

Congruence-injective acts over monoid

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Abstract

This article introduces and studies the concept of congruence-injective S -act over monoids S as a new generalization of injectivity in act theory. This approach is based on congruence and extends the notions of quasi-injective and pseudo-injective S -acts. The main outcome of this study is a characterization theorem for congruence-injective S -acts with respect to perfect congruence. From this result, we obtain a new characterization of quasi-injective acts, which parallels in L. Fuchs's classical result for module theory. These contributions enrich the theory of injectivity in the category of acts.

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1. Introduction

Research in algebra continues to explore generalizations of injectivity across various algebraic categories including the category of acts. Berthiaume

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[1] introduced in the year 1976 a concept of injective acts. Lopez and Luedeman [2] proposed in the year 1979 the quasi-injective act which is an generalization of the injective act. And to this day research continues to explore further generalizations of injectivity as seen in [3], [4], [5], [6]. This motivates to introduce a new generalization called congruence - injective acts which depends on two parameters, this approach facilitates the development of new results that extend to quasi-injective acts. Its characterization can be used to find a characterization of quasi-injective acts, analogous to Fuchs's [7] characterization of quasi-injective modules presented in (Lemma 1).

2. Preliminaries

Let V_S and B be a right S -acts over a monoid S with zeros 0_V and 0_B , respectively, the S -congruence $R_S(a)$ of the S -act is defined by $R_S(a) = \{(t_1, t_2) \in S \times S \mid at_1 = at_2\}$ is the annihilator of element $a \in S$. Define $\text{Mon}_S(B, V) = \{\gamma \mid \gamma : B_S \rightarrow V_S \text{ and } \gamma \text{ is an } S\text{-monomorphism}\}$. Berthiaume [1] introduced a concept of essential (or large) S -subact. A S -subact F of V_S is said to be essential of V_S if and only if for any $\gamma \in \text{Hom}_S(F, U)$ (where U_S any S -acts), the condition $\gamma|_F \in \text{Mon}_S(F, U)$, implies that $\gamma \in \text{Mon}_S(F, U)$. Accordingly, Feller and Gantos [8] introduced a generalization concept of essential S -subact, A nonzero S -subact F of V_S is said to be intersection large (or $\cap$ - large) of V_S if and only if $R \cap F \neq 0_V$ for every nonzero S -subact R of V_S . Moreover if $\gamma \in \text{Hom}_S(B, V)$ then $\gamma^{-1}(G)$ is $\cap$ - large of B for any $\cap$ - large G of V (see Theorem 2.11, [9]). Furthermore (proposition 4.7, [8]) establishes that every essential S -subact is $\cap$ - large. Related to this, Kim and Kim [9] introduced the concept of weakly singular congruence Ψ_V for an S -act, defined as $\Psi_V = \{(e_1, e_2) \in S \times S \mid \text{there exists an } \cap \text{ - large } S\text{-subact } D \text{ of } S \text{ such that } e_1d = e_2d \text{ for all } d \in D\}$. In (theorem 2.17, [9]) it is proved that if $\Psi_V = I_V$, then any $\cap$ - large S -subact of V_S is essential. The set of all S - congruence of V_S , denoted by $\text{Con}_S(V) = \{\Theta \mid \Theta \text{ is a } S \text{ - congruence of } V_S\}$. For a fixed $\Theta \in \text{Con}_S(V)$, we define the subset $\text{Hom}_S^\Theta(B, V)$ of $\text{Hom}_S(B, V)$ as $\text{Hom}_S^\Theta(B, V) = \{\alpha \in \text{Hom}_S(B, V) \mid \ker(\alpha) \subseteq \Theta\}$, For some Θ a subset of (V) of $\text{End}(V)$ is define by $(B) = \{\alpha \in \text{End}(V) \mid \ker(\alpha)\}$. In this article, all S -acts is assumed to be right S -acts with a zero element over a monoid S .

3. Congruence - Injective Acts

Let B_S be S -acts and S -acts $\theta \in \text{Con}_S(B)$.

Definition 3.1: An S -acts V_S is said to be (B, θ) - injective, if for any S -subact G of B and $\alpha \in \text{Hom}_S^\Theta(G, V)$ there exists an extension $\bar{\alpha} \in \text{Hom}_S(B, V)$ to α . If an S -acts V_S is (V, ρ) - injective for some $\rho \in \text{Con}_S(V)$, then V_S is said to be ρ - injective.

Examples and Remarks (3.2).

1. An S -acts V_S is quasi - injective if and only if V_S is Δ_V - injective, where Δ_V is the universal S - congruence ($\Delta_V = V \times V$).

2. An S -acts V is pseudo – injective if and only if V is I_V – injective, where I_M is the identity S – congruence.
3. If V is (B, θ) – injective, then V_S is (B, ω) – injective for any $\omega \in \text{Cons}(V)$ such that $\omega \subseteq \theta$, V_S is $(B, \theta \cap \sigma)$ – injective for any $\sigma \in \text{Cons}(V)$.
4. Any (V, θ) – injective is $(F, \theta|_F)$ – injective for any S -subact F of V_S , where $\theta|_F = \theta \cap \Delta_F$.
5. A retract of (B, θ) – injective S -acts is also (B, θ) – injective.
6. Let $S = \{d, e, 0, 1\}$ be a monoid such that $de = d^2 = d$ and $ed = e^2 = e$. Then S is θ – injective S -acts for any $\theta \in \text{Cons}(S)$.
7. The $\mathbb{Z}$ -act $\mathbb{Z}$ (under the usual multiplication of integers) is not σ – injective S -acts for any $\sigma \in \text{Con}_{\mathbb{Z}}(\mathbb{Z})$.

Definition 3.3: Let F be an S -subact of V_S , and $\theta \in \text{Cons}(V)$, Then, F is said to be θ – **invariant** if for any $\alpha \in \text{End}_S^{\theta}(V)$ implies that $\alpha(F)$ is S -subact of F .

For $\theta \in \text{Cons}(E(B))$ define $H_S = \text{Hom}_S(E(B), E(V))$ and $H_S^{\theta} = \text{Hom}_S^{\theta}(E(B), E(V))$. Recalled in [2]. For an element $w \in V$. $\lambda_v \in \text{Hom}_S(S, V)$ define by $\lambda_v(s) = vs$ for all $s \in S$, and $J(V) = \{v \in S \mid \lambda_v \notin \text{Mon}_S(H, V) \text{ for any nonzero } S\text{-subact } H \text{ of } S_S\}$.

Lemma 3.4: Let $\alpha, \beta \in \text{Hom}_S(B, V)$. Then

- (i) For any $\cap$ - large S -subact G of V_S and any $t \in G$, then the set $[G : t] = \{x \in S \mid tc \in G\}$ is $\cap$ - large S -subact of S -acts S_S .
- (ii) If $\Psi_B = I_B$, and $\alpha|_G = \beta|_G$ for some $\cap$ - large S -subact of B_S , then $\alpha = \beta$.
- (iii) If $J(V) = \emptyset$, and $\bar{G}$ is largest S -subact of B such $\alpha|_{\bar{G}} \in \text{Mon}_S(\bar{G}, V)$. Then $\alpha^{-1}(\alpha(\bar{G}))$ is $\cap$ - large S -subact of B_S .

Proof:

- (i) Let $\gamma \in \text{Hom}_S(S, V)$ be defined by $\gamma(c) = tc$ for all $c \in S$. Then $[G : t] = \gamma^{-1}(G)$ is $\cap$ - large of S -acts S .
- (ii) Let $t \in B$. By (i) we have $[G : t] = \{x \in S \mid tc \in G\}$ is $\cap$ - large S -subact of S . Now, for all $z \in [G : t]$ we have $\alpha(t)z = \alpha(tz) = \beta(tz) = \beta(t)z$. Thus $(\alpha(t), \beta(t)) \in \Psi_V$. Since $\Psi_V = I_V$, this implies $\alpha(t) = \beta(t)$ all $t \in B$.
- (iii) Suppose $T = \alpha^{-1}(\alpha(\bar{G}))$ and it is not $\cap$ - large S -subact of B , then there exist a nonzero S -subact H of B such that $T \cap H = \emptyset$. Consequently, there exists $h \neq 0 \in H$, $\alpha(h) = 0$, that is $\alpha(h) \notin J(V)$. Thus, there exists a nonzero right ideal S of S such that $\lambda_{\alpha(h)} \in \text{Mon}_S(S, B)$. Observe that hS is a nonzero S -subact of H , and since $\bar{G} \subseteq T$, it follows that $\bar{G} \cap hS = \emptyset$. Moreover, $\bar{G} \subseteq \bar{G} \cup hS$. By maximality of $\bar{G}$, it must be the case that $\alpha|_{\bar{G} \cup hS} \notin \text{Mon}_S(\bar{G} \cup hS, V)$. Hence, there exist $w \in \bar{G}$ and $hs \in hS$ for some $s \in S$, such that $w \neq hs$ with $\alpha(w) = \alpha(hs)$. This implies $hs \in \alpha^{-1}(\alpha(\bar{G})) = T$. However, since $\bar{G} \cap hS = \emptyset$, it follows that $hs = 0$ and so $\alpha(w) = 0$ and hence $w = 0$. Which contradicts the assumption that $w \neq hs$. Therefore, T must be a $\cap$ - large S -subact of B .

Proposition 3.5: Let V_S and B_S be S -acts, and let $\Theta \in \text{Con}_S(E(B))$. If V is $(B, \Theta|_B)$ – injective and $\Psi_B = I_B$, then $\alpha(B)$ is S -subact of V for any $\alpha \in H^\Theta$.

Proof: Let $\alpha \in H^\Theta$, and consider $G = \{b \in B \mid \alpha(b) \in V\}$. Define a mapping $\beta : G \rightarrow V$ by $\beta(g) = \alpha(g)$. Clear, β is S -homomorphism. Since $\ker(\beta) \subseteq \ker(\alpha) \subseteq \Theta|_G$, the $(B, \Theta|_G)$ – injectivity of V implies that there exists an extension $\gamma \in \text{Hom}_S(B, V)$ of β . The Injectivity of $E(V)$ ensures the existence of an extension $\delta \in H_S$ of γ . Consequently, we have $\delta(g) = \gamma(g) = \beta(g) = \alpha(g)$ for all $g \in G$. Since V is $\cap$ – large of $E(V)$, Since $\Psi_B = I_B$ by essentiality of B in $E(B)$ get $\Psi_{E(B)} = I_{E(B)}$. It then follows, by (theorem 2.11, lemma 2.4) in [9] that $\alpha^{-1}(V)$ and $L = \alpha^{-1}(V) \cap B$ are $\cap$ – large of $E(B)$. Furthermore, by lemma 2.4 ii, we conclude that $\delta = \alpha$, Hence, $\alpha(B) = \delta(B)$ is an S -subact of V .

Corollary 3.6: Let V_S be $\Theta|_V$ – injective S -acts V for some $\Theta \in \text{Con}_S(E(B))$ and $\Psi_V = I_V$. Then V_S is a Θ – invariant S -subact of $E(V)$.

Corollary 3.7: Let V be an S -act with $\Psi_V = I_V$. Then

- (i) If V_S is quasi – injective, then V_S is Δ_V - invariant S -subact of $E(V)$, Consequently, V is an invariant S -subact of $E(V)$. (Theorem 2.4 , [2]).
- (ii) If V_S is pseud – injective, then V_S is I_V – invariant S -subact of $E(V)$. That is, for any $\alpha \in \text{Mon}_S(E(B), E(V))$ it follows that $\alpha(B)$ is S -subact of $E(V)$.

4. Built a Characterization of Congruence Injectivity

Let V be S -act. For any $\Theta \in \text{Con}_S(V)$ and any $w \in V$. Define $\Theta_S^w = \{(z_1, z_2) \in \Delta_S \mid (wz_1, wz_2) \in \Theta\}$. Now for a right ideal D , define $\Omega_\Theta^D = \{\mu \in \text{Hom}_S(D, V) \mid \exists w \in V \text{ such that } R_S(w)|_D \subseteq \ker(\mu) \subseteq \Theta_S^w\}$.

Definition 4.1: An S – congruence τ of an S -acts V is said to be perfect for V if τ is the upper bound, with respect to inclusion, of the set $\text{RC}(V) = \{ \rho_F \mid F \text{ is cyclic } S\text{-subact of } V \}$, (V denotes the upper bound).

Proposition 4.2: Let Θ be a perfect S – congruence on an S -acts V . If for any right ideal D of S and any $\alpha \in \Omega_\Theta^D$, there exists an extension $\tilde{\alpha} \in \text{Hom}_S(S, V)$, then V is Θ – injective.

Proof: Let F be an S -subact of V and $\mu \in \text{Hom}_S^\Theta(F, V)$. Now consider the collection $\mathcal{F}$ defined by $\mathcal{F} = \{(R, \beta) \mid F \leq R \leq V, \beta \in \text{Hom}_S(R, V) \text{ such that } \beta|_F = \mu\}$. clearly $\mathcal{F}$ is a nonempty ordered set. By Zoren's lemma, $\mathcal{F}$ has a maximal element, denoted by $(\bar{R}, \bar{\beta})$. Now, if $\bar{R} = V$, then proof is complete. Otherwise, let $r_0 \in V$ and $r_0 \notin \bar{R}$. Consider the right ideal $Z = \{x \in S \mid r_0 x \in \bar{R}\}$ and define a $\gamma : Z \rightarrow V$ by $\gamma(z) = \bar{\beta}(r_0 z)$ for all $z \in Z$. Clearly $\gamma \in \text{Hom}_S(Z, V)$. To show that $\gamma \in \Omega_\Theta^Z$, let $(x_1, x_2) \in R_S(r_0)|_Z$ then $\bar{\beta}(r_0 x_1) = \bar{\beta}(r_0 x_2)$, which implies $(x_1, x_2) \in \ker(\gamma)$, and if $(z_1, z_2) \in \ker(\gamma)$, $\bar{\beta}(r_0 z_1) = \bar{\beta}(r_0 z_2)$, which implies $(r_0 z_1, r_0 z_2) \in \rho_{\bar{R}}$, where $\rho_{\bar{R}}$ is the Rees factor congruence of $\bar{R}$. Since $\rho_{\bar{R}} \subseteq \Theta$, it follow that

is $(z_1, z_2) \in \Theta_S^{r_0}$. Thus $\gamma \in \Omega_\Theta^Z$. By hypotheses there exists an extension $\bar{\gamma} \in \text{Hom}_S(S, V)$ to γ . Now, define a mapping $\bar{\beta}_0 : \bar{R} \cup r_0 S \rightarrow V$ by

$$\bar{\beta}_0(r) = \begin{cases} \bar{\beta}(r) & \text{if } r \in \bar{R}, \\ \bar{\gamma}(c) & \text{if } r = r_0 c \text{ with } c \in S \end{cases} \quad (1)$$

To show that $\bar{\beta}_0$ is well-defined, let $\bar{r} = r_0 c \in \bar{R} \cap r_0 S$. Then $c \in Z$, and we have $\bar{\beta}_0(r_0 c) = \bar{\gamma}(c) = \gamma(c) = \bar{\beta}(r_0 c) = \bar{\beta}(\bar{r}) = \bar{\beta}_0(\bar{r})$. Thus $\bar{\beta}_0$ is well-defined. It is straightforward to verify that $\bar{\beta}_0 \in \text{Hom}_S(\bar{R} \cup r_0 S, V)$. Consequently, $(\bar{R}, \bar{\beta}) \leq (\bar{R} \cup r_0 S, \bar{\beta}_0)$, which contradicts the maximality of $(\bar{R}, \bar{\beta})$.

Proposition 4.3: Let Θ be a S – congruence on an S -acts V . If V is Θ – injective, then for any right ideal D of S and any $\alpha \in \Omega_\Theta^D$, there exists an extension $\bar{\alpha} \in \text{Hom}_S(S, V)$ of α .

Proof: Let D be a right ideal of S , and let $\alpha \in \text{Hom}_S(D, V)$ such that $\alpha \in \Omega_\Theta^D$. Then under the given assumption there is $t_0 \in V$ such that $R_S(t_0)|_D \subseteq \ker(\alpha) \subseteq \Theta_S^{t_0}$. Now, let $F = t_0 D$ and define a mapping $\beta : F \rightarrow V$ by $\beta(t_0 d) = \alpha(d)$ for all $d \in D$. To check β is well-defined, suppose $t_0 d_1 = t_0 d_2$ for some $d_1, d_2 \in D$. Then $(d_1, d_2) \in R_S(t_0)|_D$, which implies that $\alpha(d_1) = \alpha(d_2)$. Therefore, $\beta(t_0 d_1) = \beta(t_0 d_2)$, confirming that β is well-defined. Now suppose $(t_0 e_1, t_0 e_2) \in \ker(\beta)$. Then $\beta(t_0 e_1) = \beta(t_0 e_2)$, which implies that $\alpha(e_1) = \alpha(e_2)$. Therefore $(e_1, e_2) \in \ker(\alpha) \subseteq \Theta_S^{t_0}$. Implies that $(t_0 e_1, t_0 e_2) \in \Theta$, so $\ker(\beta) \subseteq \Theta$. By Θ – injectivity of V , there exists an extension $\bar{\beta} \in \text{End}_S(V)$ of β . Now, define a mapping $\bar{\alpha} : S \rightarrow V$ by $\bar{\alpha}(c) = \bar{\beta}(t_0 c)$ for all $c \in S$. It is straightforward to verify that $\bar{\alpha} \in \text{Hom}_S(S, V)$ and that $\bar{\alpha}$ is an extension of α .

Now we can derive an important theorem that provides a characterization of the concept of Θ – injectivity for a perfect S – congruence Θ .

Theorem 4.4: Let $\theta \in \text{Con}_S(V)$ a perfect S – congruence. Then V is θ – injective if and only if every right ideal D of S and every $\alpha \in \Omega_\Theta^D$, there exists an extension $\bar{\alpha} \in \text{Hom}_S(S, V)$ for α .

In theorem 3.4, if we take $\theta = \Delta_V$, obtain a characterization of quasi-injective.

Corollary 4.5: An S -acts V_S is quasi-injective if and only if, every right ideal D of S and any $\alpha \in \Omega_{\Delta_V}^D$, there exists an extension $\bar{\alpha} \in \text{Hom}_S(S, V)$.

Conclusion

This research successfully introduces a novel framework that bridges the gap between quasi-injective and pseudo-injective acts. Driven by two main parameters, the S - act b and the S - congruence θ this concept dynamically interpolates between both extremes. Expanding θ TO universal S -congruence

($= V V$) recovers quasi-injectivity. While minimizing it to the identity S – congruence yields pseudo-injectivity.

This parameter-driven approach has led to the discovery of a completely new equivalent characterization for quasi-injective acts that was previously unknown, offering a powerful new tool for the homological classification.

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