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Featured FZ-Modules with studying some related results

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Abstract

In this research, a new type of fuzzy modules, namely the featured fuzzy modules, has been studied. This concept is defined in terms of the fuzzy ideal derived from the ring. It is characterized by the condition that the fuzzy ideal of the ring equals zero. This model represents a novel approach to fuzzy modules, which has not been previously explored. All the results presented are original and were derived from foundational concepts, particularly fuzzy ideals in the commutative ring, to establish and develop this new concept.

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1. Introduction

Suppose R is the symmetric rings with unit. The fuzzy modules (short FZ-Modules) is defined in definition (2.2) [1]. The concept of a featured FZ-Modules is introduced in Definition (3.1) [2-3], and the fuzzy ideal (we call

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it an FZ-id.) is detailed in Definition (2.3). In this work, we focused on studying feature fuzzy modules, the new type for fuzzy modules. Numerous results related to this concept have been presented and linked to other fuzzy concepts. One of the most significant outcomes is property (3.3), which played a pivotal role in connecting these results to their counterparts in non-featured FZ-Modules. Additionally, new results were introduced, leading to a deeper understanding of the featured fuzzy modules concept, particularly properties (3.11) and (3.12), which were considered distinctive as they serve as equivalents to the definition. Other opposing fuzzy concepts were also examined, showing that their negation does not yield a featured FZ-Module. Among the notable results are properties (3.9) and (3.10). To enhance the Darity and applicability of our work, several examples were provided, offering further explanation and insight into this study.

2. Basics

Definition 2.1: Let X be any set. X named FZ-sets where X: M→I, and I=[0,1] and M is any nonempty set [2].

Definition 2.2: A set X is named an FZ-Modules over a modules M with the ring R iff:- 1- X(x-y) ≥Min{X(x), X(y)}, forx, yinM. 2- X(rx)≥ $X(x)$, ∀r∈R. 3- $X(0)1$, 0∈M. If Y & X are two FZ-Modules of a modules M over R, then Y is an FZ-submodules if and only if Y⊆X . As a special case when X(x)=1,∀x∈M, then Y is called an FZ-submodules [3].

Definition 2.3: An FZ-subset ring R is FZ-ideals for R, where $x, y \in$R

1- k$(x-y)$≥ Min{k (x), k(y)}. 2- k(xy)≥Max {k (x), k(y)} [4].

Definition 2.4: Let 0_1:S→[0,1] is FZ-sets in S, for x∈S, t∈[0,1] define by :X_t(y)=$\begin{cases} t \; if \; x = y \\ 0 \; if \; x \neq y \end{cases}$, is called FZ-singleton in$S$. for $x = 0$ & t=1,so 0_1(x)= $\begin{cases} 1 \; if \; x = 0 \\ 0 \; if \; x \neq 0 \end{cases}$, If x = 1, then 1_t:R→[0,1] where R is a ring such that:1_t(y) = $\begin{cases} t \; if \; y = 1 \\ 0 \; if \; y \neq 1 \end{cases}$.[5]

Definition 2.5: Let A be an FZ-id. a ring R, an AFZ-Maximal if A is a non-empty set and any FZ-id. B, if A⊆B , so we have either A_t=B_t or B_t =λ_R ,∀tin the [0,1] and λ_R=1,∀x∈R. [6]

Remark 2.6: $X$is FZ-Modules a module M, then F-ann$_R$X= ann$_e$x ,∀t∈(0,1]. [7]

Definition 2.7: An FZ-Modules $X$of R-modules $M$is FZ-torsion-free if $(X) = 0_1$, where $T(X) = \{x_t \subseteq X, r_\lambda x_t = 0_1, \forall \lambda, t \in [0,1]$ and for some $\forall 0_1 \neq x_t \in X\}$. [8]

Author in [7] define FZ-simple R-modules in another way we need that.

3. Featured FZ-Modules

Definition 3.1: FZ-Modules X of an R-modules $R$is named FZ-Featured if F-$ann_X A \neq 0_1$ where A is an FZ-ideal of R.

Remark 3.2: An FZ-Modules X is FZ-Featured$\Leftrightarrow$ F-$ann_x A\neq 0_1$ for each proper FZ-ideals A for the rings R.

Proof: (-) If X be the Featured FZ-Modules. So be it A is a proper FZ-id. of R, then $\exists$ FZ-maximal FZ-id. B of R we note [9]. Therefore F-$ann_x B\neq 0_1$, but F-$ann_x B\subseteq$F-$ann_x A$ [10]. So F-$ann_x A\neq 0_1$.

($\Leftarrow$) It is obvious.

Proposition 3.3: - X is a Featured FZ-Modules. Xt be the Featured modules.

Proof: We show X be Featured FZ-Modules. We know X: M$\rightarrow$[0,1] , we note X(x)=$\begin{cases} 1\ for\ x\ in\ M \\ 0\ otherwises \end{cases}$, let A: M$\rightarrow$[0,1] define as follows:

$A(X)=\begin{cases} t\ for\ x\ in\ I \\ 0\ otherwises \end{cases}$, $\forall$ t in(0,1], where I is maximal ideals for rings R.

Decided X_t=m, At=I .(F-$ann_x A)_t$=$ann_x A_t$ and F-$ann_x A\neq 0_1$ implies that (F-$ann_x A)_t \neq 0_t \leq 0_1$, but (F-$ann_x A)_t$=$ann_x A_t$. Therefore, $ann_x A_t$ =$ann_x I\neq 0$. Thus X_t is Featured modules.

($\Leftarrow$) If X_t Featured modules, show that X Featured FZ-modules? Since(F-$ann_x A)_t$=$ann_x A_t\neq 0$ [9], then (F-$ann_x A)_t\neq 0_t\leq 0_1$ therefore F-$ann_x A\neq 0_1$, [10].

Hence, X is a Featured FZ-modules.

Example 3.4: Let M=Z=R , X : M $\rightarrow$ [0,1] s.t X(x)=$\begin{cases} 1 \text{ for } x \text{ in the } M \\ 0 \text{ otherwises} \end{cases}$, clear that X is an FZ-modules and X_t=M=Z .But Z is not featured, since there is no ideal in Z such that the annihilator is not equal to zero and X_t=Z by remark(3.2). Therefore, X is not FZ -Featured by proposition (3.3).

Example 3.5: So be it M = Q $= R$. Let X : M$\rightarrow$[0,1] , as X(x) = $\begin{cases} 1 \text{ where } x \text{ belong to } M \\ 0 \text{ otherwises} \end{cases}$.

X If FZ-Modules and Xt=Q, Q is a field, then it has one proper ideal zero and the annihilator of the zero ideal is not equal to zero, then Q=Xt is featured. Therefore, X is a Featured FZ-Modules by proposition (3.2).

Example 3.6: If $M = Q$ & $R = Z$, X : M $\rightarrow$ [0, 1] as X(x) = $\begin{cases} 1 \text{ if } x \in Q \\ o \text{ otherwise} \end{cases}$, X is an FZ-Modules and X_t = Q and Q is not a featured modules over the ring Z [11]. Then X is not FZ -Featured.

Lemma 3.7: *Let X be FX-modules. If X is FZ-torsion –free, then F-ann$_x$A=0$_1$, for each non-empty FZ-ideal of R.*

Remark 3.8: X is an FZ-Modules. If X is FZ-torsion free, then X is not FZ – Featured.

Example 3.9: Let R = Z, M = Z_{P^∞} Here, p is a prime number. DefineX : M $\rightarrow$ [0,1] as follows:- X(x) = $\begin{cases}1 \text{ if } x \in Z_{P^\infty} \\ 0 \text{ otherwise}\end{cases}$. X is an FZ-Modules and X_t=M =Z_{P^∞} It is not a featured module. Therefore, X not FZ -Featured FZ-Modules from (3.2).

Remark 3.10: If X is FZ-Modules. If X is not FZ-simple, then X is not FZ - Featured.

Proof: If X : M $\rightarrow$ [0, 1] define as: X(x) = $\begin{cases}1 \text{ if } x \in M \\ 0 \text{ otherwise}\end{cases}$, X is an modules M, X_t=M Suppose that X is any FZ-simple and A is FZ-maximal ideals in R. Now, If X is FZ -Featured, then F-annxA01 and hence F-ann$_x$A=X (since X is FZ-simple). Hence AX=0$_1$, therefore A$\subseteq$F-ann$_x$X impels that F-ann$_R$X=R [9]. But (F-ann$_R$X)$_t$=ann$_R$X$_t$. Thus ann$_R$X$_t$=ann$_R$M=R. Hence, M=Xt =0, which is a contradiction. Then M=X$_t$ not featured modules, therefore X not Featured FZ-Modules from (3.2).

Proposition 3.11: If X is FZ-Modules for the $R-$modules M. X is FZ - Featured$\Leftrightarrow$ F-ann$_R$(ann$_x$A)=A for each FZ-maximal FZ-ideal A of R.

Proof: ($\Rightarrow$) Let X be an FZ -Featured and let A be an FZ-maximal FZ-ideals for R. so A$\subseteq$$F$-ann$_R$(ann$_x$A). Since X is FZ -Featured, then F-ann$_x$A$\neq$0$_1$ implies that F-annR(annxA) =R, and since cap A is an FZ-maximal FZ-ideal of cap R, we give F-ann$_R$(ann$_x$A)=A.($\Leftarrow$) F-ann$_R$(ann$_x$A)=A, for each FZ-maximal FZ-ideal A of R. If X is not FZ -Featured, then F-annxA=01, for some FZ-maximal FZ-ideal A of R, which implies that F-annR(annxA)=R, and hence A=R, which is a contradiction. Therefore, X is FZ –Featured.

Proposition 3.12: Let X be an FZ-Modules .X is FZ-Featured $\Leftrightarrow$for each FZ-maximal ideal A of R, and $\exists x_t \subseteq$ X s. t A = F $- ann_R(x_t)$.

Proof: Suppose that X is a Featured FZ-Modules, A is an FZ-maximal id. of R, so we get F-ann_RA $\neq 0_1$.

Let $0_1 \neq x_t \subseteq$ F $-$ ann$_x$ A , $\forall$t $\in (0,1]$, hence x_tA $= 0_1$, then A$\subseteq$F $-$ ann$_R$(x$_t$). But A is an FZ-maximal ideal for R, & xt01, we get F-annR(xt) = A. On the other side: Let Abe an FZ-maximal ideal of R, then A=F-ann$_R$(x$_t$) for some 0$_1\neq$x$_t\subseteq$X. Therefore x$_t$A= 0$_1$, which implies that xt F-annxA and hence F-annxA = 0$_1$.

4. Discussion and Conclusion

Through our study of this type of fuzzy submodules, we developed and introduced a new concept, namely, Featured FZ-Modules, by building on related results. It has been shown that it cannot be separated from other fuzzy submodules. Some of the results presented in this research support and confirm this finding; however, this study has also opened the door to many new insights and terms that may be developed in the future upon the completion of our research on this concept.

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