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M-polynomial of prime graph of commutative ring Z_{p^2} , Z_{2p} and Z_{pq}

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Abstract

This study examines the Mpolynomial defined by $M(G; \alpha, \beta) = \sum_{i \leq j} m_{ij}(G)\alpha^i \beta^j$of the prime graph associated with certain Commutative rings. The M-polynomial is a tool in graph theory that encodes details about the degrees of edges in a graph. The study defines the primegraph structure of a Commutative ring Z_{p^2}, Z_{2p} and Z_{pq} and calculates its M polynomial for several specific rings. The results include detailed theorems and proofs demonstrating how the M polynomial can be derived according to the degree sequences of the vertices in the prime graph. Examples are provided for different rings to illustrate the computation method. The work contributes to the ongoing research connecting algebraic structures, particularly Commutative rings, with graph-theoretic concepts.

Subject Classification: *05C25, 13A99.*

Keywords: *Prime graph, M polynomial, Commutative ring, Graph theory.*

Introduction

Graph theory has become a potent instrument for studying and examining algebraic structures, especially rings. One prominent method is to use graphs to explore algebraic objects' properties from a graphical perspective. The authors looked at one of them, the prime graph of a ring with particular graph characteristics [1]. Authors have established a pseudo-VonNeumann regular graph of a Commutative ring [2], while other authors have defined an associated graph of a Commutative ring [3]. The M polynomial of certain graph operations and graphs linked to cycles is calculated by the author [4]. The Schultz and Modified Schultz Polynomials of Certain Cog-Special Graphs were examined by the authors [5]. M polynomial and degree-based topological indices were examined by another author [6]. In this study, we focus on computing the prime

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graphs of specific Commutative rings' M polynomial. The author examined the topological indices of prime graphs of the Commutative ring Zn and the Hosoya, Schultz, and modified Schultz polynomials [5], We concentrate on calculating the prime graphs of particular Commutative rings' M polynomial. We construct explicit equations for the M polynomials by examining the edge structures and identifying the degree sequences. The groundwork for further research in algebraic graph theory is laid by this work, which also advances our understanding of the interaction between ring theory and graph theory.

1. Basic concepts :

Definition 1.1: [7], [8] A *primegraph* of a ring R (represented by $G(R)$) is a graph with $V = R$ and $E = \{\overline{\mu\lambda} / \mu R\lambda = 0 \, or \, \lambda R\mu = 0, \& \, \mu \neq \lambda\}$.

Definition 1.2: [6] The formula for the M polynomial of a graph G is $M(G;\alpha,\beta) = \sum_{i\leq j} m_{ij}(G)\alpha^i\beta^j$, where $m_{ij}, i, j \geq 1$, is the quantity of edges uv of G such that $\{d(u), d(v)\} = \{i, j\}$.

Definition 1.3: [9], [10] Any graph is considered complete if there is an edge connecting each pair of graph vertices. The symbol for the entire graph with n graph vertices is K_n.

2. Primary Results:

2.1 *M polynomial of PG ($\boldsymbol{Z_{\omega^2}}$):*

Theorem 2.1.1: [11], [12] *For aprimegraph of a Commutative ring Z_{ω^2} where ω is prime number, Then*

$$M(PG(Z_{\omega^2});\alpha,\beta) = \omega^2 - \omega \, \alpha^{\omega^2-1}\beta + (\omega-1)\alpha^{\omega^2-1}\beta^{\omega-1} + \binom{\omega-1}{2}\alpha^{\omega-1}\beta^{\omega-1}$$

Proof: Since $PG(Z_{\omega^2})$ contain the sequence degree $\{\omega^2 - 1, \omega - 1, 1\}$ where the degree $\omega^2 - 1$ for the vertex 0 and the degree ω for the vertices set $k = \{\omega, 2\omega, \dots, \omega^2 - \omega\}$ and $|k| = \omega - 1$ that is which will form a complete graph with the vertex 0 complete graph K_ω. Now for the two degrees $\{\omega^2 - 1, 1\}$ we have edges $\omega^2 - \omega$, the two degrees $\omega^2 - 1, \omega - 1\}$ the number of edges $|k| = \omega - 1$ finally for the degree $\omega - 1$ in K_ω is $\binom{\omega-1}{2}$, Consequently, the prime graph's M Polynomial of the Commutative ring Z_{ω^2} is

$$M(PG(Z_{\omega^2});\alpha,\beta) = \omega^2 - \omega \, \alpha^{\omega^2-1}\beta + (\omega-1)\alpha^{\omega^2-1}\beta^{\omega-1} + \binom{\omega-1}{2}\alpha^{\omega-1}\beta^{\omega-1}$$

Example: For $PG(Z_{25})$ then $M(PG(Z_{25});\alpha,\beta) = 20\alpha^{24}\beta + 4\alpha^{24}\beta^4 + 6\alpha^4\beta^4$, as shown in Figure 1.

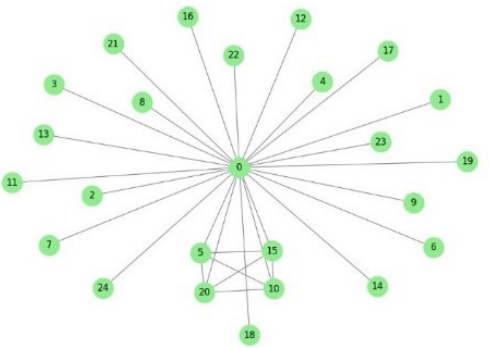


Figure 1
$\boldsymbol{PG(Z_{25})}$

2.2 *M polynomial of PG* ($\boldsymbol{Z_{2\omega}}$)*:*

Theorem 2.2.1: [13] *For a primegraph of a Commutative ring* $Z_{2\omega}$ *then* $M(PG(Z_{2\omega}); \alpha, \beta) = (\omega-1)\alpha\beta^{2\omega-1} + (\omega-1)\alpha^2\beta^{2\omega-1} + \alpha^{\omega}\beta^{2\omega-1} + (\omega-1)x^2y^{\omega}$.

Proof: Since $PG(Z_{2\omega})$ contain the sequence degree $\{\ 2\omega-1\ ,\omega, 2,1\}$ where the degree $2\omega-1$ for the vertex 0 and the degree ω for the vertex ω , the degree 2 to the vertex set $k = \{\ 2,4,\ldots,2\omega-2\}$ and $|k| = \omega-1$ finally the degree 1 for the vertices in the Units set, for the two degrees $\{\ 2\omega-1\ ,\omega\}$ we have one edge, the two-degree $\{\ 2\omega-1\ ,2\}$ the number of edges $|k| = \omega-1$, for the two degrees $\{2\omega-1,1\}$ we have $|U| = \omega-1$ of edges, finally for two degrees $\{\omega, 2\}$. The number of edges is $\omega-1$, then the prime graph's M Polynomial of the Commutative ring $Z_{2\omega}$

$$M(PG(Z_{2\omega}); \alpha, \beta) = (\omega-1)\alpha\beta^{2\omega-1} + (\omega-1)\alpha^2\beta^{2\omega-1} + \alpha^{\omega}\beta^{2\omega-1} + (\omega-1)\alpha^2\beta^{\omega}$$

Example: For $PG(Z_{10})$ then $M(PG(Z_{10}); \alpha, \beta) = 4\alpha\beta^9 + 4\alpha^2\beta^9 + \alpha^5\beta^9 + 4\alpha^2\beta^5$, as shown in Figure 2.

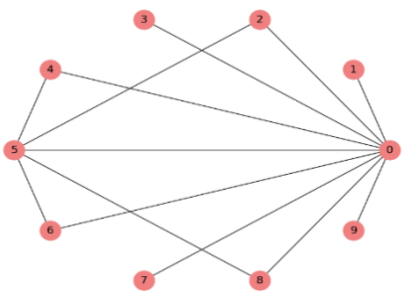


Figure 2
$\boldsymbol{PG(Z_{10})}$

2.3 *M polynomial of PG* ($\mathbf{Z}_{pq}$)*:*

Theorem 2.3.1: *For aprime graph of a Commutative ring* $Z_{\omega\varphi}$*.Then*

$$M\left(PG\left(Z_{\omega\varphi}\right);\alpha,\beta\right)=(\omega-1)(\varphi-1)\alpha\beta^{\omega\varphi-1}+(\varphi-1)\alpha^{\omega}\beta^{\omega\varphi-1}+(\omega-1)\alpha^{\varphi}\beta^{\omega\varphi-1}+(\omega-1)(\varphi-1)\alpha^{\omega}\beta^{\varphi}.$$

Proof: Since $PG\left(Z_{\omega\varphi}\right)$ contain the sequence degree $\{\ \omega\varphi-1\ ,\omega,\varphi,1$. The Table 1 below shows the number of edges between vertices for the above degrees.

Table 1

Number of edges between vertices

Degree pair (i , j)	Count	Term in M polynomial	Description
$(\mathbf{1},\boldsymbol{\omega\varphi}-\mathbf{1})$	$(\omega-1)$ $(\varphi-1)$	$(\omega-1)(\varphi-1)$ $\alpha\beta^{\omega\varphi-1}$	Edges from 0 to each unit
$(\boldsymbol{\omega},\boldsymbol{\omega\varphi}-\mathbf{1})$	$\varphi-1$	$(\varphi-1)$ $\alpha^{\omega}\beta^{\omega\varphi-1}$	Edges from 0 to each element divisible by ω only
$(\boldsymbol{\varphi},\boldsymbol{\omega\varphi}-\mathbf{1})$	$\omega-1$	$(\omega-1)$ $\alpha^{\varphi}\beta^{\omega\varphi-1}$	Edges from 0 to each element divisibleby φ only
$(\boldsymbol{\omega},\boldsymbol{\varphi})$	$(\omega-1)$ $(\varphi-1)$	$(\omega-1)(\varphi-1)$ $\alpha^{\omega}\beta^{\varphi}$	Edges between $\omega-divisible$ and $\varphi-divisible$ elements

Example: For $PG(Z_{21})$ then $M(PG(Z_{21});\alpha,\beta)=12\alpha\beta^{20}+6\alpha^{3}\beta^{20}+2\alpha^{7}\beta^{20}+12\alpha^{3}\beta^{7}$, as shown in Figure 3.

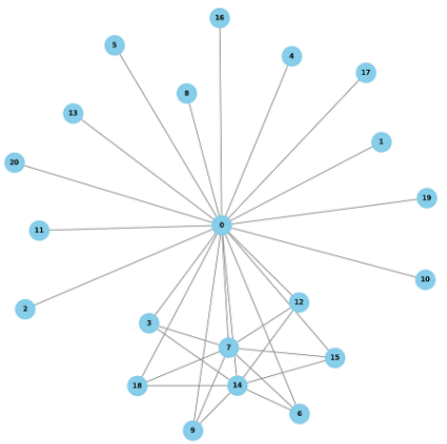


Figure 3
$\boldsymbol{PG(Z_{21})}$

3. Conclusion

In this study, we have explored the structure of prime graphs associated with Commutative rings and computed their corresponding M polynomials. +the graph's edges, offering insights into the underlying algebraic structure. Through several theorems and illustrative examples, we demonstrated how the M polynomial can be systematically derived based on the degree sequences of the vertices in the prime graph. These results not only enhance the understanding of the connection between graph theory and ring theory but also provide a basis for future investigations into topological indices and other graph-based invariants of algebraic systems.

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