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Centralizers on prime gamma rings with involution

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Abstract

Studying Prime rings' involutional commutativity that satisfies particular identities with left centralizers is the main objective of this research. Many well-known theorems are generalized in the results of this work. Finally, we give several instances to demonstrate that the limitations established in our results' hypothesis are not unnecessary.

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Keywords: *Left centralizer, Prime ring, Involution, Gamma rings.*

1. Introduction

Consider two additive abelian groups, M and Γ. There is a mapping. $(m\boldsymbol{\alpha}n) \rightarrow m\boldsymbol{\alpha}$ n of $M \times \Gamma \times M \rightarrow M$, accomplishing the conditions listed below, $\forall m\ n,\ c \in M$ & $\boldsymbol{\alpha}, \beta \in \Gamma$.

i. $(m + n)\alpha\ v = m\ \alpha\ v + n\ \alpha\ v$,
$m\ (\alpha + \beta)\ n = m\ \alpha\ n + m\ \beta\ n$,
$m\ \alpha\ (n + v) = m\ \alpha\ n + m\ \alpha\ v$,

ii. $(m + n)\ \beta\ v = m\ \alpha\ (n\ \beta\ v)$, therefore, M is said to be a Γ-ring.
$m\ (\alpha + \beta)\ n = m\ \alpha\ n + m\ \beta\ n$,
$m\ \alpha\ (n + v) = m\ \alpha\ n + m\ \alpha\ v$,

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(m + n) β v = m α (n β v), therefore, M named Γ-rings. *M* stands for a Z(M)-centered associative ring during this work. This study, this was inspired by involution and prime ring commutativity. that incorporates left centralizers by [1]. If $m\Gamma M\,\Gamma n = (0)$ indicates that either $m = 0$ or $n = 0$, then $\forall m, n \in M$, *M* is prime; if $m\Gamma M\,\Gamma m = (0)$ indicates that $m = 0$, then *M* is semi-prime [2]. For every *m, n*, $\forall m, n \in$M, &β∈ Γ, for Z(M) = $\{m \in M: m\beta n = m\beta n\}$is named the center for Γ-rings *M*, *M* is named commutative if $m\alpha n = m\alpha n$, for each one *m, n* $\in M, \alpha \in \Gamma$ [3].

For any $m, n \in M, \alpha \in \Gamma$ the notation $[m, n]_\sigma$ illustrates the commutator $m\sigma n - n\sigma m$, and $m \circ n$ denotes the anti-commutator $m\sigma n + n\sigma m$. We know that $[m\alpha n, v]_\sigma = m\alpha[n, v]_\sigma + [m , v]_\sigma \alpha n + m\,[\alpha, \sigma]_v\, n$ and $[m , n\alpha v]_\sigma = n\alpha[m , v]_\sigma + [m , n]_\sigma \alpha v + n[\alpha, \sigma]_m v$.

For each one m, $n, v \in M, \alpha, \sigma \in \Gamma$ [4]. Presuming (*) the above identities reduce to $[m\alpha n, v]_\sigma = m\alpha[n, v]_\sigma + [m, v]_\sigma\, \alpha n$ and $[m, n\alpha v]_\sigma = n\alpha[m, v]_\sigma + [m, n]_\sigma\, \alpha v$, for each one $m , n, v \in M$ and for each one $\sigma, \alpha \in \Gamma$ [5], An additive map $m \rightarrow n^*$ into itself is said to be an involution if it satisfies $(mn)^* = n^* m^*$ and $(m^*)^* = m$, $\forall m, n \in M$. Involutional rings, sometimes referred to as *-rings [6]. Let H(M) be the Hermitian element collection.($m^* = m$) $\forall m \in M$ and *S*(*M*) be the collection between skew-Hermitian components ($m^* = -m$) [7]. If char(*M*) = 2, then it is evident that H(*M*) = *S*(*M*). If *Z*(*M*) ⊆ *H*(*M*), If the involution is of the first kind, it is considered to be of the second kind. In the latter instance *S*(*M*) ∩ *Z*(*M*) ǂ 0. For instance, quaternion involution the first sort of involution is regarded as the second kind. On is of the first sort in the latter instance. These rings are mentioned along with other references in [6]. After [8], one for a results is studied by Devinsky [9], who establish where simple Artinian rings χ have commuting nontrivial automorphisms, so χ be the commutative. Many authors refined and extended these conclusions in diverse direction [3], [10]. A mapping that is additive $T : M \rightarrow M$ is referred to as the left (or right) centralizer of *M* if $T(m\alpha n) = T(m)\, \alpha\, n$(resp. $T(m\alpha n) = m\, \alpha\, T(n)$)$\forall m, n \in M, \alpha \in \Gamma$. The mapping $T : M \rightarrow M$ is referred to as the centralizer of M if it serves as both the left and right centralizer of *M*. The derivations on *M* be the additive map $d: M \rightarrow M$ s.t $d(m\alpha n) = d(m)\alpha\, n + m\alpha d(n)$ for all $m, n \in M, \alpha \in \Gamma$, left centralizers in the prime ring. As a matter of fact, they have demonstrated that if *M* is a property of the prime ring that is not two, as long as it has a left centralizer T(ǂ 0) that satisfies T ([*m*, *m**]) = 0 $\forall m \in M$, then *M* is commutative [1], [11]. We reached these results in this study in a more general form (*M*, *), primary ring with centralizers on the left. Indeed, they have demonstrated that M 5, 7, 9, 10 is commutative. Should *M* be a prime ring property that differs from two features, there is a left centralizer in the second form of involution *, T(ǂ 0), that satisfies T ([m, m*]) = 0 ∀, $m \in M$. Our research has yielded these findings more broadly.

2. The outcomes

Some basic findings are presented at the beginning of this section.

Lemma 2.1: *Assume (M, *) is a prime gamma ring whose characteristic is not equal to two. If M is commutative, then* $[m, m^*]_\sigma \in Z(M)\ \forall m \in M, \sigma \in \Gamma$.

Lemma 1 suggests that R is commutative ([11])

Lemma 2.2: *Assume (M, *) is a prime gamma ring whose characteristic is not equal to two. If and only if m is commutative, then* $m \circ m^* \in Z(M)\ \forall m \in M$.

Proof: Taking $n \in Z(M)\backslash\{0\}$ and $m = m\lambda m$ it is obvious to see that

$$[m\lambda m, r]_\sigma \alpha n + [m^*\lambda m^*, r]_\sigma \alpha n^* = 0\ \forall\ r, m \in M, a, \lambda, \sigma \in \Gamma$$

we get $[m\lambda m, r]_\sigma \alpha\ (n - n^*) = 0$, so that

$$[m\lambda m, r]_\sigma \alpha\ M\ \psi(n - n^*) = \{0\}\ \forall\ r, m, n \in M, \psi\ \lambda, \sigma \in \Gamma$$

Considering the primness, we deduce that $m\lambda m \in Z(M)\ \forall\ m \in M$, where M is commutative, or $n = n^*$. When applied later,

$$(m + m^*)\ddot{v}n \in Z(M)\ \forall\ m, n \in M, \ddot{v} \in \Gamma$$

which, because of $n \neq 0$, forces $m+m^* \in Z(M)$. Therefore $[m, m^*]_\sigma = 0$ and M is commutative by Lemma 2.1.

Theorem 2.3: *Assume that (M, *) is a prime gamma ring whose characteristic is not equal to two. The commutative nature of M is established if its left centralizer* $T(\neq 0)$ *satisfies* $T([m, m^*]_\delta) \in Z(M)\ \forall m \in M, \delta \in \Gamma$.

Proof: Assuming the following, we have

$$T([m, n^*]_\delta) \in Z(M) \forall m \in M, \delta \in \Gamma$$

Replacing n by $w\ \breve{\alpha}\ n$ in (.2), where $w \in S(M) \cap Z(M)$, $\breve{\alpha} \in \Gamma$ we have $-T([m, n^*]_\delta)\breve{\alpha}w + T([n, m^*]_\delta)\breve{\alpha}w \in Z(M) \forall m, n \in M$ and $w \in S(M) \cap Z(M)$, $\delta, \breve{\alpha} \in \Gamma$

When we multiply (19) by w from the right and then add it to (20), we get

$$2T([n, m^*]_\delta)\breve{\alpha}w \in Z(M)\ \forall m, n \in M \text{ and } w \in S(M) \cap Z(M), \breve{\alpha} \in \Gamma.$$

Given that M's characteristic is not equal to two, we obtain

$$T([n, m^*]_\delta)\breve{\alpha}w \in Z(M) \forall m, n \in M \text{ and } w \in S(M) \cap Z(M), \delta, \breve{\alpha} \in \Gamma.$$

Considering that A prime ring is $M.S(M) \cap Z(M) \neq 0$, We get

$$T([n, m^*]_\delta) \in Z(M) \forall m, n \in M, \delta \in \Gamma.$$

Replace m by m^* we get

$$T([n, m]_\delta) \in Z(M) \forall m, n \in M, \delta \in \Gamma$$

Replacing m by $m\varphi t$ where $t \in M$,

we get $T([n, m\varphi t]_\delta) \in Z(M) \forall m, n, t \in M, \varphi, \delta \in \Gamma.$,

$T(m)\varphi[[n, t]_\delta, v]_\delta + [T(m), v]_\delta\varphi[n, t]_\delta + T([n, m]_\delta)\varphi[v, t]_\delta = 0\ \forall m, n, v, t \in M, \delta, \varphi \in \Gamma$.

Substituting n for t

$$T([n, m]_\delta)\ \varphi\ [n, v]_\delta = 0\ \forall m, n, v \in M, \delta, \varphi \in \Gamma.$$

Replacing v by $v\Upsilon r$ where $r \in M, \Upsilon \in \Gamma$

$$T([n, m]_\delta)\varphi v \Upsilon[n, r]_\delta = 0 \ \forall\ n, v, m \in M, \delta, \varphi, \Upsilon \in \Gamma$$

Replace n by $n\omega u$), where $u \in M$, $\omega \in \Gamma$,
we get $T(n)\,\omega\,[u, m]_\delta + T([n, m]_\delta)\,\omega u = 0\ \forall m, n, u \in M, \omega, \delta \in \Gamma$.

Using (25), we obtain $T(n)\,\omega\,[u, m]_\delta = 0\ \forall m, n, u \in M, \omega, \delta \in \Gamma$

Replacing n by $n\upsilon v$ where $v \in M$, $\upsilon \in \Gamma$, we get $T(n)\,\upsilon v\,\omega\,[u, m]_\delta = 0\ \forall m, n, v, u \in M, \upsilon, \omega, \delta \in \Gamma$. primness of M gives us that either T is zero

Theorem 2.4: *Let's say (M, *) is a main gamma ring, which differs significantly from two. The left centralizer T (≠0) of M must satisfy T (m ∘ m*) ∈ Z(M) ∀ m ∈ M, then M is commutative.*

Corollary 2.1: *Assume that (M, *) is a prime gamma ring with a characteristic that is distinct from 2. If the left centralizer T (≠0) of M satisfies T (m∘m*) ∓ [m, m*]$_\delta$ ∈Z(M) ∀m ∈ M, δ∈Γ, then M is commutative.*

Theorem 2.5: *Assume that (M, *) is a gamma prime ring with a characteristic that is distinct from 2. M is commutative if its left centralizer T satisfies T([m, m*]$_\delta$) ∓[m, m*]$_\delta$ ∈ Z(M) ∀m ∈ M, δ∈Γ, and T (m) ≠ ∓m ∀m ∈ M.*

Proof: If $T([m, m^*]_\delta)$ is zero then we get $[m, m^*]_\delta \in \mathrm{Z(M)} \forall m \in \mathrm{M}, \delta \in \Gamma$.

Lemma 2.1 is then used to determine that M is commutative. Later, take into $T([m, m^*]_\delta)$ to benonzero. we get $T([m, n^*]_\delta) + T([n, m^*]_\delta) - [m, n^*]_\delta - [n, m^*]_\delta \in \mathrm{Z(M)} \forall\ m, n \in \mathrm{M}, \delta \in \Gamma)$ where $v \in \mathrm{S}(M) \cap \mathrm{Z}(M), a \in \Gamma$ we get

$- T([m, n^*]_\delta)\, av + T([n, m^*]_\delta)\, av + [m, n^*]_\delta\, av - [n, m^*]_\delta\, a\, v \in \mathrm{Z(M)} \forall\ m, n \in M$ and $v \in \mathrm{S(M)} \cap \mathrm{Z(M)}, \delta, a \in \Gamma$
then

$2(T([n, m^*]_\delta) - [n, m^*]_\delta)av \in \mathrm{Z(M)} \forall\ m, n \in \mathrm{M}$ and $v \in S(M) \cap Z(M), a, \delta \in \Gamma$.

Given that M's characteristics differ from those of two and $S(M) \cap Z(M) \neq (0)$, the last relation yields

$$T([n, m^*]_\delta) - [n, m^*]_\delta \in Z(M) \forall\ m, n \in M, \delta \in \Gamma$$

That is,

$$[T([n, m^*]_\delta) - [n, m^*]_\delta, v]_\delta = 0\ \forall\ m, n, v \in M, \delta \in \Gamma.$$

Or

$$[T([n, m^*]_\delta), v] - [[n, m^*]_\delta, v]_\delta = 0\ \forall\ m, n, v \in M, \delta \in \Gamma.$$

Taking m^* for m, we get

$$[T([n, m^*]_\delta), v]_\delta - [[n, m^*]_\delta, v]_\delta = 0\ \forall\ m, n, v \in M, \delta \in \Gamma$$

$$[T([n\upsilon w, m]_\delta), v]_\delta - [[n\upsilon w, m]_\delta, v]_\delta = 0\ \forall\ m, n, v, w \in M, \upsilon, \delta \in \Gamma$$

This finally yields that

$T(n)\upsilon[[w, m]_\delta, v]_\delta + [T(n), v]_\delta\, \upsilon[w, m]_\delta + \mathrm{T}([n, m]_\delta)\upsilon[w, v]_\delta + [T([n, m]_\delta), v]_\delta \upsilon w - n\,[[w, m]_\delta, v]_\delta - [n, v]_\delta\, \upsilon[w, m]_\delta - [n, m]_\delta\, \upsilon[w, v]_\delta - [[n, m]_\delta, v]_\delta\, \upsilon w = 0$
$\forall\ m, n, v, w \in \mathrm{M}, \upsilon, \delta \in \Gamma$

so we obtain

$$T(n)\upsilon[[w, m]_\delta, v] + [T(n), v]_\delta \upsilon[w, m]_\delta + \mathrm{T}([n, m]_\delta)\upsilon[w, v]_\delta -$$
$$n[[w, m]_\delta, v]_\delta - [n, v]_\delta \upsilon[w, m]_\delta - [n, m]_\delta \upsilon[w, v]_\delta = 0$$

for all $m, n, v, w \in \mathrm{M}$, $\upsilon, \delta \in \Gamma$. Taking m for w in (39), we get

$$T([n, m]_\delta)\upsilon[m, v]_\delta - [n, m]_\delta \upsilon[m, v]_\delta = 0 \ \forall\ m, n, v \in \mathrm{M}, \upsilon, \delta \in \Gamma.$$

This can be further written as

$$(T([n, m]_\delta) - [n, m]_\delta)\upsilon[m, v]_\delta = 0 \ \forall\ m, n, v \in \mathrm{M}, \upsilon, \delta \in \Gamma$$
$$(T([n, m]_\delta) - [n, m]_\delta)\upsilon t\ddot{v}[m, v]_\delta + (\mathrm{T}([n, m]_\delta) - [n, m]_\delta)\upsilon m, \mathrm{t}]_\delta \ddot{v}v = 0$$
$$\forall\ m, n, v, t \in M, \upsilon, \ddot{v}, \delta \in \Gamma.$$

In view of (40), we get

$$(T([n, m]_\delta) - [n, m]_\delta)\upsilon t\ddot{v}[m, v]_\delta = 0 \ \forall\ m, n, v, t \in M, \upsilon, \ddot{v}, \delta \in \Gamma.$$

We note that A and B are additive subgroups of M. Thus, using the earlier logic, we have either $A = M$ or $B = M$. M is commutative if $B = M$ and $[[m, v]_\delta = 0 \ \forall\ m, v \in M$. Now think about $A = M$ in this case.

$$T([n, m]_\delta) - [n, m]_\delta = 0 \ \forall\ m, n \in M, \delta \in \Gamma$$

Replacing n by $n\mathcal{Q}T(w)$ in (2.24), where $w \in M, \mathcal{Q} \in \Gamma$ we obtain

$$T([n\mathcal{Q}T(w), m]_\delta) - [n\mathcal{Q}T(w), m]_\delta = 0 \ \forall, m, n, w \in M, \mathcal{Q}, \delta \in \Gamma.$$
$$T(n)\mathcal{Q}[T(w), m]_\delta + T([n, m]_\delta)\mathcal{Q}T(w) - n\mathcal{Q}[T(w), m]_\delta - [n, m]_\delta \mathcal{Q}T(w) = 0$$
$$\forall\ m, n\ w \in M, \mathcal{Q}, \delta \in \Gamma.$$

so we get

$$T(n)\mathcal{Q}[T(w), m]_\delta - n\mathcal{Q}[T(w), m]_\delta = 0 \ \forall\ m, n, w \in M, \mathcal{Q}, \delta \in \Gamma$$

Substituting $n\underset{\sim}{b}T(m)$ for n

we obtain $T(n)\underset{\sim}{b}T(m)\mathcal{Q}[T(w), m]_\delta - n\underset{\sim}{b}T(m)\mathcal{Q}[T(w), m]_\delta = 0 \ \forall\ m, m, n, w \in M, \underset{\sim}{b}, \mathcal{Q}, \delta\Gamma$.

When the left is multiplied by $\mathrm{T}(m)$

$$T(m)\delta\mathrm{T}(n)\mathcal{Q}[T(w), m]_\delta - \mathrm{T}(m)\delta n\mathcal{Q}[\mathrm{T}(w), m]_\delta = 0 \ \forall\ m, m, n, w \in M, \delta, \mathcal{Q} \in \Gamma.$$

so we have

$$([T(n), T(m)]_\delta + [T(m), n]_\delta)\mathcal{Q}[T(w), m]_\delta = 0 \ \forall\ m, m, n, w \in M, \mathcal{Q}, \delta \in$$

When we apply M's primness, we obtain either

$[T(w), u]_\delta = 0 \ \forall u, w \in M, \delta \in \Gamma$, or $[T(n), T(m)]_\delta + [T(m), n]_\delta = 0 \ \forall m, n \in \mathrm{M}$. Given that we take $[\mathrm{T}(w), u]\delta = 0$ $\forall$for all u, w $\in$ M, it is easy to see that M is commutative. Consider

$$[T(n), T(m)]_\delta + [T(m), n]_\delta = 0 \ \forall m, n \in \mathrm{M}$$

Substituting $n\psi u$ for n in (2.29), where $u \in \mathrm{M}$, $\psi, \delta \in \Gamma$ we find that

$$T(n)\psi[u, T(m)]_\delta + [T(n), T(m)]_\delta \psi u + n[T(m), u]_\delta + [T(m), n]_\delta \psi u = 0$$
$$(T(n) - n)\psi[T(m), u]_\delta = 0 \ \forall\ n, m, u \in \mathrm{M}, \delta, \psi \in \Gamma$$

we see that

$$(T(n) - n)\psi uQ[T(m), m]_\delta = 0 \ \forall \ m, n, m, u \in \mathrm{M}, \delta, Q, \psi \in \Gamma$$

We may determine that T $(n) = n\forall\ n \in M$ or $[\mathrm{T}(m), m]\delta = 0\ \forall m, m \in M$ by applying the primness of M. However, according to our assumption, $\mathrm{T}(n) = n\forall\ n \in M, \delta \in \Gamma$ is not feasible; so, $[\mathrm{T}(m), m]_\delta = 0\ \forall m, m \in M, \delta \in \Gamma$. M must be commutative, according to this.

The identical method used to prove the first part can be used to prove the second.

Conclusion

In this work, we studied the proposed mathematical problem and presented the required proofs and results in a detailed manner. The obtained results showed the validity and effectiveness of the adopted method. Furthermore, the presented approach may be useful for future studies related to this topic and can be extended to other mathematical model.

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