

Endo-greatly bounded modules and module homomorphisms

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Abstract

The class of *modules* is introduced in this paper. We study and discuss the classes for the *module*, named Endo-Greatly Bounded module, which is briefly introduced as (Endo-G.B. *modules*) and *modules* Homomorphism. It also gives some examples and properties. Moreover, we studied relations among this type of *modules* and certain *modules*, such as Prime *modules*, compressible *modules*, monoform *modules*, retractable *modules*, and quasi-Dedekind *modules*. We present results that are proven to explain this relationship. In this paper, compressible and monoform *modules* play a major role in order to obtain Endo-G.B. *modules*. Also, retractable *modules* and a quasi-Dedekind *module* are both related to Endo-G.B. *modules*, we shall show this relationship.

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1. Introduction

Assume S is the *rings*, and it is commutative contains identities & H a unitary (left) S -*modules*. A bounded *module* notion in terms for its annihilator was introduced by Carl Faith in 1976, when S -*module* M named bounded if $\exists r \in M$ s.t $ann_S(r) = ann_S(M)$ [1].

In this paper, we consider several approach to connect these classes of *modules* with others kinds for the *modules*, like as retractable *module*, compressible *module*, critically compressible *module*, and quasi-Dedekind *module*. This work introduces several new conditions and establishes properties and corollaries. This paper is considered another solution to Zelmanowitz's

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problem [1]. Actually, the Endo - G.B. *S-module* plays major role in our work, which we needed for the definitions for the Endo-G.B. *module*. Moreover, an Endo-G.B. *modules* gives us directly three different *modules*: a bounded *module*, a finitely annihilated *module*, and a retractable *module*.

2. Endo - Greatly Bounded *module*

Definition 2.1: The *S-modules* K named Endo-G.B. *modules* where proper's *submodule* for the K be the Endo-G.B., where K called Endo-G.B. *submodule* if $\exists \rho, \gamma \in \text{End}(K)$ s.t $\rho(x) \in Q$ or $\gamma(x) \in Q, x \in K$, implies that $\text{ann}_S((\rho\gamma)(x)) = \text{ann}_S(Q)$.

Example 2.2: Let $H = \mathbb{Z}_4 \oplus \mathbb{Z}_2$ as the $\mathbb{Z}$ -*module*,

Hence, N be the Endo-G.B. *submodule*, H not Endo-G.B. *modul*.

3. Results

Proposition 3.1: If H the compressible *S-module*, so H the Endo-G.B.*S-module*.

Proof: If $\rho, \gamma \in \text{End}(H)$ & defined $\rho, \gamma: H \rightarrow H$ as $\rho(x) = x, \gamma(x) = x, \forall x \in H$. Consider the following Figure 1.

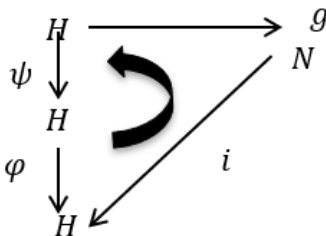


Figure 1
Identity endomorphisms on H .

Since H is a compressible *module*, for all nonzero *submodules* N for the H there is monomorphism $g: H \rightarrow N$ such that $iog = \rho\gamma$ when i be inclusion maps.

Thus $\rho(x) = \rho\gamma(x) = iog(x) = g(x) \in N$. Therefore $\rho(x) \in N$ if $t \in \text{ann}_S((\rho\gamma)(x))$ implies that $t(\rho\gamma(x)) = 0, t\rho(\gamma(x)) = 0$ means that $t\rho(x) = 0$. Then $t\rho(x) = \rho(0) \forall x \in H$ and hence $tg(x) = g(0), \forall x \in H$, therefore $g(tx) = g(0) \forall x \in H$ implies that $tx = 0 \forall x \in H$, since g is a monomorphism. Therefore, $t \in \text{ann}_S(H) = \text{ann}_S(N)$, since H is a prime *module*. Hence, $\text{ann}_S(\rho\gamma(x)) = \text{ann}_S(N)$.

Proposition 3.2: Assume H the multiplication-injective *S-module*. So H be the Endo-G.B. *module* if H be the compressible *module*.

Proof: If H the Endo-G.B. module, so H be the finitely annihilated module [2-3]. H is finitely generated. Thus, we conclude that H be the scalar module [3-4]. Now, let $N < H$, Then N be the scalar S -submodule, thus $\exists \rho, \gamma \in \text{End}(H)$ such that $\rho, \gamma: H \rightarrow H$ defined by $\rho(x) = rx, \gamma(x) = x, \forall x \in N$. Consider the following Figure 2.

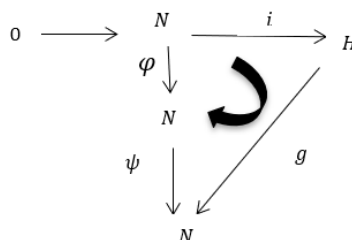


Figure 2
Endomorphisms $\rho, \gamma \in \text{End}(H)$ acting on the submodule $N \subset H$.

Then there exists a S -homomorphism g like that $goi = \rho$. It remains to prove that g is a monomorphism. $(goi) = g(x) = (\rho\gamma)(x)$.

Let $g(x_1) = g(x_2)$, then $(\rho\gamma)(x_1) = (\rho\gamma)(x_2)$ implies that $rx_1 = rx_2$ so $x_1 = x_2$. Therefore, g is an S -monomorphism, and we conclude that H is a compressible module. Conversely, using proposition (3.1).

Next, we discuss relations among monoform modules & Endo-G.B. modules. Also, from the mono form module, we infer that H is prime uniform and $\text{ann}_S(H)$ is a prime ideal of S [5, 6].

Proposition 3.3: Every monoform S -module be the Endo-G.B. S -modules

Proof: Since H is a monoform module, it is a uniform prime module, & each nonzero submodule A for the H be the dense. if $\rho, \gamma \in \text{End}(H)$ & defined $\rho, \gamma: H \rightarrow H$ by $\rho(x) = rx \neq 0, \gamma(x) = x \neq 0, 0 \neq x \in H, 0 \neq r \in S$. Then $rx \in A$, since H is a uniform module. Thus, we have that $\text{ann}_S(A) \subseteq \text{ann}_S(\rho\gamma)(x)$.

If $t \in \text{ann}_S((\rho\gamma)(x))$ implies that $t(rx) = 0$ and $tr(x) = 0$,

But $r \notin \text{ann}_S(x)$ since A is a dense submodule & $\text{Ann}_S(H)$ be the prime ideals for the S , so $t \in \text{ann}_S(x) = \text{ann}_S(H) \subseteq \text{ann}_S(A)$.

Therefore, $\text{ann}_S((\rho\gamma)(x)) = \text{ann}_S(A)$ so H be the Endo-G.B. module.

Proposition 3.4: Assume H the cyclic quasi-Dedekind S -module, so H be the Endo-G.B. S -modules.

Proof: For $\rho, \gamma \in \text{End}(H)$ & $\rho, \gamma: H \rightarrow H$ defined by $\rho(x) = rx, \gamma(x) = x, x \in H$.

if A the proper submodule for the H , so A the cyclic submodule such that $A = Tx$ for some $x \in H$ since H is cyclic. Therefore, $\rho(x) \in A$

implies that $\text{ann}_S(A) \subseteq \text{ann}_S((\rho\gamma)(x))$.

if $t \in \text{ann}_S((\rho\gamma)(x))$ so we have: $t((\rho\gamma)(x)) = 0$, then $t((\rho(\gamma))(x))$

$=0$, and $\rho(tx) = 0$, But H the quasi-Dedekind *module* [5]. So we have $tx = 0$. So we conclude that $t \in \text{ann}_S(x) = \text{ann}_S(H) = \text{ann}_S(A)$.

Proposition 3.5: If H the Endo-G.B. *S-modules*. so for each nonzero submodules N for the H , gotten $\text{Hom}(H, N) \neq 0$.

Proof: Assume N the nonzero submodules for the End-G.B. *modules*, so

$\exists 0 \neq \rho \in \text{End}(H)$ & S -homomorphism $\rho, \gamma: H \rightarrow H$ defined by $\rho(k) = rk, \gamma(k) = k, 0 \neq k \in H$ s.t $\rho(k) \in N$ & $\text{ann}_S((\rho\gamma)(k)) = \text{ann}_S(N)$.

Suppose that $\text{Hom}(H, N) = 0$ and consider the inclusion map $i: N \rightarrow H$.

Then $\rho = i \circ f$ where $f: H \rightarrow N, f \neq 0$.

Therefore, $\rho(k) = (i \circ f)(k) = i(f(k)) = 0$ implies that

$\text{ann}_S(\rho\gamma(k)) \neq \text{ann}_S(N)$ which is a contradiction. Thus $\text{Hom}(H, N) \neq 0$

For each of nonzero submodules for the H .

Proposition 3.6: An Endo-G.B. *S-modules* be the retractable *S-modules*.

Proof: By the previous proposition, we have for any nonzero submodules N for the H , $\text{Hom}(H, N) \neq 0$. Thus, H is a retractable *S-module*.

The next proposition is considered to be an equivalent of proposition (3.2) by using a retractable *module* in the proof.

Proposition 3.7: If H the multiplications torsion-free *S-modules* where

(S be the integrals domain). So, statement in below equivalents:

- 1- H be the Endo-G.B. *modules*.
- 2- H be the compressible *modules*.

Proof: (1) $\Rightarrow$ (2)

If H the Endo-G.B. *modules*, so by [4], H be the scalar *module* and every $\rho \in \text{End}(H)$ is monomorphism.

Now, by the assumption, H is a retractable *module*, then for nonzero submodules N for the H , $\exists$ the nonzero homomorphisms $h: H \rightarrow N$ and let $i: N \rightarrow H$ be an inclusion map. Thus, the composition $i \circ h = \rho$ is a monomorphism, and clearly h is a monomorphism. Thus, H is a compressible *module*.

(2) $\Rightarrow$ (1), By proposition (3.1).

4. Discussion and Conclusion

This article introduces the novel classes for the *S-modules*, which named Endo-Greatly bounded *modules*, and their *module* homomorphisms. We established a behavior for a Endo-G.B. *module* with respect to prime and quasi-Dedekind *modules*, we derive some propositions and corollaries.

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