

Generalization geodetic number idealization ring

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Abstract

This paper introduces m-geodetic graphs as a new concept and gives complete characterizations of perfectly m-geodetic zero-divisor graphs. Additionally, it provides

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necessary and sufficient conditions in various cases. Results of this kind are important in deepening our understanding of the structural properties of any given zero divisor graph associated with an idealization ring. numbers in these structures.

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1. Introduction

Suppose that K is a finite ring and it is commutative with identity 1_K . Also, assume that T is K -module. The concept of the idealization ring (IR) is defined as $(+)T = \{(r_1, n_1) : r_1 \in K, n_1 \in T\}$, If (r_1, n_1) & (r_2, n_2) are belong to $K(+)T$, s.t $(r_1, n_1) + (r_2, n_2) = (r_1 + r_2, n_1 + n_2)$ & $(r_1, n_1)(r_2, n_2) = (r_1 r_2, r_1 n_2 + r_2 n_1)$. Under this operation, $K(+)T$ forms a commutative ring with unity becomes the commutative rings with unit. Axtell and Stickles [1] discussed the concept of the zero-divisor graphs (ZDGs) of the idealization ring (IR). While other authors have investigated various properties of these graphs, including divisor graph structures [2-3] planarity [2], and general structural of algebras properties [3-4]. The (ZDG) of (IR) $K(+)T$ is $\Gamma(K(+)T)$ and with vertices it is the set of all non-zero divisor of $K(+)T$ two different vertices (a, b) , $(0, 0)$ and (c, d) are adjacent if $(a, b)(c, d) = (0, 0)$. We consider in this article all graph G is a simple un directed graph with vertices symbolized by $V(G)$. Let v_1 and v_2 be two vertices in the $V(G)$, the $v_1 - v_2$ geodesic is a shortest path between the two vertices v_1 and v_2 . The $IG[v_1, v_2]$ is the set of all vertices in $V(L)$ that lie on the $v_1 - v_2$ geodesic and contains v_1 and v_2 . The geodetic set of (G, S) L is any subset H of L if $I[H] = V(L)$ where $I[H] = \cup_{v_1, v_2 \in H} I_L[v_1, v_2]$, where $g(L)$ be geodetic number (G, N) for the graph L which be minimum number for the geodetic sets (G, S) of the graph L . For simplicity we use the notation Int.D to the ring is an integral domain. Moreover, several basic results and lemmas that needed in this article to study the geodetic number (G, N) are presented. The interplay between algebraic structures and graph theory has led to significant developments in both fields. (ZDGs) of rings provide a powerful visualization tool that reveals structural properties that might otherwise remain hidden in purely algebraic treatments. This approach was pioneered by author [5] and later refined by other authors [6], who established fundamental characterizations of (ZDGs). In recent years, the study of idealization rings $K(+)T$ and their associated zero-divisor graphs $\Gamma(K(+)T)$ has attracted significant attention. This paper introduces the novel concept of m-geodetic graphs, which extends traditional geodetic theory by focusing on the multiplicity of shortest paths between vertices. We provide complete characterizations of perfectly m-geodetic (ZDGs) and establish necessary and sufficient conditions for various algebraic structures. This approach offers a refined classification of (ZDGs) based on their path distribution characteristics, deepening our understanding of how algebraic

properties manifest in graph-theoretic terms. Now, $V\left(\Gamma(Z_{r_1}(+)Z_{r_2})\right) = \{(0, t): t \in Z_{r_2} \setminus \{0\}\} \cup \{(a, l): a \in Z_{r_1} \setminus \{0\}, l \in Z_{r_2}, \gcd(a, r_1) \neq 1 \text{ or } \gcd(a, r_2) \neq 1\}$.

In this paper, we define a new definition: If for any two vertices v_1, v_2 there exist exactly m distinct shortest paths between them, then a graph $\Gamma(K(+)T)$ is called m -geodetic. In network applications, multiple shortest paths between vertices provide redundancy and resistance to failures. An m -geodetic graph with $m > 1$ ensures alternative optimal routes exist between nodes. The m -geodetic property provides a refined classification of (ZDGs), revealing subtle differences in their underlying algebraic structures that might otherwise remain hidden. The perfectly m -geodetic property further refines this concept by requiring that every induced subgraph maintains the same m -geodetic character. This hereditary property has significant implications for understanding the structure of (ZDGs) at multiple scales and provides deeper insights into the algebraic structure of the corresponding rings.

2. Path Distribution Numbers for the Zero-Divisor Graph $\Gamma(Z_{r_1}(+)Z_{r_2})$

In this section, we introduce the method of path distribution number and evaluate its worth as an indicator for understanding structural features of (ZDGs) Idealization rings $Z_{r_1}(+)Z_{r_2}$. These invariant records the distribution of distances between vertices and shows local as well as global connectivity characteristics everywhere along the graph. We will begin by considering the cases where r_1 be the prime power and r_2 be prime, as these basic instances reveal critical patterns of path distribution. Further work will then be done on the general case of mutual products prime numbers. We are certain that this work will show how the algebraic structure of the idealization ring, in particular the primary decomposition of r_1 and its relationship with r_2 , determines path distribution patterns that occur in their connected (ZDGs). These results show that there is a close relationship between the arithmetic properties of the ring and the geometric properties of paths in its graph, revealing an entirely new aspect that combines algebraic and graph-theoretical structures. In the following, we define a new definition and characteristics that enhance our understanding of $\Gamma(K(+)T)$.

Definition 2.1: Let J be the set of all numbers of distinct paths between any two vertices v_1 and v_2 of the graph L . Then J is called the path distribution index set.

Definition 2.2: The maximum number of the path distribution index set in the graph L is called the path distribution number, symbolized by $\hat{r}(L)$.

Definition 2.3: The graph G named m -geodetic, where not adjacent vertices v_1, v_2 there exist exactly m distinct shortest paths between them. If the graph is a complete graph, then the graph is 1-geodetic.

Theorem 2.4: Let $\Gamma(K(+)T)$ be a (ZDG) of (IR) . Then

- (1) If K is integral domain and $ann(T) = 0$, then $\hat{r}(\Gamma(K(+)T)) = 1$.
- (2) If $r_1 = p^r$ and $r_2 = p$, then

$$J = \begin{cases} 1, & \text{if } d(v_1, v_2) = 1 \\ p\phi(p^n) + p - 1, & \text{if } d(v_1, v_2) = 2 \text{ and } v_1, v_2 \in L_i, i < \lceil \frac{r}{2} \rceil \end{cases}$$

$$\hat{r}(\Gamma(K(+)T)) = p\phi(p^n) + p - 1 \text{ and } (p\phi(p^n) + p - 1) - \text{geodetic.}$$
- (3) If $r_1 = p_1 p_2, r_2 = p_1$ and $p_1 < p_2$, then

$$J = \begin{cases} 1, & \text{if } d(v_1, v_2) = 1 \\ p_1 - 1, & \text{if } v_1 \in L_{1^*} \text{ and } v_2 \in L_{1^*}, \text{ and } \hat{r}(\Gamma(K(+)T)) = p_2 - 1 \\ p_2 - 1, & \text{if } v_1 \in L_{1^{**}} \text{ and } v_2 \in L_{1^{**}} \end{cases}$$
- (4) If $ann(T) \neq 0$, then $\hat{r}(\Gamma(K(+)T)) \leq p - 1$.

Proof: Let $\Gamma(K(+)T)$ be a (ZDG) of (IR) . Then

- (1) If K is integral domain and $ann(T) = 0$, so $\Gamma(K(+)T)$ complete graph. So, $\hat{r}(\Gamma(K(+)T)) = 1$.
- (2) If $r_1 = p^r$ and $r_2 = p$, then

Case 1: If $d(v_1, v_2) = 1$, then for any two vertices v_1 and v_2 there exists a unique path, then $J = 1$.

Case 2: If $d(v_1, v_2) = 2$ and $v_1, v_2 \in L_i, i < \lceil \frac{r}{2} \rceil$, then $v_1 = (k_1 p^{i_1}, f)$ and $v_2 = (k_2 p^{i_2}, t)$, where k_1, k_2 is a unit in $Z_{p^{r-\{i_1, i_2\}}}$, $f, t \in Z_p$ and $i_1, i_2 \in L_i$ such that $v_1 v_2 \neq (0, 0)$. Any path between these vertices goes through an intermediate vertex in L_j where $j \geq \lceil \frac{r}{2} \rceil$ for possible intermediate vertices are of the form $(k_1 p^{r-n}, t_1)$ where $n = \min\{i_1, i_2\}$, $\gcd(k_1, p^n) = 1, t_1 \in Z_p$ and all vertices of the form $(0, t)$ where $t \in Z_p$. Therefore, the number of intermediate vertices in the path between v_1 and v_2 is $J = p\phi(p^n) + p - 1$.

- (3) If $r_1 = p_1 p_2$ and $r_2 = p_1$, then

Case 1: If $d(v_1, v_2) = 1$, then $v_1 \in L_{0^*}$ and $v_2 \in L_{0^*}$, or $v_1 \in L_{0^*}$ and $v_2 \in L_{1^*}$ that is, $J = 1$.

Case 2: If $v_1 \in L_{1^*}$ and $v_2 \in L_{1^*}$, then any path between these vertices goes through an intermediate vertex in L_{0^*} . Therefore, the number of intermediate vertices in the path between v_1 and v_2 is $p_1 - 1$, that is, $J = p_1 - 1$.

Case 3: If $v_1 \in L_{1^{**}}$ and $v_2 \in L_{1^{**}}$, then any path between these vertices goes through an intermediate vertex in L_{1^*} of the form $(kp_1, 0)$. Therefore, the number of intermediate vertices in the path between v_1 and v_2 is $\phi(p_2)$, that is, $J = p_2 - 1$.

- (4) If $\text{ann}(T) \neq 0$, then for any two different vertices in different components $v_1 = (a_1, m_1)$ and $v_2 = (a_2, m_2)$ where $a_1, a_2 \in \text{ann}(T) \setminus \{0\}$. Therefore, any path between these vertices goes through an intermediate vertex in the set $\{(0, m) : m \in Z_p^*\}$ that is, $J \leq p - 1$.

The concept of perfectly m -geodetic graphs extends the basic idea of m -geodetic definitions to geometric structures with induced subgraphs. This is a stronger property than the original and can offer more detailed information about the overall structure of these (ZDGs) associated with idealization Rings. As m -geometric anatomy graphs guarantee that for any two vertices, there is exactly one path of all lengths, so perfectly m -Geodetic graphs must go further and satisfy this property, even restricting oneself to some of the vertices that are chosen in a general configuration. In understanding the path structures arrangement is hereditary. Since local structure in a hierarchy can impose customary treatment conditions along stretches of line, this singular genetic change is particularly important for noting which relations exist and how they propagate throughout graphs, thereby providing path structures after all! The definition that follows formalizes, in mathematical language, what is no more than a careful consideration of concepts familiar from ring theory. This will be one key step toward characterizing specific classes of (ZDGs).

Definition 2.5: A (ZDG) $\Gamma(K(+)T)$ is perfectly m -geodetic if:

- (1) It is m -geodetic graph.
- (2) Every induced subgraph is also m -geodetic.

Theorem 2.6: Let $\Gamma(K(+)T)$ be a (ZDG) of (IR). Then

- (1) If K be the integrals domain and $\text{ann}(T) = 0$, hence the graph is perfectly 1-geodetic.
- (2) If $r_1 = p^r$ and $r_2 = p$, then the graph is not perfectly $(p\phi(p^n) + p - 1)$ -geodetic.
- (3) If $r_1 = p_1p_2, r_2 = p_1$ and $p_1 < p_2$, then the graph is not perfectly m -geodetic,
- (4) If $\text{ann}(T) \neq 0$, then the graph is not perfectly m -geodetic.

Proof: Let $\Gamma(K(+)T)$ be a (ZDG) of (IR). Then

- (1) If K be integral domains and $\text{ann}(T) = 0$, so we get $\Gamma(K(+)T)$ be complete graph, and any induced subgraph is also 1-geodetic. So, the graph is perfectly 1-geodetic.
- (2) If $r_1 = p^r$ and $r_2 = p$, then by the same procedure in the previous theorem graph is not perfectly $(p\phi(p^n) + p - 1)$ -geodetic.
- (3) If $r_1 = p_1p_2$ and $r_2 = p_1$, so by the same procedure in the previous theorem graph is not perfectly m -geodetic.

Case 1: If $d(v_1, v_2) = 1$, then $v_1 \in L_0^*$ and $v_2 \in L_0^*$, or $v_1 \in L_0^*$ and $v_2 \in L_1^*$ that is, $J = 1$.

Case 2: If $v_1 \in L_0^*$ and $v_2 \in L_1^*$, then any path between these vertices goes through an intermediate vertex in L_0 . Therefore, the number of intermediate vertices in the path between v_1 and v_2 is $p_1 - 1$, that is, $J = p_1 - 1$.

Case 3: If $v_1 \in L_1^{**}$ and $v_2 \in L_1^{**}$, then any path between these vertices goes through an intermediate vertex in L_1^* of the form $(kp_1, 0)$. Therefore, the number of intermediate vertices in the path between v_1 and v_2 is $\phi(p_2)$, that is, $J = p_2 - 1$. But, any two non-adjacent vertices v_1, v_2 There are no exactly m distinct shortest paths between them. That is not m -geodetic graph.

- (4) If $\text{ann}(T) \neq 0$, then for any two different vertices in different components $v_1 = (a_1, m_1)$ and $v_2 = (a_2, m_2)$ where $a_1, a_2 \in \text{ann}(T) \setminus \{0\}$. Therefore, any path between these vertices goes through an intermediate vertex in the set $\{(0, m) : m \text{ belong to } Z_p^*\}$ that is, $J \leq -1 + p$. But, any two non-adjacent vertices v_1, v_2 There are no exactly m distinct shortest paths between them. That is not m -geodetic graph.

Conclusion

This paper introduces m -geodetic graphs and establishes complete characterizations of perfectly m -geodetic zero-divisor graphs of idealization rings $K(+)T$. The results reveal a deep connection between the algebraic structure of the idealization ring and the path distribution properties of its associated zero-divisor graph.

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