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Expended translation BIF ψ-ideals of ψ-algebra

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Abstract

This is our first start ψ- algebra, ψ- ideals of ψ- algebra. We explain BIF-set, BIF-ψ-ideals of ψ-algebra, expended translation BIF-ψ-ideals of ψ- algebra, we discuss some results of images and pre-images of expended translation BIF-ψ- ideal of ψ- algebra and we studied the Cartesian product of expended translation BIF-ψ- ideal of ψ- algebra. Symbolize $\xi = [0, 1]$.

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1. Introduction

Fuzzy set theory has acted as the building block since its inception. However, as real-world problems become more complex and involve both fuzziness and randomness, there is a need for a more advanced system. One of these developments is the Bipolar Intuitionistic Fuzzy Set (BIF-set), which generalizes the bipolar fuzzy notion to model situations involving vagueness, hesitation, and probabilistic variability. This theory was proposed by Gerstenkorn and Mańko [1] to develop a more robust mathematical model that defines the positive, negative, and non-membership components. Fundamentally, this dual aspect provides a better representation of uncertainty, especially in situations involving partial membership and elimination in a system based on partial or conflicting information. Other advancements have continued over the years, and among them, Jabir and Hameed [2] added in 2023 presented revised details or results on BIF-sets in behavior and certain uses, which strengthened the subject's theoretical base and provided new research directions. In parallel to this kind of work, ψ-algebra has been defined as an

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extension of the conventional algebraic structures to incorporate fuzzy operations. Jabir and Hameed [2] first established an environment for ψ-algebras to perform algebraic work in fuzzy and intuitionistic settings and laid the theoretical groundwork for the same. After this, the same authors further developed the theory by defining and studying ψ-subalgebras [3], which are necessary for decomposing more complex algebraic systems into simpler subalgebras that retain all relevant properties. The authors in [4] introduced translation ψ-subalgebras and further developed the theory by investigating how algebraic transformations behave under translation mappings. Additionally, the authors in [5] expanded the understanding of translation ψ-ideals, providing a deeper perspective on ideal theory within the framework of generalized fuzzy algebraic systems. To further generalize this work, the present study builds upon recent advancements in the field, as discussed in [6] and [7]. Altogether, the proposals in this series of papers make a significant contribution by providing scholars with a comparative quantitative methodology that allows uncertainty to be handled more nuanced when it arises in the mathematical modeling of reality, especially in areas such as decision theory, artificial intelligence, and computational logic. In the following sub-section of this paper, we introduce the definitions of the sets, and we point out some obvious properties that are as follows: ψ- algebra and fuzzy ψ- ideal of ψ- algebra, any important notation and definitions that are necessary for this work are presented here.

2. Basic concepts

Definition 2.1: [2] The algebraic system $(\Upsilon; +, -, 0)$, with both processes (+), (−), and unchanged (0) is named ψ- algebra, if the following conditions are met: $\forall\ \mu, \lambda, \kappa\ \in \Upsilon$,$(\psi_1)\ \mu + \mu^- = \mu^- + \mu = 0$, $(\psi_2)\ (0 + \mu^-) + \mu = 0$,

$(\psi_3)\ (\mu + \lambda^-) + \kappa^- = \mu + (\kappa + \lambda)^-$,$(\psi_4)\ (\lambda + \mu) + (\mu + \kappa^-)^- = \lambda + \kappa$.

Definition 2.2: [2] $\mathcal{T} \neq \emptyset$ of an $\psi - a(\Upsilon; +, -, 0)$ is name an ψ- ideal of Υ if it provided: all $\mu, \lambda, \kappa\ \in\ \Upsilon$,

(I_1) $0 \in \mathcal{T}$,

(I_2) $(v + \kappa) \in \mathcal{T}$ and $(\mu + \kappa^-) \in \mathcal{T}$ imply $(\lambda + \mu) \in \mathcal{T}$.

Example 2.3: [2] Let Y = {0, 1, 2, 3}, and consider the operations (+, −) as shown in Table 1:

Table 1

Cayley tables defining the operations (+) and (−) on the set Y = {0, 1, 2, 3}.

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

−	0	1	2	3
0	0	3	2	1
1	1	0	3	2
2	2	1	0	3
3	3	2	1	0

Then $(\Upsilon; +, -, 0)$ is an ψ- algebra. Defined $I = \{0,2\}$ is ψ- ideal of Υ.

Definition 2.4: [8] A BIF-set is B of ψ-algebra $(\Upsilon; +, -, 0.)$ having the formula $B = \{(\mu, \tau_B(\mu), \upsilon_B\,(\mu)) \mid \mu \in \Upsilon\}$ where the functions $\tau_B: \Upsilon \to \xi$ indicate the level involvement(specifically $\tau_B(\mu)$) and $\upsilon_B: \Upsilon \to \xi$ indicate the extent of not being a member(specifically $\upsilon_B(\mu)$) which is called af function and $0 \leq \tau_B\,(\mu) + \upsilon_B\,(\mu) \leq 1$, for all $\mu \in \Upsilon$.

Definition 2.5: [5] if $(\Upsilon; +, -, 0.)$ is ψ- algebra, fuzzy subsets υ for Υ be name anti- fuzzy ψ- ideal for Υ this corresponds to here are the conditions, for all $\mu, \lambda, \kappa \in \Upsilon$,

(AFI$_1$) $(0) \leq \upsilon(\mu)$,

(AFI$_2$) $\upsilon\,(\lambda + \mu) \leq \max\{\upsilon\,(\lambda + \kappa), \upsilon\,(\mu + \kappa^-)\}$.

Definition 2.6: [5] Let $B = \{(\mu, \tau_B(\mu), \upsilon_B\,(\mu)) \mid \mu \in \Upsilon\}$ be a bi s of an ψ-al $(\Upsilon; +, -, 0.)$. B it's called to be BIF-ψ- ideal of Υ if $\forall\, \mu, \lambda, \kappa \in \Upsilon$,

$(B\,\psi_1)$ $\tau_B(0) \geq \tau_B(\mu), \upsilon_B(0) \leq \upsilon_B(\mu)$,

$(B\,\psi_2)$ $\tau_B(\lambda + \mu) \geq \min\{\tau_B(\lambda + \kappa), \tau_B(\mu + \kappa^-)\}$ and

$(B\,\psi_3)$ $\upsilon_B(\lambda + \mu) \leq \max\{\upsilon_B(\lambda + \mu), \upsilon_B(\mu + \kappa^-)\}$.

τ_B is a fuzzy ψ- ideal for ψ- algebra & υ_B its anti- fuzzy of ψ- algebra.

Definition 2.7: [9] if $B = \{(\mu, \tau_B(\mu), \upsilon_B(\mu)) \mid \mu \in \Upsilon\}$ is bi-s of ψ-algebra $(\Upsilon; +, -, 0.)$, τ_B is fuzzy subset for Υ, where $R = 1 - \sup\{\tau(\mu) \mid \mu \in \Upsilon\}$ and $S = \inf\{\upsilon(\mu) \mid \mu \in \Upsilon\}$, such that $a \in [0, R]$ and υ_B be fuzzy subset of Υ such that $b \in [0, S]$. A mapping $(\tau_B)_{\mathrm{a}}^{\mathrm{R}} : \Upsilon \to \xi$ and $(\upsilon_B)_{\mathrm{b}}^{\mathrm{S}} : \Upsilon \to \xi$,

$$B_{(a,b)}^{(R,S)} = \{(\mu, (\tau_B)_{\mathrm{a}}^{\mathrm{R}}(\mu), (\upsilon_B)_{\mathrm{b}}^{\mathrm{S}}\,(\mu)) \mid \mu \in \Upsilon\} \text{ i}$$

s called **a translation of *B*** if it corresponds to: $(\tau_B)_{\mathrm{a}}^{\mathrm{R}}(\mu) = \tau_B(\mu) + a$ and $(\upsilon_B)_{\mathrm{b}}^{\mathrm{S}}\,(\mu) = \upsilon_B(\mu) - b$, $\forall\, \mu \in \Upsilon$.

Definition 2.8: [10] Let $B = \{(\mu, \tau_B(\mu), \upsilon_B(\mu)) \mid \mu \in \Upsilon\}$ BIF-set of ψ-al and $a \in [0, R]$, $b \in [0, S]$ of $(\Upsilon; +, -, 0)$, then $B_{(a,b)}^{(R,S)} = \{(\mu, (\tau_B)_{\mathrm{a}}^{\mathrm{R}}(\mu), (\upsilon_B)_{\mathrm{b}}^{\mathrm{S}}\,(\mu)) \mid \mu \in \Upsilon\}$ is called a translation BIF$\boldsymbol{\psi}$-ideal of Υ, for all $\mu, \lambda, \kappa \in \Upsilon$,

$1 - (\tau_B)_{\mathrm{a}}^{\mathrm{R}}(0) \geq (\tau_B)_{\mathrm{a}}^{\mathrm{R}}(\mu)$ and $(\upsilon_B)_{\mathrm{b}}^{\mathrm{S}}(0) \leq (\upsilon_B)_{\mathrm{b}}^{\mathrm{S}}(\mu)$,

$2 - (\tau_B)_{\mathrm{a}}^{\mathrm{R}}(\lambda + \mu) \geq \min\{(\tau_B)_{\mathrm{a}}^{\mathrm{R}}\,(\lambda + \kappa), (\tau_B)_{\mathrm{a}}^{\mathrm{R}}(\mu + \kappa^-)\}$, and

$3 - (\upsilon_B)_{\mathrm{b}}^{\mathrm{S}}\,(\lambda + \mu) \leq \max\{(\upsilon_B)_{\mathrm{b}}^{\mathrm{S}}\,(\lambda + \kappa), (\upsilon_B)_{\mathrm{b}}^{\mathrm{S}}(\mu + \kappa^-)\}$. i.e,

$1 - \tau_B\,(0) + a \geq \tau_B\,(\mu) + a$ and $\upsilon_B\,(0) - b \leq \upsilon_B\,(\mu) - b$,

$2 - \tau_B\,(\lambda + \mu) + a \geq \min\{\tau_B\,(\lambda + \kappa) + a, \tau_B(\mu + \kappa^-) + a\}$

$= \min\{\tau_B\,(\lambda + \kappa), \tau_B(\mu + \kappa^-)\} + \alpha$, and

$3 - \upsilon_B\,(\lambda + \mu) - b \leq \mathrm{m}a\mathrm{x}\{\upsilon_B\,(\lambda + \kappa) - b, \upsilon_B(\mu + \kappa^-) - b\}$

$= \mathrm{m}a\mathrm{x}\{\upsilon_B\,(\lambda + \kappa), \upsilon_B(\mu + \kappa^-)\} - b$

3. Expended translation BIFψ-ideal of ψ-algebra

We investigate a novel idea named expended translation BIF-ψ-ideal of ψ-algebra, and examine various basic qualities of them.

Definition 3.1 [11]: If $(\Upsilon; +, -, 0)$ is ψ- a, $B^M = \{(\mu, {\tau_B}^M(\mu), {\upsilon_B}^M(\mu)) \mid \mu \in \Upsilon\}$ is said to be an expended translation BIF-ψ-ideal of Υ if the following will be achieved: for all $\mu, \lambda, \kappa \in \Upsilon$,

(MTBI$_1$) ${\tau_B}^M(0) = D.\tau_B\,(0) + a \geq D.\tau_B\,(\mu) + a = {\tau_B}^M\,(\mu)$, and

$${\upsilon_B}^M\,(0) = D.\upsilon_B\,(0) - b \leq D.\upsilon_B\,(\mu) - b = {\upsilon_B}^M(\mu)$$

(MTBI$_2$) ${\tau_B}^M\,(\lambda + \mu) = D.\tau_B\,(\lambda + \mu) + a \geq \min\{D.\ \tau_B\,(\lambda + \kappa) + a, D.\tau_B(\mu + \kappa^-) + a = \min\{{\tau_B}^M\,(\lambda + \kappa), {\tau_B}^M(\mu + \kappa^-)\}$, and

(MTBI$_3$) ${\upsilon_B}^M\,(\lambda + \mu) = D.\upsilon_B\,(\lambda + \mu) - b \leq \max\{D.\upsilon_B\,(\lambda + \kappa) - b, D.\upsilon_B(\mu + \kappa^-) - b\} = \max\{{\upsilon_B}^M\,(\lambda + \kappa), {\upsilon_B}^M(\mu + \kappa^-)\}$.

Proposition 3.2: Let $B = \{(\mu, \tau_B(\mu), \upsilon_B(\mu)) \mid \mu \in \Upsilon\}$ be a BIF-setof ψ-algebra $(\Upsilon; +, -, 0.)$ and $a \in [0, R], b \in [0, S]$ so that τ_B be a f ψ-ideal of Υ and υ_B be af ψ-ideal of Υ, following that $B^M = \{(\mu, {\tau_B}^M(\mu), {\upsilon_B}^M(\mu)) \mid \mu \in \Upsilon\}$ is expended translation BIF-ψ-ideal of Υ.

Proof: Since τ_B be a f ψ-ideal, υ_B be anti- fuzzy ψ-ideal, for all $\mu \in \Upsilon$ implies that $\tau_B\,(0) \geq \tau_B\,(\mu) \Rightarrow \tau_B\,(0) + a \geq \tau_B\,(\mu) + a \Rightarrow$

$$D.\tau_B\,(0) + a \geq D.\tau_B\,(\mu) + a \Rightarrow {\tau_B}^M(0) \geq {\tau_B}^M\,(\mu),$$
$$\upsilon_B\,(0) \leq \upsilon_B\,(\mu) \Rightarrow \upsilon_B\,(0) - b \leq \upsilon_B\,(\mu) - b$$
$$D.\upsilon_B\,(\mu) - b \leq D.\upsilon_B\,(\mu) - b \Rightarrow {\tau_B}^M(0) \leq {\tau_B}^M(\mu).$$

For all $\mu, \lambda, \kappa \in \Upsilon$,

$$\tau_B\,(\lambda + \mu) \geq \min\{\ \tau_B\,(\lambda + \kappa), \tau_B(u + \kappa^-)\} \Rightarrow$$
$$\tau_B\,(\lambda + \mu) + a \geq \min\{\ \tau_B\,(\lambda + \kappa) + a, \tau_B(\mu + \kappa^-) + a\} \Rightarrow$$

$D.\tau_B\,(\lambda + \mu) + a \geq \min\{D.\ \tau_B\,(\lambda + \kappa) + a, D.\ \tau_B(\mu + \kappa^-) + a\}$, implies that ${\tau_B}^M\,(\lambda + \mu) \geq \min\{{\tau_B}^M\,(\lambda + \kappa), {\tau_B}^M(\mu + \kappa^-)\}$, and

$$\upsilon_B\,(\lambda + \mu) - b \leq \max\{\upsilon_B\,(\lambda + \kappa) - b, \upsilon_B(\mu + \kappa^-) - b\} \Rightarrow$$
$$D.\upsilon_B\,(\lambda + \mu) \leq \max\{D.\upsilon_B\,(\lambda + \kappa) - b, D.\upsilon_B(\mu + \kappa^-) - b\} \Rightarrow$$
$${\upsilon_B}^M\,(\lambda + \mu) \leq \max\{{\upsilon_B}^M\,(\lambda + \kappa), {\upsilon_B}^M(\mu + \kappa^-)\}.$$

Hence $B^M = \{(\mu, {\tau_B}^M(\mu), {\upsilon_B}^M(\mu)) \mid \mu \in \Upsilon\}$ is expended translation BIF ψ-ideal of Υ.

Proposition 3.3: Let $B = (\tau_B(\mu), \upsilon_B(\mu))$ be a BIFof ψ-algebra such that $B^M = \{(\mu, {\tau_B}^M(\mu), {\upsilon_B}^M(\mu)) \mid \mu \in \Upsilon\}$ is expended translation BIF-ψ-ideal of Υ, for some $a \in [0, R], b \in [0, S]$. Then $B = (\tau_B(\mu), \upsilon_B(\mu))$ BIF-set ψ-ideal of Υ.

Proof: Suppose that B^M is expended translation BIF-ψ-ideal of Υ, for some $a \in [0, R], b \in [0., S]$, with $\mu, \lambda, \kappa \in \Upsilon$,

$$D.\tau_B\ (0) + a \geq D.\tau_B\ (\mu) + a, D.\upsilon_B\ (\mu) - b \leq D.\upsilon_B\ (\mu) - b$$

which implies ${\tau_B}^M(0) \geq {\tau_B}^M\ (\mu)$ and ${\tau_B}^M(0) \leq {\tau_B}^M(\mu)$. Now, we have

$${\tau_B}^M\ (\lambda + \mu) \geq \min\{{\tau_B}^M\ (\lambda + \kappa), {\tau_B}^M(\mu + \kappa^-)\}$$
$$D.\tau_B\ (\lambda + \mu) + a \geq \min\{D.\ \tau_B\ (\lambda + \kappa) + a, D.\ \tau_B(\mu + \kappa^-) + a\}$$
$$\tau_B(\lambda + \mu) \geq \min\{\ \tau_B\ (\lambda + \kappa), \tau_B(\mu + \kappa^-)\}, \text{ and}$$
$${\upsilon_B}^M\ (\lambda + \mu) \leq \max\{{\upsilon_B}^M\ (\lambda + \kappa), {\upsilon_B}^M(\mu + \kappa^-)\}$$
$$D.\upsilon_B\ (\lambda + \mu) - \mathrm{b} \leq \max\{D.\upsilon_B\ (\lambda + \kappa) - b, D.\upsilon_B(\mu + \kappa^-) - b\}$$
$$\upsilon_B\ (\lambda + \mu) \leq \max\{\upsilon_B\ (\lambda + \kappa), \upsilon_B(\mu + \kappa^-)\}.$$

4. Homomorphism of expended translation BIFψ-ideal of ψ-algebra

Here, we give results for image and pre-image of expended translation BIF-ψ-ideal of ψ-algebra.

Theorem 4.1: *Let* $f: (\Upsilon; +, -, 0) \to (\Upsilon';\ +', -', 0')$ *be a homomorphism of two* ψ- $algebra$ *and put* $B^M = \{(\mu, {\tau_B}^M(\mu), {\upsilon_B}^M(\mu)) \mid \mu \in \Upsilon\}$ *is expended translation BIF-*ψ*-ideal of* Υ'*, then the pre-image* $f^{-1}(B^M) = \{(\mu, (\tau_{f^{-1}(B)})^M$ $(\mu), (\upsilon_{f^{-1}(B)})^M(\mu) \mid \mu \in \Upsilon\}$ is a BIF-ψ-ideal of Υ.

Proof: Let $\mu, \lambda, \kappa \in \Upsilon$, ${\tau_B}^M(f(0)) = {\tau_B}^M(0')$ and ${\upsilon_B}^M(f(0)) = {\upsilon_B}^M(0')$

$$(\tau_{f^{-1}(B)})^M(0) = {\tau_B}^M(f(0)) \geq {\tau_B}^M(f(\mu)) = (\tau_{f^{-1}(B)})^M(\mu), (\upsilon_{f^{-1}(B)})^M(0)$$
$$= {\upsilon_B}^M(f(0)) \leq {\upsilon_B}^M(f(\mu)) = (\upsilon_{f^{-1}(B)})^M(\mu).$$
$$(\tau_{f^{-1}(B)})^M(\lambda + \mu) = {\tau_B}^M(f(\lambda + \mu)) \geq \min\{{\tau_B}^M\ f(\lambda + \kappa), {\tau_B}^M f(\mu + \kappa^-)\}$$
$$= \min\{(\tau_{f^{-1}(B)})^M(\lambda + \kappa), (\tau_{f^{-1}(B)})^M(\mu + \kappa^-).$$

Also, $(\upsilon_{f^{-1}(B)})^M(\lambda + \mu) = {\upsilon_B}^M(f(\lambda + \mu)) \leq$

$$\max\{{\upsilon_B}^M\ f(\lambda + \kappa), {\upsilon_B}^M f(\mu + \kappa^-)\}$$
$$= \max\{(\upsilon_{f^{-1}(B)})^M(\lambda + \kappa), (\upsilon_{f^{-1}(B)})^M(\mu + \kappa^-).$$

Then $f^{-1}(B^M) = \{(\mu, (\tau_{f^{-1}(B)})^M(\mu), (\upsilon_{f^{-1}(B)})^M(\mu) \mid \mu \in \Upsilon\}$ is a BIF-ψ-ideal of Υ.

Theorem 4.2: *Give* $f: (\Upsilon; +, -, 0) \to (\Upsilon';\ +', -'0'.)$ *be an epimorphism of two* ψ*-algebras and let* $B^M = \{(\mu, {\tau_B}^M(\mu), {\upsilon_B}^M(\mu)) \mid \mu \in \Upsilon\}$ *be expended translation BIF-*ψ*-ideal of* Υ *using sup-inf trail, this image* $f(B^M) = \{(u, (\tau_{f(B)})^M(\mu), (\upsilon_{f(B)})^M(\mu) \mid \mu \in \Upsilon\}$ *is expended translation BIF-*ψ*-ideal of* Υ.

Proof: Let r, s, t $\in \Upsilon'$ with $\mu = f^{-1}(\mathrm{r})$, $\lambda = f^{-1}(\mathrm{s})$ and $\kappa = f^{-1}(\mathrm{g})$ such that

$${\tau_B}^M(\mu) = \sup_{g \in f^{-1}(\mathrm{r})} {\tau_B}^M(g), {\upsilon_B}^M(\lambda) = \sup_{g \in f^{-1}(\mathrm{s})} {\tau_B}^M(g)$$
$$\text{and } {\tau_B}^M(\kappa) = \sup_{g \in f^{-1}(\mathrm{g})} {\tau_B}^M(g),$$
$${\upsilon_B}^M\ (\mu) = \inf_{g \in f^{-1}(\mathrm{r})} {\upsilon_B}^M(g), {\upsilon_B}^M(\lambda) = \inf_{t \in f^{-1}(\mathrm{s})} {\upsilon_B}^M\ (g) \text{ and}$$
$${\upsilon_B}^M(\kappa) = \inf_{t \in f^{-1}(\mathrm{t})} {\upsilon_B}^M\ (g).$$

$$(\tau_{f(A)})^M(0) = \sup_{g\in f^{-1}(0')}\tau_B{}^M(g) = \tau_B{}^M\big(f(0)\big) \geq \tau_B{}^M\big(f(\mu)\big) = (\tau_{f(A)})^M(\mu),$$

$$(\upsilon_{f(A)})^M(0) = \inf_{g\in f^{-1}(0')}\upsilon_B{}^M(g) = \upsilon_B{}^M\big(f(0)\big) \leq \upsilon_B{}^M\big(f(\mu)\big) = \upsilon_B{}^M(\mu).$$

$$\begin{aligned}(\tau_{f(A)})^M(\mathrm{s}+\mathrm{r}) &= \sup_{g\in f^{-1}(\mathrm{s}+\mathrm{r})}\tau_B{}^M(g)\\ &= (\tau_{f(A)})^M(\lambda+\mu)\\ &\geq \min\{\tau_B{}^M(\lambda+\kappa), \tau_B{}^M(\mu-\kappa)\}\\ &= \min\{\sup_{g\in f^{-1}(\mathrm{s}+\mathrm{t})}\upsilon_B{}^M(g), \sup_{t\in f^{-1}(\mathrm{r}-\mathrm{t})}\upsilon_B{}^M(g)\}\\ &= \min\{(\tau_{f(A)})^M(\mathrm{s}+\mathrm{t}), (\tau_{f(A)})^M(\mathrm{r}-\mathrm{t})\}.\end{aligned}$$

Also, $(\upsilon_{f(A)})^M(\mathrm{s}+\mathrm{r}) = \inf_{g\in f^{-1}(\mathrm{s}+\mathrm{r})}\upsilon_B{}^M(g)$

$$\begin{aligned}&= \upsilon_B{}^M(\lambda+\mu)\\ &\leq \max\{\upsilon_B{}^M(\lambda+\kappa), \upsilon_B{}^M(\mu-\kappa)\}\\ &= \max\{\inf_{g\in f^{-1}(\mathrm{s}+\mathrm{t})}\upsilon_B{}^M(g), \inf_{g\in f^{-1}(\mathrm{r}-\mathrm{t})}\upsilon_B{}^M(g)\}\\ &= \max\{(\upsilon_{f(A)})^M(\mathrm{s}+\mathrm{t}), (\upsilon_{f(A)})^M(\mathrm{r}-\mathrm{t})\}.\end{aligned}$$

Then the image $f(B^M) = \{(\mu, (\tau_{f(B)})^M(\mu), (\upsilon_{f(B)})^M(\mu) \mid \mu \in \Upsilon\}$ is expended translation BIF-ψ-ideal of Υ.

5. Cartesian Product of expended translation BIFψ-ideal of ψ-algebra

Here, we present the concepts of cartographic products of expended translation BIF-ψ-ideal of ψ-algebra.

Definition 5.1: Let $B^M = \{(\mu, \tau_B{}^M(\mu), \upsilon_B{}^M(\mu)) \mid \mu \in \Upsilon\}$ be expended translation BIFsubset of Υ and $C^M = \{(\vartheta, \tau_C{}^M(\vartheta), \upsilon_C{}^M(\vartheta)) \mid \vartheta \in \beta\}$ be a translation BIFsubset of Υ. The cartographic product of B and C is described

as $B \times C = (\Upsilon \times \beta, \tau_B{}^M \times \tau_C{}^M, \upsilon_B{}^M \times \upsilon_C{}^M$, where $\tau_B{}^M \times \tau_C{}^M$: $B \times C \to [0,1]$ and

$\upsilon_B{}^M \times \upsilon_C{}^M$: $B \times C \to [0,1]$, $\forall\ \mu \in \Upsilon$, $\vartheta \in \beta$, such that $(\tau_B{}^M \times \tau_C{}^M)(\mu, \vartheta) = \min\{\tau_B{}^M(\mu), \tau_C{}^M(\vartheta)\}$ and $(\upsilon_B{}^M \times \upsilon_C{}^M)(\mu, \vartheta) = \max\{\upsilon_C{}^M(\mu), \upsilon_C{}^M(\vartheta)\}$

Theorem 5.4.4: *Let $B^M = \{(\mu, \tau_B{}^M(\mu), \upsilon_B{}^M(\mu)) \mid \mu \in \Upsilon\}$ be expended translation BIF ψ-subset of Υ and $C^M = \{(\vartheta, \tau_C{}^M(\vartheta), \upsilon_C{}^M(\vartheta)) \mid \vartheta \in \beta\}$ be expended translation BIF-ψ-subset of β. If B^M is expended translation BIF-ψ-ideal of $\Upsilon$and C^M is expended translation BIF-ψ-i of β, then $B^M \times C^M$ is expended translation BIF-ψ-ideal of $\Upsilon \times \beta$.*

Proof: For any $\Upsilon = (\Upsilon_1, \Upsilon_2) \in \Upsilon \times \Upsilon$, we have

1- $(\tau_B{}^M \times \tau_C{}^M)(0) = (\tau_B{}^M \times \tau_C{}^M)(0,0)$
$= \min(\tau_B{}^M(0), \tau_C{}^M(0)) \geq \min(\tau_B{}^M(\Upsilon_1), \tau_B{}^M(\Upsilon_2)$
$= (\tau_B{}^M \times \tau_C{}^M)(\Upsilon_1, \Upsilon_2) = (\tau_B{}^M \times \tau_C{}^M(\Upsilon)$.
$(\upsilon_B{}^M \times \upsilon_C{}^M)(0) = (\upsilon_B{}^M \times \upsilon_C{}^M)(0,0)$
$= \max(\upsilon_B{}^M(0), \upsilon_C{}^M(0)) \leq \max(\upsilon_B{}^M(\Upsilon_1), \upsilon_C{}^M(\Upsilon_2))$
$= (\upsilon_B{}^M \times \upsilon_C{}^M)(\Upsilon_1, \Upsilon_2) = (\upsilon_B{}^M \times \upsilon_C{}^M)(\Upsilon)$.

2- Let $\Upsilon = (\Upsilon_1, \Upsilon_2)$, $\beta = (\beta_1, \beta_2)$ and $\delta = (\delta_1, \delta_2)$
$(\tau_B{}^M \times \tau_C{}^M)(\beta + \Upsilon) = (\tau_B{}^M \times \tau_C{}^M)((\beta_1, \beta_2) + (\Upsilon_1, \Upsilon_2))$
$= (\tau_B{}^M \times \tau_C{}^M)(\beta_1 + \Upsilon_1), (\beta_2 + \Upsilon_2))$
$= \min\{\tau_B{}^M(\beta_1 + \Upsilon_1), \tau_C{}^M((\beta_2 + \Upsilon_2)\}$
$\geq \min\{\min\{\tau_B{}^M(\beta_1 + \delta_1), \tau_B{}^M(\Upsilon_1 - \delta_1)\}, \min\{\tau_C{}^M(\beta_2 + \delta_2),$
$\tau_C{}^M(\Upsilon_2 - \delta_2)\}\}$
$= \min\{\min\{\tau_B{}^M(\beta_1 + \delta_1), \tau_C{}^M(\beta_2 + \delta_2)\}, \min\{\tau_B{}^M(\Upsilon_1 - \delta_1),$
$\tau_C{}^M(\Upsilon_2 - \delta_2)\}\}$
$= \min\{\tau_B{}^M \times \tau_C{}^M((\beta_1, \beta_2) + (\delta_1, \delta_2)),$
$(\tau_B{}^M \times \tau_C{}^M)((\Upsilon_1, \Upsilon_2) - (\delta_1, \delta_2))\}$
$= \min\{(\tau_B{}^M \times \tau_C{}^M)(\beta + \delta), (\tau_B{}^M \times \tau_C{}^M)(\Upsilon - \delta)\},$
$(\upsilon_B{}^M \times \upsilon_C{}^M)(\beta + \Upsilon) = (\upsilon_B{}^M \times \upsilon_C{}^M)((\beta_1, \beta_2) + (\Upsilon_1, \Upsilon_2))$
$= (\upsilon_B{}^M \times \upsilon_C{}^M)((\beta_1 + \Upsilon_1), (\beta_2 + \Upsilon_2))$
$= \max\{\upsilon_B{}^M(y_1 + x_1), \upsilon_C{}^M((\beta_2 + \Upsilon_2)\} \leq \max\{\max\{\upsilon_B{}^M(\beta_1 + \delta_1),$
$\upsilon_B{}^M(x_1 - z_1)\}, \max\{\upsilon_C{}^M(\beta_2 + \delta_2), \upsilon_C{}^M(\Upsilon_2 - \delta_2)\}\}$
$= \max\{\max\{\upsilon_B{}^M(\beta_1 + \delta_1), \upsilon_C{}^M(\beta_2 + \delta_2)\}, \max\{\upsilon_B{}^M(\Upsilon_1 - \delta_1),$
$\upsilon_C{}^M(\Upsilon_2 - \delta_2)\}\}$
$= \max\{(\upsilon_B{}^M \times \upsilon_C{}^M)((\beta_1, \beta_2) + (\delta_1, \delta_2)), (\upsilon_B{}^M \times \upsilon_C{}^M)$
$((\Upsilon_1, \Upsilon_2) - (\delta_1, \delta_2))\}$
$= \max\{(\upsilon_B{}^M \times \upsilon_C{}^M)(y + z), (\upsilon_B{}^M \times \upsilon_C{}^M)(\Upsilon - \delta)\}.$
Hence $B^M \times C^M$ is expended translation BIF-ψ-ideal of $\Upsilon \times \beta$.

6. Conclusions

In this paper, we introduced expanded translation BIF-ψ-ideals of ψ-algebra, we reached the relation between fuzzy ideals of ψ-algebra and expanded translation BIF-ψ-ideals of ψ-algebra, as well as have homomorphism of expended translation BIF-ψ-ideal of ψ-algebra and the Cartesian product of expended translation BIF-ψ-ideal of ψ-algebra.

References

[1] T. Gerstenkorn and J. Mańko, "Bifuzzy probabilistic sets," *Fuzzy sets and systems*, vol. 71, no. 2, pp. 207–214 (1995).

[2] N. H. Jabir and A. T. Hameed, "On the φ-algebra," *Journal of Interdisciplinary Mathematics*, vol. 26, no. 6, pp. 1-8 (2023), doi: 10.47974/JIM-1626.

[3] N. H. Jabir and A. T. Hameed, "On the ψ-subalgebras of ψ-algebra," *International Journal of Academic and Applied Research*, vol. 7, no. 4, pp. 8-11 (2023).

[4] A. T. Hameed and N. H. Jabir, "On the Translation Bifuzzy ψ-subalgebra of ψ-algebra," *Journal of Kufa for Mathematics and Computer*, vol. 8, no. 8, pp. 129-139 (2024).

[5] N. H. Jabir and A. T. Hameed, "On the translation bifuzzy ψ-ideal of ψ-algebra," *Journal of Kufa for Mathematics and Computer,* vol. 10, no. 2, pp. 140-160 (Aug. 2023), doi: 10.31642/JoKMC/2018/100223.

[6] N. H. Jabir, "The cyclic decomposition of the factor group $C\ell F(2^{2k} \times D_4, Z)/\bar{\rho}(2^{2k} \times D_4)$ when κ is a prime number," *Journal of Physics: Conference Series*, vol. 1818, no. 1, p. 012018 (Mar. 2021).

[7] N. H. Jabir and R. H. Abass, "The circular division of the factor group when r is an odd number," *Journal of Discrete Mathematical Sciences and Cryptography*, vol. 24, no. 6, pp. 1-12 (2021), doi: 10.1080/09720529.2020.1861777.

[8] K. T. Atanassov, "On the Modal Operators defined over the intuitionistic Fuzzy Sets," *Notes on Intuitionistic Fuzzy Sets*, vol. 10, no. 1, pp. 7–12 (2004).

[9] M. A. A. Ansari and M. Chandramouleeswaran, "Fuzzy translations of fuzzy β-ideals of β-algebras," *International Journal of Pure and Applied Mathematics*, vol. 92, no. 5, pp. 657–667 (2014).

[10] I. S. Ahmed, A. Al-Fayadh, H. H. Ebrahim, and L. S. Abdalbaqi, "Implementation of the neutrosophic sets in measurable space with respect to neutrosophic ring," *Int. J. Neutrosophic Sci.*, vol. 25, no. 3, pp. 187-193 (2025).

[11] Z. T. Abdulqader, R. B. Esmaeel, and N. E. Arif, "L-open set and some related concepts," *Journal of Discrete Mathematical Sciences and Cryptography,* vol. 27, no. 5, pp. 1715–1721 (2024).

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