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Annihilator large- superfluous, endomorphism small-essential homomorphisms and Morita duality

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Abstract

The dual notions of endomorphism small-essential submodules (ESE) and annihilator large-superfluous submodules (ALS) are suggested and justified. Using these notions, we present a general characterization of generalized Hopfian and weakly co-Hopfian couples of modules and show that they are dual to each other under Morita duality.

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1. Introduction

In the main, the notation D represents an associative ring with unit element, Y a right unitary module over D, and T the ring of endomorphisms of Y. We put out all notions, notations, and past work conclusions freely [1-12].

It is well-known that the concepts of superfluous (small) and large (essential) submodules are dual in Module (Semi module) theory [1-2].

In [3], the authors connect these duplex concepts. In fact, they operate superfluous (resp. large) submodules by large (resp. Superfluous) once. A submodule V of Y is large-superfluous (for short, LS), denoted by $\boldsymbol{V} \ll_{\boldsymbol{L}} \boldsymbol{Y}$, if for each essential submodule K of Y, V+K=Y hints that K=Y, on the contrary, a submodule V of Y is small-essential (for short SE), denoted by $\boldsymbol{V} \trianglelefteq_{\boldsymbol{S}} \boldsymbol{Y}$, if for each small submodule K of Y, V$\cap$ $\boldsymbol{K}$=0 hints that K=0. An epimorphism (resp. a monomorphism) $\boldsymbol{\phi}: \boldsymbol{Y} \rightarrow \boldsymbol{X}$ is called small (resp essential) if Ker ($\boldsymbol{\phi}$) (resp. Im($\boldsymbol{\phi}$)) is small (resp. essential) in Y (resp. X). These notions were regarding semi modules [4].

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As a generalization of superfluous submodules, the authors in [5] studied the notion of annihilator superfluous submodules. A submodule V of Y is called annihilator superfluous if for any submodule K of Y, V+K=Y hints that $\boldsymbol{\ell}_{\mathbf{T}}(\mathbf{K}) = \mathbf{0}.$ The author considers its dual notion in [6], A submodule V of Y is called T-essential, if for each submodule K of Y ,V∩ $\boldsymbol{K}$=0 hints that $(\boldsymbol{K}:_{\boldsymbol{T}} \quad \boldsymbol{Y} \quad) = \{\boldsymbol{f} \in$T | f(Y) ⊆K }=0. More commonly, the author in [5], [8] introduced and studied the notions of annihilator, large superfluous submodules, and Endo -essential submodules relative to small submodules. A submodule V of Y is an annihilator, large, and superfluous. (for short ALS), denoted by $\boldsymbol{V} \ll_{\boldsymbol{AL}} \boldsymbol{Y}$, if V+K=Y hints that$\boldsymbol{\ell}_{\mathbf{T}}(\mathbf{K}) = \mathbf{0}$ for all large submodules K of Y. While V is called Endo small essential (for short ESE), denoted by $\trianglelefteq_{\boldsymbol{ES}} \boldsymbol{Y}$, if for each small submodule K of Y, V∩K=0 hints that $(\boldsymbol{V}: \quad _{\boldsymbol{T}} \quad \boldsymbol{Y} \quad) = \mathbf{0}$.

In [9], [12], a module Y is a generalized Hopfian (for short, gH) if every surjective endomorphism of Y has a small kernel. In the dual case, a module Y is weakly co-Hopfian (for short, wcH) if every injective endomorphism of Y has an essential image.

In [3], the authors call a pair (M, N) of D-modules M and N, where e-gH is the set of all essential small epimorphisms from M to N. Dually, a pair (M, N) is s-wcH if every monomorphism from M to N is small-essential.

In section two, we introduce ESE and ALS submodules and homomorphisms using the notions of ESE and ALS submodules. An epimorphism θ between two modules Y and X is called ALS if θ has an ALS kernel, and a monomorphism θ is called ESE if $\boldsymbol{lm}(\boldsymbol{\theta})$ is ESE submodule. First, we give a characterization of such homomorphisms and study some of their properties under pullback and pushout. It is proved that ALS-epimorphisms and ESE homomorphisms under Morita duality are dual to each other.

These notions have been used in section three as an application. We suggest ALS-gH (resp.ESE-wcH) as a generalization of both e-gH and gH (resp. s-wcH and wcH). We consider assertions under which a couple of modules satisfy one of ALS-gH or ESE-wcH; then, their duals satisfy the other. Selectively, if $(\mathbf{D_2},\mathbf{D_1})$ -bimodule V defines Morita duality with two right V-reflexive $\boldsymbol{D_1}$-module Y and X, then a couple (Y, X) is ESE-wcH (resp. ALS-gH) if and only if $(\boldsymbol{X}^*, \boldsymbol{Y}^*)$ is ALS-gH (resp. ESE-wcH).

2. ESE and ALS homomorphisms

Definition 2.1: Presume that A and B are two D-modules. Then An epimorphism $\theta: A \to B$ is called annihilator large- superfluous (for short, ALS), if $\ker(\theta) \ll_{AL} A$.

A monomorphism $\phi: A \to B$ is called endo small-essential (shortly, ESE), if $lm(\phi) \trianglelefteq_{ES} B$.

Proposition 2.2: Presume that A and B are D-modules, then

1. An epimorphism $\theta: A \to B$ is ALS if and only if for each essential monomorphism h, if θoh is an epimorphism, then $\ell_T(lm(h)) = 0$, where $T = End(A)$.
2. A monomorphism $\phi: A \to B$ is ESE if and only if for each small epimorphism h, if $ho\phi$ is monomorphism, then $\left(Ker(h):_{T'} B\ \right) = 0$ where $T' = End(B)$.

Proof:

(1) First of all, presume that $\theta: A \to B$ is an epimorphism and $K = \ker(\theta)$. Then there is a unique isomorphism $u: A/K \to B$ with $uo\pi = \theta$ where $\pi: A \to A/K$ is the natural epimorphism. Hence, it follows that for each homomorphism ω, $uo\pi o\omega = \theta o\omega$ is an epimorphism if and only if $\pi o\omega$ is an epimorphism.

If θ is an ALS epimorphism, then $K \ll_{AL} A$. Let h be an essential monomorphism. Then $lm(h) \leq_e A$.

If θ oh is an epimorphism, then πoh is epimorphism and hence $lm(h) + K = A$, thus $\ell_T(lm(h)) = 0$. Conversely, let $L \leq_e A$ with $K + L = A$ and $i_L: L \to A$ be the inclusion map. Then i_L is essential monomorphism, so πoi_L is an epimorphism. $\ell_T(L) = \ell_T(lm(i_L)) = 0$. This shows that θ is ALS.

(2) Let $\phi: A \to B$ be a monomorphism and $W = lm(\phi)$. Then there is an isomorphism $\omega: W \to W'$ for some submodule W' of B with $i_{w'}0\omega = \phi$ where $i_{w'}: W' \to B$ is the inclusion map. It follows that for each homomorphism h, $hoi_{w'}o\omega = ho\phi$ is monomorphism if and only if $hoi_{w'}$ is monomorphism. Presume that $\phi: A \to B$ is ESE monomorphism. Let h be a small epimorphism. Then $\ker(h) \ll B$. If $ho\phi$ is monomorphism then $hoi_{w'}$ is momomorphism and hence $\ker(h) \cap W = 0$. Thus $(\ker(h) :_T A) = 0$. For the converse direction, let $L \ll B$ with $W \cap L = 0$. $\pi: B \to B/L$ is a small epimorphism, so $\pi oi_{w'}$ is monomorphism, then $(\ker(\pi): T'B)) = 0 = (L: T'B)$ and thus ϕ is ESE.

Proposition 2.3: Regarding the following commutative Figure 1 of modules and homomorphisms with exact rows.

Let α be an epimorphism and β preserve ALS submodules. If ϕ is ALS, then so is $\phi'2$. Let γ be a monomorphism and the inverse image of β preserves ESE submodules. If θ' is ESE, then so is θ.

$$\begin{tikzcd}
O \arrow[r] & A \arrow[r, "\theta"] \arrow[d, "\alpha"'] & B \arrow[r, "\phi"] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"] & O \\
O \arrow[r] & A' \arrow[r, "\theta'"'] & B' \arrow[r, "\phi'"'] & C' \arrow[r] & O
\end{tikzcd}$$

Figure 1
Commutative of modules and homomorphisms

Proof:

(1) Presume that ϕ is ALS, then $Ker(\phi) \ll_{AL} B$. By the hypothesis, $B\big(Ker(\phi)\big) \ll_{AL} B'$. We see that $Ker(\phi') \subseteq \beta\big(Ker(\phi)\big)$ and hence ϕ' is ALS, ([7], lemma 2.3). Let $b' \in Ker(\phi')$. The exactness of the bottom row hints that there is $a \in A$ such that $\alpha(a) = b'$. By the commutativity of the diagram and the exactness of the top row, for the element $a \in A$, $\theta' o\alpha(a) = b' = \beta\big(\theta(a)\big)$ and $(\phi o\theta)(a) = 0$. Thus $\theta(a) \in Ker(\phi)$ with $\beta(\theta(a) = b'$ and hence $b' \in \beta(Ker(\phi))$.

(2) Presume that θ' is ESE, then $lm(\theta') \trianglelefteq_{ES} B'$ and hence, by the hypothesis, $\beta^{-1}\big(lm(\theta')\big) \trianglelefteq_{ES} B$. We claim that $\beta^{-1}(lm(\theta')) \subseteq lm(\theta)$, and hence, θ is ESE, ([8], lemma 2.3). if $b \in \beta^{-1}(lm(\theta'))$, then $\beta(b) \in lm(\theta') = Ker(\phi')$. There is $b' \in B$ with $\beta(b) = b'$ and $\phi'(b') = 0$. Hence $b' \in lm(\theta)$ and $0 = (\phi' o\beta)(b) = (\gamma o\phi)(b)$ and so $\phi(b) = 0$ which hints that $b \in Ker(\phi) = lm(\theta)$.

Corollary 2.4: *Regarding Figure 2 below:*

$$\begin{tikzcd} A \arrow[r, "\theta_1"] \arrow[d, "\theta_2"'] & B \arrow[d, "\phi_2"] \\ C \arrow[r, "\phi_1"'] & D \end{tikzcd}$$

Figure 2
Preservation of ESE monomorphisms under pullbacks and ALS epimorphisms under pushouts.

Then

1. *Presume that the diagram is a pullback of ϕ_1 and ϕ_2 and the inverse image of ϕ_2 preserves ESE submodules. If ϕ_1 is ESE monomorphism, then so is θ_1.*
2. *Presume that Figure 3 is a pushout of θ_1 and θ_2 and ϕ_2 preserves ALS submodules. If θ_1 is an ALS epimorphism, then so is ϕ_1.*

Proof: We give the proof of (1) and that of (2) by dual manner. Suppose that the diagram is a pullback of ϕ_1 and ϕ_2. By Proposition 5.1 in [10], we have the following commutative diagram with the two rows exact.

$$\begin{tikzcd} O \arrow[r] & A \arrow[r, "\theta_1"] \arrow[d, "\theta_2"'] & B \arrow[r, "\pi_1"] \arrow[d, "\phi_2"] & L \arrow[r] \arrow[d, shift right] & O \\ O \arrow[r] & C \arrow[r, "\phi_1"'] & D \arrow[r, "\pi_2"'] & L \arrow[r] \arrow[u, shift right] & O \end{tikzcd}$$

Figure 3
Pushout of $\boldsymbol{\theta_1}$ and $\boldsymbol{\theta_2}$ and $\boldsymbol{\phi_2}$

by proposition (2.3), θ_1 is ESE monomorphism.

Two categories C and D are called dual (to each other), if there are contravariant factors $F: C \to D$ and $G: D \to C$ with functorial isomorphisms $GF \simeq id_C$ and $FG \simeq id_D$. The functors F and G are called dualities. G is called the inverse of F.

Let D_1 and D_2 be two rings and $C \subseteq D_1 - MOD$, $D \subseteq MOD - D_2$ full subcategories which is closed under finite products, sub-and factor modules. Presume that $F: C \to D$ is a duality with inverse $G: D \to C$. In case $S \in D$, there is a D_2-module $V \in C$ with $S = End(V)$ and satisfy $F \simeq Hom_{D2}(-, \ {}_{D2}V \ )$ and $G \simeq Hom_{D1}(-, D_{D1})$, ${}_{D2}V$ is an injective cogenerator in σ [V] (and C). V_{D1} is an injective cogenerator in $MoD - D_1$ [1], [11] involves a nice proposal of Morita duality. For two rings D_1 and D_2, if Y is a right D_1-module and left D_2-module, we let $Y^* =_{D2} Hom_{D1}(Y, V)$, Y is called V-reflexive if the evaluation homomorphism $e_Y: Y \to Y^{**}$ is an isomorphism.

Theorem 2.5: *Let* ${}_{D2}V_{D1}$ *defines a Morita duality and* $\theta: Y \to X$. *If Y and X are V-reflexive, then*

1. *θ is an ALS epimorphism if and only if $\theta^*: X^* \to Y^*$ is ESE monomorphism.*
2. *θ is an ESE monomorphism if and only if $\theta^*: X^* \to Y^*$ is an ALS epimorphism.*

Proof: (1) Presume that θ is ALS epimorphism, then by [1], $\theta^*: X^* \to Y^*$ is a monomorphism. To show that $\theta^*$is ESE monomorphism. let $h: Y^* \to H$ be a small epimorphism with $Ho\theta^*$ is monomorphism. Then $(ho\theta^*)^* = \theta^{**}oh^*: H^* \to X^*$ is an epimorphism and h^* is essential monomorphism. Since Y and X are V-reflexive, then the evaluation homomorphism $e_Y: Y \to Y^{**}$ and $e_X: X \to X^{**}$ are isomorphisms, that is the following Figure 4 is commutative.

$$\begin{tikzcd} Y \arrow[r, "\theta"] \arrow[d, "e_Y"'] & X \arrow[d, "e_X"] \\ Y^{**} \arrow[r, "\theta^{**}"'] & X^{**} \end{tikzcd}$$

Figure 4
Commutative diagram for V-reflexive modules

Since θ is an ALS epimorphism, Corollary (2.4) hints that θ^{**} is an ALS epimorphism. By proposition (2.2) we have $\ell_{T^{**}}\big(lm(h^*)\big) = 0$ where $T^{**} = End(Y^{**})$. We claim that $(Ker(h):_{T^*} Y^*) = 0$ where $T^* = End(Y^*)$. Let $\omega \in (Ker(h):_{T^*} Y^*)$. Then $\omega(Y^*) \subseteq Ker(h)$ and hence $ho\omega(Y^*) = 0$, so $ho\omega = 0$, Now $0 = (ho\omega)^* = \omega^* oh^*$. Thus $0 = \omega^*(h^*(H))$, then! $\omega^* \in \ell_{T^{**}}\big(lm(h^*)\big) = 0$, so $\omega = 0$. For the converse direction, presume that $\theta^*: X^* \to Y^*$ is ESE monomorphism. By Corollary 2.4 in [1]. $\theta: Y \to X$ is an epimorphism. We show that θ is an ALS epimorphism, let $h: H \to Y$ be an essential monomorphism with θoh is an epimorphism, then $(\theta oh)^* = h^* o\theta^*$ is monomorphism and h^* is a small epimorphism. Since θ^* If ESE is a monomorphism, then by Corollary 2.4, θ is an ESE monomorphism. By proposition (2.2), $(Ker(h^*):_{T^*} Y^*) = 0$ where $T^* = End(Y^*)$. We claim that $\ell_T(lm(h)) = 0$. For this, let $u \in \ell_T(lm(h))$. Then $uoh(Y) = 0$, so $uoh = 0$. Now, $0 = (uoh)^* = h^*ou^*$, so $(h^*ou^*)(Y^*) = 0$ and hence $u^*(Y^*) \in Ker(h^*)$, This hints that $u^* \in (Ker(h^*):_{T^*} Y^*) = 0$, so $u^* = 0$, and hence $u = 0$. (2) Can be proved by duality.

3. ALS-generalized Hopfian and ESE-weakly co-Hopfian modules

Definition 3.1: For two D-modules Y and X.

1. A couple (Y, X) is called an ALS generalized Hopfian couple of modules, if every surjective $\theta: Y \to X$ possesses an ALS kernel; we denote this couple ALS-gH.
2. A couple (Y, X) is called ESE weakly co-Hopfian, denoted by ESE-wcH, if every monomorphism $\emptyset: Y \to X$ possesses an ESE image in X. Evidently, every e-gH (resp. s-wcH) couple is ALS-gH (resp. ESE-wcH).

Let D_1 and D_2 be two rings and ${}_{D_2}V_{D1}$ a bimodule. For right D_1-module Y and left D_2-module X, the V-duals, $Hom_{D1}(Y, V)$ and $Hom_{D2}(X, V)$ are signified by Y^* and X^*.

Proposition 3.2: Let Y and X be two right D-modules with the property, for each $W <_e Y$ with nonzero $\ell_T(W)$, there is a nonzero θ in Y^* such that $T\theta \ll Y^*$ and $\theta(W) = 0$. Then. If (X^*, Y^*) If ESE-wcH is a couple, then (Y, X) is ALS-gH couple.

Proof: Presume that $\phi: Y \to X$ is an epimorphism. Then $\phi^*: X^* \to Y^*$ is a monomorphism where $\phi^*(f) = fo\phi$ for all $f \in X^*$. Let $W <_e Y$with $Ker(\phi) + W = Y$. Then $\phi(W) = \phi(Y) = X$. Since the couple (X^*, Y^*) is ESE-wcH, then $lm(\phi^*)$ is ESE submodule in Y^*. For each $h(\neq 0) \in Y^*$ with $Th \ll Y^*$. By proposition (2.2) in [8], there is $t \in T^*$ such that $th(\neq 0) \in lm(\phi^*)$ where $T^* = End(Y^*)$, then there is $f \in X^*$ with $0 \neq th = \phi^*(f) = fo\phi$. Hence $th(W) = fo\phi(W) = f\phi(Y) \neq 0$. Thus, for every nonzero $h^* \in Y^*$ with $Th \ll Y^*$, we have $h(W) \neq 0$. By the hypothesis, $\ell_T(W) \neq 0$. This hints that $Ker(\phi)$ is ALS in Y and hence the couple (Y, X) is ALS-gH.

Corollary 3.3: *Let ${}_{D_2}V_{D1}$defines a Morita duality, Y a reflexive right D_1-module, and X is a right D_1-module. If the couple (X^*, Y^*) is ESE-wcH, then (Y, X) is an ALS-gH couple.*

Proof: Let $W <_e Y$. Since V_{D1} is an injective congenator, then there is a mapping $f: Y/W \to V$ such that $fo\pi(\neq 0) \in Y^*$, where $\pi: Y \to Y/W$ in the natural epimorphism. Thus $fo\pi(W) = 0$ and hence $\ell_T(W) \neq 0$. Since $W <_e Y$, then by EX. 24.5 in [1], we have $\pi^*((Y/W)^*) \ll Y^*$. Observe that, $T(fo\pi) =$

$T\pi^*(f) \subseteq \pi^*((Y/W)^*)$, thus $T(fo\pi) \ll Y^*$. Since (X^*, Y^*) is an ESE-wcH couple, then by Proposition (3.2), we have (Y, X) is an ALS-gH couple.

Proposition 3.4: Let ${}_{D_2}V_{D1}$ defines a Morita duality, Y is a right D_1-module and X be a reflexive D_1-module. If (X^*, Y^*) is ALS-gH couple of left D_2-modules, then (Y, X) is an ESE-wcH couple of right D_1-modules.

Proof: Let $\phi: Y \to X$ be a monomorphism and $h: X \to H$ a small epimorphism with $ho\phi$ is monomorphism. Since V_{D1} is an injective cogenerator, $\phi^*: X^* \to Y^*$ is an epimorphism where $\phi^*(f) = fo\phi$ for all $f \in X^*$. By EX 24.5 in [1], $h^*: H^* \to X^*$ is an essential monomorphism and hence $\phi^* oh^*$ is an epimorphism. But (X^*, Y^*) is an ALS-gH couple, it hints that $Ker(\phi^*)$ is ALS submodule in X^*. By proposition (2.2), $\ell_{T^*}(lm(h^*)) = 0$ where $T^* = End(X^*)$ and hence $(Ker(h):_{T^{**}} Y^*) = 0$ (see the proof of Theorem (2.5)). Then proposition (2.2) hints that ϕ is ESE and hence (Y, X) is an ESE-wcH couple.

Theorem 3.5: *Let ${}_{D_2}V_{D1}$ define a Morita duality, Y and X be right V-reflexive D_1-modules. Then*

1. (Y, X) *is ESE-wcH if and only if* (X^*, Y^*) *is ALS-gH.*
2. (Y, X) *is ALS-gH if and only if* (X^*, Y^*) *is ESE-wcH.*

Proof: Morita duality property of ${}_{D_2}V_{D1}$ hints that ${}_{D_2}V_{D1}$ is a faithfully balanced bimodule, V_{D1} and ${}_{D_2}V$ are injective cogenerators.

1. Let (Y, X) be ESE-wcH. V-reflexivity of both Y and X gives that $Y \cong Y^{**} and\ X \cong X^{**}$, then (Y^{**}, X^{**}) is ESE-wcH. Corollary (3.3) implies that (X^*, Y^*) is ALS-gH. The other direction follows from proposition (3.4).
2. Let (Y, X) be ALS-gH. Then $Y \cong Y^{**}$ and $X \cong X^{**}$ hints that the couple (Y^{**}, X^{**}) is an ALS-gH couple. By part (1), we have (X^*, Y^*) is ESE-wcH. The converse follows from Corollary (3.3).

4. Conclusion

Confident sort of homomorphisms have been regarded. In fact, we regarded an epimorphism (resp. a monomorphism) whose kernel (resp. image) is annihilator large – superfluous (resp. endomorphism small-essential). We suggested a criterion for these homomorphisms: their properties hold under pushouts and pullbacks, and they are dual under Morita duality. In applications, these homomorphisms are used to introduce the notions of ALS generalized Hopfian and ESE weakly co-Hopfian, as a conception of essential generalized Hopfian and small weakly co-Hopfian, respectively. Finally, we regarded an affirmation under which a couple of modules satisfied one of the ALS: generalized Hopfian or ESE weakly co-Hopfian. Then the dual satisfies the other.

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