

Direct-injective and direct-projective semimodules

Khitam Sahib Aljebory *

*General Directorate of Education Babylon
Ministry of Education
Babylon
Iraq*

Rasha Najah Mirza [†]

*Department of Mathematics
Faculty of Computer Science and Mathematics
University of Kufa
Kufa
Iraq*

Abstract

In this paper, new generalizations of injective (projective) semimodules are introduced. Some of its characterizations are discussed, as well as some properties of quasi-injective and quasi-projective semimodules are extended for direct-injective (projective) semimodule. If a semimodule W is named direct-injective symbolize (Di-injective), if for any direct summand V of W , a monomorphism $\phi: V \rightarrow W$, there is a homomorphism $\alpha: W \rightarrow W$ such that $\alpha\phi = i_V$. A semimodule $\wp$ is said to be direct-projective symbolize (Di-projective), if for any direct summand U of $\wp$ with projection $v: \wp \rightarrow U$ and epimorphism $\mu: \wp \rightarrow U$, there is an endomorphism $\alpha: \wp \rightarrow \wp$ such that $\mu\alpha = v$.

Subject Classification: Primary 16D50, Secondary 16L99.

Keywords: Semirings, Semimodules, Injective (projective) semimodules, Quasi-injective (projective) semimodules.

1. Introduction

There are authors who have studied some generalizations of injective and projective modules as [1], etc.; while some others deal with semimodules, some generalizations of injective and projective semimodules were studied by several authors. This paper deals with another generalization of them, the notions direct-injective and direct-projective semimodule introduced, some results of module are expended for semimodule. Also, some characteristics of these notions are

* E-mail: khitamsahib@gmail.com (Corresponding Author)

[†] E-mail: rasha.mirzah@uokufa.edu.iq

discussed. S will be commutative semiring with identity and all semimodule are unitary. To start, a few definitions will be provided. A semiring is a set $S \neq 0$ defined with operations (addition & multiplications) satisfying the conditions as; (1) $(S, +)$ is an abelian semigroup with identity 0_S . (2) $(S, \cdot)$ is semigroup with identity $1_S \neq 0$. (3) $s(s'+s'') = s s' + s s''$ and $(s'+s'') s = s' s + s'' s$; for $s, s', s'' \in S$. (4) $0 s = s 0 = 0, \forall s \in S$ [2]. If the monoid $(S, \cdot)$ is abelian, then a semiring S is abelian. A semiring S is called semi-domain if $s s' = 0$, so either $s = 0$ or $s' = 0$ for $s, s' \in S$ [3]. $(W, +)$ is left S -semimodule if $(W, +)$ is commutative additive semigroup with identity 0_W and a function $S \times W \rightarrow W$ indicated by $(s, w) \mapsto s w$ is called scalar multiplication, where the following roles holds, $\forall s, s', s'' \in S$ and $w, w' \in W$. (1) $(s s') w = s(s' w)$. (2) $s(w + w') = s w + s w'$. (3) $(s + s') w = s w + s' w$. (4) $s 0_W = 0_W = 0_S w$. Where the condition $1 w = w$ is hold for each w in W , for semimodule W , hence W is called unitary [4].

A subsemimodule is a nonempty subset X of an S -semimodule W which satisfied closed under addition and scalar multiplication, symbolized by $X \leq W$ [5]. If a subsemimodule X of W satisfied for $x, y \in W$, and $x + y, x \in X$, then $y \in X$, its called subtractive subsemimodule [6]. W is a subtractive semimodule if all of its subsemimodules are subtractive [7]. We named a semimodule W is semisubtractive, if $t, k \in W$ there is $z \in W$ with $t + z = k$ for $t \in W$ with $k + t = t$ [8]. If w is an element of left S -semimodule W is named cancellable if $w + v = w' + u$ implies $v = u$. If all element of W is cancellable then the S -semimodule W is named cancellative [8].

A semimodule W is named direct sum of sub semimodules X, Y indicated by $W = X \oplus Y$ if for $w \in W$ distinctly expressed as $w = x + y$ for $x \in X, y \in Y$, then X and Y are called direct summand of W , indicated by $X \leq_{\oplus} W$ [4]. A S -semimodule V is identified as W -injective (Z is injective with regard to W) if any S -homomorphism from U to Z can be expanded to V . The semimodule V is considered injective if it is injective with regard to each left S -semimodule [6]. An S -semimodule W is regarded as V -projective, if for each S -homomorphism $\emptyset$ from V onto U where U is any S -semimodule, and each homomorphism δ from W into X , there exists a homomorphism $\theta: W \rightarrow V$ satisfy $\emptyset \theta = \delta$. If it is projective for each S -semimodule A, W is called projective [8]. A simple semimodule is a nonzero semimodule W with no nonzero proper subsemimodule [9]. A semimodule W is semisimple where it is a direct sum simple subsemimodules of W (every subsemimodule is simple) [10].

A semimodule W is an indecomposable if only direct summands of its are $\{0\}$ and W [10]. This paper consists of two parts. Part one dealt with the concept Di-injective. The second part focused on Di-projective for semimodule.

2. Direct-injective semimodule

Definition 2.1: A semimodule W is named direct-injective (Di-injective), if for any direct summand V of W , a monomorphism $\emptyset: V \rightarrow W$, there is a homomorphism $\alpha: W \rightarrow W$ with $\alpha \emptyset = i_V$; where i_V is the inclusion map, as shown in Figure 1.

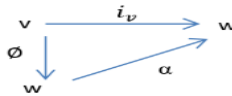


Figure 1

Diagram representing the Di-injectivity condition of a semimodule

Remarks 2.2:

1. Each quasi-injective semimodule is Di-injective semimodule.
2. Each semi-simple semimodule is Di-injective.
3. If W is indecomposable semimodule, then α either identity map or zero map.

Proposition 2.3: Let W be Di-injective semimodule and N is any direct summand of W . Then N is Di-injective also, as shown in Figure 2.

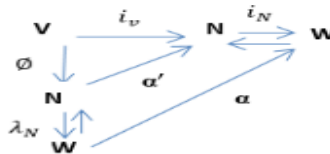


Figure 2

Diagram illustrating that every direct summand of a Di-injective semimodule is Di-injective

Proof: Let $N \leq_{\oplus} W$ where W is Di-injective semimodule this means for $0: V \rightarrow N$ is monomorphism and $i_V: V \rightarrow W$ is the inclusion map, hence there is $\alpha: W \rightarrow W$ such that $\alpha \lambda_N 0 = i_N i_V$, can be defined the projection map from W into N and a homomorphism $\alpha': N \rightarrow N$ such that $\alpha' = \pi_N \alpha \lambda_N$, then $\alpha' 0 = \pi_N \alpha \lambda_N 0 = \pi_N i_N i_V = i_V$. Therefore, N is Di-injective.

Proposition 2.4: Let W be Di-injective and $W \cong V$, then V is Di-injective.

Proof: Assume that W is Di-injective and $\gamma: W \rightarrow V$ an isomorphism. Let N be direct summand of V , by Di-injective of W , for the inclusion map $i_N: N \rightarrow V$ and monomorphism $0: N \rightarrow V$, there is $\alpha: W \rightarrow W$ such that $\alpha \gamma 0 = \gamma i_N$, can be defined the map γ^{-1} from V into W and a homomorphism $\alpha': V \rightarrow V$ such that $\alpha' = \gamma^{-1} \alpha \gamma$, then $\alpha' 0 = \gamma^{-1} (\alpha \gamma 0) = \gamma^{-1} \gamma i_N = i_N$. Therefore, V is Di-injective.

Proposition 2.6: Let W_1 and W_2 be Di-injective semimodules, then $W_1 \oplus W_2$ is Di-injective.

Proof: Assume that W_1 and W_2 is Di-injective semimodules and N is direct summand of W_i and $0: N \rightarrow W_i$ monomorphism for $i=1,2$ since W_i is Di-injective, there is homomorphism $\alpha_i: W_i \rightarrow W_i$ such that $\alpha_i 0 = i_N$, there is $\lambda_i: W_i \rightarrow W_1 \oplus W_2$ which is monomorphism and $\pi_i: W_1 \oplus W_2 \rightarrow W_i$, so defined $\alpha: W_1 \oplus W_2 \rightarrow W_1 \oplus W_2$ by $\alpha_i \pi_i = \alpha$, then $\alpha \lambda_i 0 = \alpha_i \pi_i \lambda_i 0 = \alpha_i 0 = i_N$ hence $W_1 \oplus W_2$ is Di-injective.

Corollary 2.7: Let $\{W_i\}_{i=1}^n$ be a family of semimodules, if W_i is Di-injective semimodules, then $\bigoplus_i W_i$ is Di-injective where $i=1, \dots, n$.

Proposition 2.8: If W is an indecomposable, Di-injective semimodules which is nonsingular, then $\text{End}(W)$ is semi-domain.

Proof: Let $\alpha \neq 0$, $\alpha, \emptyset \in \text{End}(W)$, then $\emptyset(W) \subseteq \ker(\alpha)$. But $\ker(\alpha)$ is closed, hence $\ker(\alpha) \leq_{\oplus} W$, so either $\emptyset = 0$ or $\alpha = 0$, since W is indecomposable.

Proposition 2.9: Assume W is an indecomposable semisubtractive, cancellative Di-injective, for $\varepsilon, \psi \in \text{End}(W) = S$, hence. If $\ker(\varepsilon) \subseteq \ker(\psi)$, then $S\psi \subseteq S\varepsilon$.

Proof: Let $\varepsilon, \psi \in S$ where $\ker(\varepsilon) \subset \ker(\psi)$. Define $\xi: \varepsilon(W) \rightarrow \psi(W)$ by $\xi(\varepsilon(w)) = \psi(w)$ is well define, if $\varepsilon(w) = \varepsilon(v)$ for $w, v \in W$, by semisubtractive of W , $\exists u \in W$ such that either $w = v + u$ or $v = w + u$ if $w = v + u \Rightarrow \varepsilon(w) = \varepsilon(v) + \varepsilon(u)$ by cancellative, $\varepsilon(u) = 0 \Rightarrow u \in \ker(\varepsilon) \subset \ker(\psi) \Rightarrow u \in \ker(\psi) \Rightarrow \psi(w) = \psi(v)$, the same result hold if $v = w + u$. By hypothesis, there is $\rho \in S$ such that $\rho \xi(\varepsilon(w)) = \rho \psi(w) = \rho \psi(w) = i\varepsilon(w) = \varepsilon(w) \forall w \in W$ implies $S\psi \subseteq S\varepsilon$.

3. Direct-projective semimodules

Definition 3.1: A semimodule $\wp$ is named direct-projective symbolize (Di-projective), if for any direct summand U of $\wp$ with projection $v: \wp \rightarrow U$ and epimorphism $\mu: \wp \rightarrow U$, there is an endomorphism $\alpha: \wp \rightarrow \wp$ such that $\mu\alpha = v$. As shown in Figure 3.

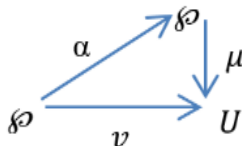


Figure 3

Diagram representing the Di-projectivity condition of a semimodule

Remarks 3.2:

- (1) Every quasi-projective semimodule is Di-projective. In [2], $\mathbb{B}$ as $\mathbb{B}$ -semimodule is Di-projective, where $(\mathbb{B}, +, \cdot)$ is a Boolean semiring.
- (2) If $\wp$ is indecomposable semimodule, then α either the identity or zero map.

Proposition 3.3: Let $\wp$ be semisimple Di-projective semimodule, if there is $\emptyset: \wp/D \rightarrow \wp/C$, then $C \subseteq D$.

Proof: Suppose $\wp$ is Di-projective semimodule. Since $\wp$ is Di-projective, there is $\varphi: \wp \rightarrow \wp$ with $\pi_C \varphi = \alpha \pi_D$ implies $\pi_C \varphi(D) = \alpha \pi_D(C) = 0$ and $\pi_D(C) = \frac{C+D}{D}$, then $\frac{C+D}{D} = 0$ implies $C+D=D$, then $C \subseteq D$, As shown in Figure 4.

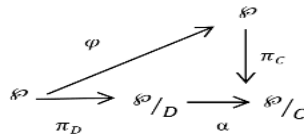


Figure 4

Diagram illustrating the proof step showing that $C \subseteq D/D=0$ and hence $C \subseteq D$ in a Di-projective projective semimodule.

Proposition 3.4: Let $\wp$ be indecomposable Di-projective semimodule, $\varphi \in \text{End}(\wp)$, then there is $\theta \in \text{End}(\wp)$, such that $\text{Hom}(\wp, \wp(\wp))\theta = \varphi \text{End}(\wp)$.

Proof: It is clear that $\delta \text{End}(\wp) \subseteq \text{Hom}(\wp, \wp(\wp))\theta$, let $\beta \in \text{Hom}(\wp, \wp(\wp))$, since $\wp$ is Di-projective, there is $f: \wp \rightarrow \wp$ such that $\delta f = \beta$, take $\theta = 1_\wp$, we have $\delta f = \beta 1_\wp = \beta$ implies $\beta \in \delta \text{End}(\wp)$.

Proposition 3.5: Let $\wp$ be indecomposable Di-projective semimodule, for $\alpha, \beta \in \text{End}(\wp)$ if $\beta(\wp) \subseteq \alpha(\wp)$, then $\beta \text{End}(\wp) \subseteq \alpha \text{End}(\wp)$.

Proof: Since $\beta(\wp) \subseteq \alpha(\wp)$, then $\text{Hom}(\wp, \beta(\wp)) \subseteq \text{Hom}(\wp, \alpha(\wp))$ and since $\wp$ is Di-projective by Proposition (3.4), then $\beta \text{End}(\wp) = \text{Hom}(\wp, \beta(\wp))\theta \subseteq \text{Hom}(\wp, \alpha(\wp))\theta = \alpha \text{End}(\wp)$ for some $\theta \in \text{End}(\wp)$, then $\beta \text{End}(\wp) \subseteq \alpha \text{End}(\wp)$.

Proposition 3.6: Let $\wp$ be Di-projective indecomposable semimodule and $\beta, \alpha \in \text{End}(\wp)$. If $\alpha(\wp)$ is an image of $\beta(\wp)$. Then $\alpha \text{End}(\wp)$ is an image of $\beta \text{End}(\wp)$.

Proof: Let $\vartheta: \beta(\wp) \rightarrow \alpha(\wp)$ be an epimorphism, Since $\wp$ is Di-projective semimodule, hence there exists $\gamma \in \text{End}(\wp)$ such that $\alpha = \vartheta\beta\gamma$. Define $\mu: \beta \text{End}(\wp) \rightarrow \alpha \text{End}(\wp)$ by $\mu(\beta\varphi) = \alpha\varphi$, $\forall \varphi \in \text{End}(\wp)$, μ is well defined since (if $\beta\varphi = \beta\varphi'$, then $\alpha\varphi(p) = \vartheta\beta\gamma\varphi(p) = \alpha\varphi(p) = \vartheta\beta\gamma\varphi'(p) = \alpha\varphi'(p)$) hence μ is well defined. Since $\wp$ is Di-projective semimodule, by proposition (3.5) $\vartheta(\beta(\wp)) \subseteq \alpha(\wp)$ implies $\vartheta\beta \text{End}(\wp) \subseteq \alpha \text{End}(\wp) \Rightarrow \vartheta\beta \text{End}(\wp) \subseteq \alpha\varphi \text{End}(\wp) \subseteq \alpha \text{End}(\wp) \Rightarrow \mu(\beta \text{End}(\wp)) \subseteq \alpha \text{End}(\wp)$, hence μ is an epimorphism.

Proposition 3.7: Let $\wp$ be an indecomposable semimodule, then $\wp$ is Di-projective if and only if for any nonzero $\alpha \in \text{Hom}(\wp, X)$, $\alpha \text{End}(\wp) = \{f: \wp \rightarrow X \mid f(\wp) \subseteq \alpha(\wp)\}$.

Proof: Suppose that $\wp$ is Di-projective semimodule let $0 \neq \alpha \in \text{End}(\wp)$, it is clear $\alpha \text{End}(\wp) \subseteq \{f: \wp \rightarrow X \mid f(\wp) \subseteq \alpha(\wp)\}$. Let $f: \wp \rightarrow X$ and $f(\wp) \subseteq \alpha(\wp)$, since $\wp$ is Di-projective semimodule, then there is $\vartheta \in \text{End}(\wp)$ such that $\alpha\vartheta = f$ then $f \in \alpha \text{End}(\wp)$. Conversely, assume $\alpha \text{End}(\wp) = \{f: \wp \rightarrow X \mid f(\wp) \subseteq \alpha(\wp)\}$. Let $\emptyset: \wp \rightarrow \wp$ be a homomorphism and $\alpha: \wp \rightarrow X$ an epimorphism, since $f: \wp \rightarrow X$ with $\alpha\emptyset = f$.

Conclusion

In this work, the concept of quasi-injective semimodules and quasi-projective have been extended to some generalizations, Di-injective and Di-projective. Also, some characterizations of these notions are discussed. 2.9, 3.5 and 3.6 show some properties of the endomorphism semiring.

References

- [1] R. N. Mirza and T. Y. Ghawi, "Radg-lifting Modules," *Journal of Interdisciplinary Mathematics*, vol. 26, no. 6, pp. 1001–1013 (2023), doi: 10.47974/JIM-1600.
- [2] J. S. Golan, *Semirings and Their Applications*, Dordrecht, The Netherlands: Springer Science Business Media, pp. 382 (1999), doi:10.1007/978-94-015-9333-5.
- [3] T. K. Dutta and M. L. Das, "Singular Radical in Semiring," *Southeast Asian Bulletin of Mathematics*, vol. 34, no. 3, pp. 405–416 (2010).
- [4] A. J. Naeemah and S. A. A-Saadi, "The class of semi-modules with direct summands are stable," *Journal of Discrete Mathematical Sciences and Cryptography*, vol. 28, no. 4-B, pp. 1281–1289 (2025), doi: 10.47974/JDMSC-2266.
- [5] J. N. Chaudhari and D. R. Bonde, "On subtractive extension of sub-semimodules of semimodules," *Journal of the Chungcheong Mathematical Society*, vol. 26, no. 1, pp. 37–44 (2013).
- [6] K. S. H. Aljebory and A. M. Alhossaini, "Almost Injective Semimodules," *Iraqi J. of Sci*, vol. 64, no. 9, pp. 5534–5543 (2023), doi: 10.24996/ij.s.2023.64.9.27.
- [7] R. R. Nazari and S. Ghalandarzadeh, "Content Semimodules," *Extracta Mathematicae*, vol. 32, no. 2, pp. 239–254 (2017).
- [8] K. S. H. Aljebory and A. M. A. Alhossaini, "Almost self-projective semimodules," *Journal of Discrete Mathematical Sciences and Cryptography*, vol. 26, no. 7, pp. 1857–1864 (2023), doi: 10.47974/JDMSC-1675.
- [9] A. J. Naeemah and S. Al-Saadi, "Strongly extending semimodules over a semiring," *Journal of Discrete Mathematical Sciences and Cryptography*, vol. 27, no. 5, pp. 1677–1687 (2024), doi: 10.47974/JDMSC-2009.
- [10] Y. Kastove, T. G. Nam, and N. X. Tuyen, "On Subtractive Semisimple Semirings," *Algebra colloquium*, vol. 16, no. 3, pp. 415–426 (2009), doi: 10.1142/S1005386709000406.

Received January, 2026