteq\ \frac{D}{H} \cap \frac{soc(\mathcal{X})}{H} \subseteq \frac{D}{H} \cap soc\left(\frac{\mathcal{X}}{H}\right)$. Therefore $\frac{D}{H}$ is $Mi.2.AB$ submodule of $\frac{\mathcal{X}}{H}$.

**Proposition 2.11:** If $\mathcal{X}$ is projective finitely generated multiplications R.$modules$ with $\mathrm{ann}_{\mathrm{R}}(\mathcal{X}) \subseteq D$, $D$ be ideal for R. so $D$ is $Mi.2.AB$ ideal iff $D\mathcal{X}$ is $Mi.2.AB$ submodule for $\mathcal{X}$.

***Proof:*** $\Longrightarrow$ If $L, W, Z\ \leq \mathcal{X}$ with $LWZ \subseteq D\mathcal{X}$. Now, $\mathcal{X}$ is a multiplication $\Longrightarrow$ $L = I\mathcal{X}, W = J\mathcal{X}\ and\ Z = P\mathcal{X}$ for some ideal $I, Jand\ P$ of R $\Longrightarrow IJP\mathcal{X} \subseteq D\mathcal{X}$. $\mathcal{X}$ finitely generated multiplication $\Longrightarrow IJP \subseteq D + \mathrm{ann}_{\mathrm{R}}(\mathcal{X})\ \subseteq D$. From our hypothesis $D$ is $Mi.2.AB$ ideal $\Longrightarrow JP \in D \cap soc(\mathcal{X})$ or $IP \in D \cap soc(\mathcal{X})$ or $IJ\mathcal{X} \subseteq D \cap soc(\mathcal{X})$. It is certain that $JP \in D\mathcal{X} \cap soc(\mathcal{X})$ or $IP \in D\mathcal{X} \cap soc(\mathcal{X})$ or $IJ\mathcal{X} \subseteq D\mathcal{X} \cap soc(\mathcal{X})$. Thus $D\mathcal{X}$ is $Mi.2.AB$.

$\Longleftarrow$ With the same assumptions in the first direction. If $IJP \subseteq D \Longrightarrow IJ(P\mathcal{X}) \subseteq D\mathcal{X}$. From our hypothesis $D\mathcal{X}$ is $Mi.2.AB \Longrightarrow JP \in D\mathcal{X} \cap soc(\mathcal{X})$ or $IP \in D\mathcal{X} \cap soc(\mathcal{X})$ or $IJ\mathcal{X} \subseteq D\mathcal{X} \cap soc(\mathcal{X})$. We have $\mathcal{X}$ is a projective $\Longrightarrow$ $JP \in D\mathcal{X} \cap soc(\mathrm{R})$ or $IP \in D\mathcal{X} \cap soc(\mathrm{R})$ or $IJ\mathcal{X} \subseteq D\mathcal{X} \cap soc(\mathrm{R}) \Longrightarrow JP \in D \cap soc(\mathrm{R})$ or $IP \in D \cap soc(\mathrm{R})$ or $IJ\mathcal{X} \subseteq D \cap soc(\mathrm{R})$. Thus $D$ is $Mi.2.AB$.

## Conclusion

In this study, the concept of Mine.2-absorbing submodules was introduced as a stronger extension of classical 2-absorbing submodules through the use of submodule intersections. The obtained results clarified several structural relationships between Mine.2-absorbing submodules and other related classes, including pseudo and nearly 2-absorbing submodules. Moreover, a collection of propositions and examples demonstrated the behavior of this concept under homomorphisms, semisimple modules, distributive modules, and multiplication modules. The findings also revealed that certain algebraic conditions play a crucial role in preserving the Mine.2-absorbing property within different module structures. Overall, the proposed framework provides a broader perspective for studying absorbing properties in module theory and may support further developments in commutative algebra research.

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