

e*-s-Goldie-extending module

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Abstract

It is known that a module T is called e^* -s-Extending if for every e^* -s-closed submodule B of T for some a complemented submodule S of T such that $B \leq (e^*s) S$, as a generalization of an e^* -Extending module, we define a e^* -s-Goldie-Extending module if every e^* -s-closed submodule B of T , for some a complemented submodule S of T for which $B \cap S$ is e^* -essential in B and S , we prove properties of e^* -s-Goldie-extending module and present characterizations of this type of modules.

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1. Introduction

Assume that R is a ring with 1, let T denote a left unitary module over R and $I(T)$ denote the injective hull of T , a submodule B of T is said to be small in T written as $B \ll T$ if for any submodule A of T such that $T = B + A$ implies that $T = A$ [1], Leonard defined a module T to denote small if T is a small submodule of of a certain over R equivalently T is small in it injective envelope [2]. It known that a sub-module B of T is it called essential or T is an essential extension of B if for every nonzero sub-module A of T , $B \cap A \neq 0$ [3].

Authors in [4] introduced cosingular sub-module of T to be the set $Z^*(T) = \{x \in T \mid Rx \ll I(T)\}$ $Z^*(T)$ is called cosingular submodule, T is called cosingular module if $Z^*(T) = T$. The authors in [5] present a extension of

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essential submodule said to be e^* -essential sub-module where B is e^* -essential sub-module of T if for any nonzero cosingular sub-module A of T , $B \cap A \neq 0$.

The singular submodule is defined as $Z(T) = \{t \text{ in } T: \text{ann}(t) \leq_e R\}$, If $Z(T) = T$, T is said to be a singular module, and if $Z(T) = 0$, T is said to be nonsingular, [2, p. 30]. In [6] A & W generalized $Z(T)$ depending on the concept of e^* -essential submodules follows: $Ze^*(T) = \{n \text{ in } T: \text{ann}(n) \subseteq e^* R\}$ it was called e^* -singular submodule. If $Ze^*(T) = T$, then T is called e^* -singular module and if $Ze^*(T) = 0$, T is called non e^* -singular module. A&W depend on $Ze^*(T)$ to define a new generalization of an essential submodule called e^* -singular essential submodule. Let B be a submodule of T , then T is e^* -singular essential extension to B , or B is e^* -singular essential in T if $B \cap A \neq 0$, for every nonzero e^* -singular submodule A of T . For short (e^* S-essential) and denoted by $B \leq_{e^*s} T$ [7]. It is known that a module is referred to as extending if any submodule is essential in a complemented submodule of T [2] e^* s-Extending module were introduced in [4] as a generalization of extending module as follows: T is referred to as e^* s-Extending if any submodule of T is e^* s-essential in a complemented submodule of T equivalently every e^* s-closed submodule of T is e^* s-essential in a complemented submodule of T . In this article characterizations of e^* s-Goldie-Extending are given. It has been proven that under certain conditions a sub-module of e^* s-Goldie-Extending module be a e^* s-Goldie-Extending and the written uniquely as sum of a e^* s-Goldie-Extending is e^* s-Goldie-Extending.

2. e^* s- essential sub-module

In this section we define the relation ξ_{e^*s} on the sub-module of T , by using the concept of e^* s-essential sub-module.

Definition 2.1: [7] Assume T be an module over R , a submodule B of T is called e^* s-essential if $B \cap A \neq 0$ for any non zero e^* -singular submodule A of T denoted by $B \leq_{e^*s} T$.

A denote a lift R -ideal of R is e^* s- essential in R iff A is e^* s-essential sub-module, and a nontrivial module T is referred to as e^* s-uniform if any nontrivial submodule in T is e^* -essential in T [5]. We will use the concept of e^* s-essential to define a relation ξ_{e^*s} on the set of sub-modules of T .

Definition 2.2: Assume T be an module over R , consider the relation on the set of submodules of T , $B \xi_{e^*s} A$ if $B \cap A$ is e^* s- essential submodule in B and A .

Proposition 2.3: ξ_{e^*s} is an equivalent relation.

Proof: Clearly that ξ_{e^*s} is reflexive and symmetric, to show ξ_{e^*s} is transitive, Let B, A and D be a submodule of T such that $B \xi_{e^*s} A$ and $A \xi_{e^*s} D$, then

$B \cap A$ is e^* s-essential submodule in B and A , $A \cap D$ is e^* s-essential submodule in A and D , Let V be a nonzero e^* -singular submodule of B , since $B \cap A$ is e^* s-essential submodule of B . Then $0 \neq B \cap A \cap V \leq_{e^*s} A$, but

$A \cap D \stackrel{\leq}{e^*s} A$, therefore $B \cap A \cap V \cap D \neq 0$, but $0 \neq B \cap A \cap D \cap V \leq B \cap D \cap V \leq B$, so $B \cap D \cap V \neq 0$, hence $B \cap D \stackrel{\leq}{e^*s} B$. Similarly, $B \cap D \stackrel{\leq}{e^*s} D$, thus ξ_{e^*s} is an equivalent relation.

Proposition 2.4: Let $f: T \rightarrow T$ be linear (over R) homomorphism and B, A be a sub-module of T if $B \xi_{e^*s} A$, then $f^{-1}(B) \xi_{e^*s} f^{-1}(A)$.

Proof: Let B and A be a submodule of T for which $B \xi_{e^*s} A$, then $B \cap A \stackrel{\leq}{e^*s} B$ and $B \cap A \stackrel{\leq}{e^*s} A$, hence by [6] $f^{-1}(B) \cap f^{-1}(A) = f^{-1}(B \cap A) \stackrel{\leq}{e^*s} f^{-1}(B)$, and $f^{-1}(B) \cap f^{-1}(A) \stackrel{\leq}{e^*s} f^{-1}(A)$. Thus $f^{-1}(B) \xi_{e^*s} f^{-1}(A)$.

Note: Let $f: T \rightarrow T$ be an R -homomorphism, let B and A be a sub-module of T such that $B \xi_{e^*s} A$, then it is not necessary that $f(B) \xi_{e^*s} f(A)$. The following example shows: Let $\pi: Z \rightarrow \frac{Z}{2Z}$ be the natural epimorphism, clearly $2Z \xi_{e^*s} Z$ but $\pi(2Z)$ is not related to $\pi(Z)$ by ξ_{e^*s} .

Proposition 2.5: Let B_1, B_2, A_1 together with A_2 be submodules of a module T , if $B_1 \xi_{e^*s} A_1$ together with $B_2 \xi_{e^*s} A_2$, then $(B_1 \cap B_2) \xi_{e^*s} (A_1 \cap A_2)$.

Proof: Assume that $B_1 \xi_{e^*s} A_1$ and $B_2 \xi_{e^*s} A_2$, then $B_1 \cap A_1$ is e^*s -essential submodule in B_1 and A_1 , $B_2 \cap A_2$ is e^*s -essential submodule in B_2 and A_2 so $B_1 \cap B_2 \cap A_1 \cap A_2 \stackrel{\leq}{e^*s} B_1 \cap B_2$, and $B_1 \cap B_2 \cap A_1 \cap A_2 \stackrel{\leq}{e^*s} A_1 \cap A_2$ by hence $(B_1 \cap B_2) \xi_{e^*s} (A_1 \cap A_2)$.

Note: By induction one can show that the previous proposition is true for any finite number of sub-modules of T .

Proposition 2.6.: Let $\{B_\alpha \mid \alpha \in \Lambda\}$ and $\{A_\alpha \mid \alpha \in \Lambda\}$ be a family of sub-modules of an module over T , for which $\{B_\alpha \mid \alpha \in \Lambda\}$ is an independent family in T and $B_\alpha \xi_{e^*s} A_\alpha, \forall \alpha \in \Lambda$, then $\{A_\alpha \mid \alpha \in \Lambda\}$ is an independent family in T and $(\bigoplus_{\alpha \in \Lambda} B_\alpha) \xi_{e^*s} (\bigoplus_{\alpha \in \Lambda} A_\alpha)$.

Proof: By induction on the number of elements of Λ ,

When the family has only two elements, i.e., $\{B_1, B_2\}$ is independent family in T .

$B_1 \xi_{e^*s} A_1$ and $B_2 \xi_{e^*s} A_2$, then by (Prop. 2.5), we obtain $(B_1 \cap B_2) \xi_{e^*s} (A_1 \cap A_2)$, but $B_1 \cap B_2 = 0$, therefore $A_1 \cap A_2 = 0$ hence $\{A_1, A_2\}$ is linear independent family, we have $(B_1 \cap A_1) \oplus (B_2 \cap A_2)$ is e^*s -essential submodule in $B_1 \oplus B_2$ and $(B_1 \cap A_1) \oplus (B_2 \cap A_2)$ is e^*s -essential submodule in $A_1 \oplus A_2$, but $(B_1 \cap A_1) \oplus (B_2 \cap A_2) \leq (B_1 \oplus B_2) \cap (A_1 \oplus A_2) \leq (B_1 \oplus B_2)$ therefore $(B_1 \oplus B_2) \cap (A_1 \oplus A_2)$ is e^*s -essential submodule in $B_1 \oplus B_2$ similarly one can prove that $(B_1 \oplus B_2) \cap (A_1 \oplus A_2)$ is e^*s -essential submodule in $A_1 \oplus A_2$ hence $(B_1 \oplus B_2) \xi_{e^*s} (A_1 \oplus A_2)$. Assume that the result is true for the situation where the index set with $n - 1$ element, now let $\{B_1, B_2, \dots, B_n\}$ and assume that $B_i \xi_{e^*s} A_i$,

$\forall i = 1, 2, \dots, n$. Since $0 = (B_n \cap (B_1 \oplus B_2 \oplus \dots \oplus B_{n-1})) \xi_{e^*s} (A_n \cap (A_1 \oplus A_2 \oplus \dots \oplus A_{n-1}))$ by (Prop. 2.5), then $\{A_1, A_2, \dots, A_n\}$ is an independent family and $(B_1 \oplus B_2 \oplus \dots \oplus B_{n-1} \oplus B_n) \xi_{e^*s} (A_1 \oplus A_2 \oplus \dots \oplus A_{n-1} \oplus A_n)$ by using the previous case, finally let $\{B_\alpha \mid \alpha \in \Lambda\}$ be an independent family and $B_\alpha \xi_{e^*s} A_\alpha, \forall \alpha \in \Lambda$. Let T be a nontrivial e^* -singular submodule of $\bigoplus_{\alpha \in \Lambda} B_\alpha$ and assume X denote a nontrivial element in T so, $x = b_1 + b_2 + \dots + b_n$, where $b_i \in B_{\alpha_i}, \forall i = 1, 2, \dots, n$ hence $T \cap (B_{\alpha_1} + B_{\alpha_2} + \dots + B_{\alpha_n}) \neq 0$ but $(\bigoplus_{i=1}^n B_{\alpha_i}) \xi_{e^*s} (\bigoplus_{i=1}^n A_{\alpha_i})$. Then $T \cap (B_{\alpha_1} \oplus B_{\alpha_2} \oplus \dots \oplus B_{\alpha_{n-1}} \oplus B_{\alpha_n}) \cap (A_{\alpha_1} \oplus A_{\alpha_2} \oplus \dots \oplus A_{\alpha_{n-1}} \oplus A_{\alpha_n}) \neq 0$ hence $T \cap (\bigoplus_{\alpha \in \Lambda} B_\alpha) \cap (\bigoplus_{\alpha \in \Lambda} A_\alpha) \neq 0$ which implies $(\bigoplus_{\alpha \in \Lambda} B_\alpha) \cap (\bigoplus_{\alpha \in \Lambda} A_\alpha) \leq_{e^*}^* (\bigoplus_{\alpha \in \Lambda} B_\alpha)$, similarly, we can prove $(\bigoplus_{\alpha \in \Lambda} B_\alpha) \cap (\bigoplus_{\alpha \in \Lambda} A_\alpha) \leq_{e^*}^* (\bigoplus_{\alpha \in \Lambda} A_\alpha)$. Thus $(\bigoplus_{\alpha \in \Lambda} B_\alpha) \xi_{e^*s} (\bigoplus_{\alpha \in \Lambda} A_\alpha)$.

Note: Let B_1, B_2, A_1 and A_2 be submodules of a module over R . If $B_1 \xi_{e^*s} A_1$ and $B_2 \xi_{e^*s} A_2$. Then it is not necessary that $(B_1 + B_2) \xi_{e^*s} (A_1 + A_2)$. Consider the $Z \oplus Z_2$ as Z -module let $B_1 = B_2 = Z(\underline{2}, 0)$ and $A_1 = Z(1, 0)$ we can prove $B_1 \xi_{e^*s} A_1$ and $B_2 \xi_{e^*s} A_2$, but $(B_1 + B_2)$ is not related to $(A_1 + A_2)$, where there exists a nonzero e^* -singular sub-module $V = \{\underline{0}\} \oplus Z_2$ of $(A_1 + A_2)$ such that $(B_1 + B_2) \cap V = \{(0, 0)\}$.

3. e^* s- Goldie-extending module

In [4] e^* s-Extending module were introduced as a generalization of extending module as follows, whenever any submodule of a module T is e^* s-essential in a complemented submodule of T , then T is called e^* s-Extending module, equivalently every e^* s-closed submodule of T is e^* s-essential in a complemented submodule of T where a sub-module B of T is said to be e^* s-closed if B has no proper e^* s-essential extension in T [1]. As a generalization of e^* s-extending module we introduce e^* s-Goldie-extending module as follows:

Definition 3.1: Let T be a module over R , T is e^* s-Goldie-Extending module if for any sub-module B of T for some a complemented submodule S of T for which $B \xi_{e^*s} S$.

Remark 3.2: Every e^* s-extending module is an e^* s-Goldie-Extending module.

Proof: Let T denote an e^* s-Extending module together with let B denote a submodule of T . For some complemented submodule S of T for which $B \leq_{e^*s} S$, then $B = B \cap S \leq_{e^*s} S$ and $B \cap S \leq_{e^*s} B$. i.e., $B \xi_{e^*s} S$, thus T is e^* s-Goldie-Extending.

Proposition 3.3: Let T be an indecomposable module over R , we obtain following are statement equivalent:

- 1- T is e^* s-extending.
- 2- T is a e^* s-Goldie-extending.

3- T is e*s- uniform.

Proof: (1) $\Rightarrow$ (2) by Remark (3.2)

(2) $\Rightarrow$ (3) Let B be a nonzero submodule of T , since T is e*s-Goldie-Extending, then for some a complemented submodule S of T for which $B \xi_{e*s} S$, but T is indecomposable, then either $V = 0$ or $V = T$, if $V = 0$, we get $B = 0$ which is contradiction, hence $B = B \cap T \leq_{e*s} T$ it is follow that T is e*s-uniform module.

(3) $\Rightarrow$ (1) Clear

The followings are characterizations of e*s-Goldie-Extending module.

Theorem 3.4: *A module T is a e*s-Goldie-Extending if and only if for each e*s-closed sub-module B of T there exist adirect summand S of T such that $B \xi_{e*s} S$.*

Proof: $\Rightarrow$) Clear

($\Leftarrow$) Let B be a sub-module of T so there exists e*s-closed submodule A of T , $B \leq_{e*s} A$ see [5] By hypothesis for some a complemented submodule S of T for which $A \xi_{e*s} S$, since $(B \cap S) \leq_{e*s} A \cap S \leq_{e*s} S$, and $A \cap S$ is a complemented submodule of S , then $B \cap S \leq_{e*s} S$ by [6].

Now, since $(B \cap S)$ is e*s-essential submodule in $A \cap S$ and $A \cap S$ is e*s-essential submodule in A , and $B \leq A$, therefore $B \cap S \leq_{e*s} B$, hence $B \xi_{e*s} S$, thus T is e*s-Goldie- Extending module.

Theorem 3.5: *An module over R T is e*s-Goldie-Extending iff for each submodule B of T , for some a complemented submodule S of T for which $B \xi_{e*s} S$ and S is a complement of B where $T = S \oplus S$, $S \leq T$.*

Proof: $\Rightarrow$) Suppose that T is e*s-Goldie-Extending and let B be a submodule of T , then for some a complemented submodule S of T for which $B \xi_{e*s} S$, let $T = S \oplus S$, $S \leq T$ since $B \cap T \leq_{e*s} B$, then $0 = B \cap S \cap S \leq_{e*s} B \cap S$ so $B \cap S = 0$, let $A \leq T$ such that $S \leq A$ and $B \cap A = 0$, since $B \cap S \leq_{e*s} S$ then $0 = B \cap A \cap S \leq_{e*s} A \cap S$ so $S \cap A = 0$, but S is a complement of S by [3]. therefore $A = S$. $\Leftarrow$) Clear

Theorem 3.6: *A module T is e*s-Goldie-Extending iff for any complemented submodule B of the injective hull $I(T)$ of, for some a complemented submodule S of for which $B \cap T \xi_{e*s} S$.*

Proof: $\Rightarrow$) Clear

($\Leftarrow$) Let B be a sub-module of T , and let A be a complement of B , then by [1] $B \oplus A \leq_{e*s} T$ since $T \leq_{e*s} I(T)$ then $B \oplus A \leq_{e*s} I(T)$, by hypothesis for some a complemented submodule S of T such that $(I(B) \cap T) \xi_{e*s} S$ and $B \leq_{e*s} I(B)$, then $(B \cap T) \leq_{e*s} (I(B) \cap T)$, hence $(B \cap T) \xi_{e*s} I(B) \cap T$ but the relation ξ_{e*s} is transitive, then $(B \cap I) \xi_{e*s} S$.

The following Propositions give conditions in which the sub-module of e*s-Goldie -Extending is e*s-Goldie-Extending.

Proposition 3.7: Assume T be a e*s-Goldie-Extending module over R and let B be a submodule of T for which an intersection of B for any complemented submodule of T is a complemented submodule of B , then B is a e*s-Goldie-Extending.

Proof: Let $X \leq B, B \leq T$, since T is a e*s-Goldie-Extending, then for which a complemented submodule S of T such that $X \xi_{e*s} S$ by assumption, $B \cap S$ is a direct summand of B hence $X = (B \cap X) \xi_{e*s} (B \cap S)$. Then B is e*s-Goldie-Extending module.

Recall that a submodule B of T is referred to as fully invariant if for any $f \in \text{End}(I)$, $f(B) \leq B$, [3] and T is distributive if $B \cap (A \oplus D) = (B \cap A) \oplus (B \cap D)$ for all submodules B, A, D , in T .

Proposition 3.8: Every fully invariant sub-module of e*s-Goldie-Extending module is e*s-Goldie-Extending module.

Proof: Assume T denote a e*s-Goldie-Extending module and B is referred to as fully invariant submodule of T , assume X denote a submodule of B , since T is e*s-Goldie-Extending, then for some a complemented submodule S of T for which $X \xi_{e*s} S$, let $T = S \oplus \bar{S}$, where $\bar{S} \leq T$, now consider the projection map $f: T \rightarrow S$ and $1-f: T \rightarrow \bar{S}$ claim that $B = (B \cap f(t)) \oplus (1-f(t)) \cap B$, to show that $x \in B$, then $x = b + a$, $b \in S$, $a \in \bar{S}$ now $f(x) = f(b + a) = b$ and $(1-f)(x) = a$, but B is a fully invariant, then $f(x) = b \in f(t) \cap B$ and $(1-f)(x) = a \in (1-f)(t) \cap B$, thus $B = (B \cap f(t)) \oplus ((1-f)(t) \cap B) = (B \cap S) \oplus (B \cap \bar{S})$, since $X \xi_{e*s} S$, then $X = (X \cap B) \xi_{e*s} (B \cap S)$, then B is e*s-Goldie-Extending.

T is multiplication module if for each sub-module B of T , for some an ideal J of R for which $B = JT$, [2].

Proposition 3.9: Let T be a multiplication e*s-Goldie-Extending module over R then every sub-module of T is a e*s-Goldie-Extending.

Proof: Since every sub-module of a multiplication e*s-Goldie-Extending module is fully invariant, then by prop. {3-8} every sub-module of T is e*s-Goldie-Extending.

Proposition 3.10: Let T be a distributive e*s-Goldie-Extending R - module, then every submodule of T is e*s-Goldie-Extending.

Proof: Let B be a submodule of T and let X be a submodule of B , then there exists a complemented submodule S of T such that $X \xi_{e*s} S$, let $T = S \oplus \bar{S}$, $S \leq T$, T is distributive, then $B = (B \cap S) \oplus (B \cap \bar{S})$, $B \cap S$ so is a direct summand of B , thus B is e*s-Goldie-Extending by Prop. (3.7).

The followings are sufficient conditions to ensure written uniquely as a sum of e*s-Goldie-Extending module is e*s-Goldie-extending.

Proposition 3.11: Let $T = T_1 \oplus T_2$ be a distributive module*s-Goldie-Extending iff B together with A are e*s-Goldie-Extending.

Proof: $\Rightarrow$) Suppose that T is a e*s-Goldie-Extending, since T is a distributive, then by Prop. (3.10) T_1 and T_2 are e*s-Goldie-Extending. $\Leftarrow$) Let T_1 and T_2 be e*s-Goldie-Extending modules, let $B \leq T$, since T is a distributive, then $B = B \cap T = B \cap (T_1 \oplus T_2) = (B \cap T_1) \oplus (B \cap T_2)$, but T_1 and T_2 are e*s-Goldie-Extending. Then there exist a direct summand S of T_1 and adirect summand S of T_2 such that $(B \cap T_1) \xi_{e*s} S$, and $(B \cap T_2) \xi_{e*s} S$ then $B = (B \cap T_1) \oplus (B \cap T_2) \xi_{e*s} (S \oplus S)$ and $S \oplus S$ is a complemented submodule of T , therefore T is Goldie -Extending.

Proposition 3.12: Let $T = \bigoplus_{i \in I} T_i$ be module over R , where T_i are submodule of T , $\forall i \in I$ if T_i is e*s-Goldie-Extending for any $i \in I$ and any e*s- closed submodule of T is fully invariant, then T is e*s-Goldie-Extending.

Proof: Let B denote e*s-closed sub-module of T and $P: T \rightarrow T_i$ be the natural projection on T_i , $\forall i \in I$, assume $x \in B$, $P_i(x) = x_i$, B is a fully invariant so $P_i(B) \leq B \cap T_i$, then $P_i(x) = x_i \in B \cap T_i$ and $x \in \bigoplus_{i \in I} (B \cap T_i)$ but $\bigoplus (B \cap T_i) \leq B$, then $B = \bigoplus B \cap T_i$, $\forall i \in I$. Since $(B \cap T_i) \leq T_i$ and T_i is e*s-Goldie-Extending there exists adirect summand S_i of T_i such that $(B \cap T_i) \xi_{e*s} S_i$ by proposition (2.6), $B = \bigoplus (B \cap T_i) \xi_{e*s} \bigoplus S_i$, $\forall i \in I$, therefore T is e*s-Goldie-Extending.

4. Conclusions

This paper introduces the class of e*s-Goldie-Extending modules as a generalization of e*s-Extending modules. Several equivalent characterizations of this class were established using e*s-closed submodules, complements, and injective hulls. It was shown that, for indecomposable modules, the notions of e*s-Extending, e*s-Goldie-Extending, together with e*s-uniform coincide. Moreover, sufficient conditions were provided under which submodules and direct sums of e*s-Goldie-Extending modules preserve this property. These results provide further insight into of extending-type structures in module theory and open the way for further investigations in this direction.

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