Journal of Discrete Mathematical Sciences & Cryptography
ISSN 0972-0529 (Print), ISSN 2169-0065 (Online)
Vol. 29 (2026), No. 5, pp. 2161–2167
DOI : 10.47974/JDMSC-2719

ANN-essentially injective modules

Zahraa Abbas Fadel *
Mahdi Saleh Nayef †
Department of Mathematics
College of Education
Mustansiriyah University
Baghdad
Iraq

Abstract

This paper investigates the concept of annihilator essential injectivity in the setting of R-modules, where R is unital ring. An R-module M_2 is termed annihilator essentially M_1-injective if every R-homomorphism from $A \leq M_1$ to M_2 with an annihilator essential kernel can be extended to M_1. We prove that this injective property is closed under taking submodules and factor modules of M_1, as well as direct products. By providing illustrative examples, we demonstrate the distinction between annihilator essential injectivity and other classical forms of injectivity. These findings provide new insights into the study of relative injectivity and the classification of modules.

Subject Classification: *Primary 16D10, Secondary 16D50.*

Keywords: *Injective modules, Essentially injective modules, Annihilator essentially injective modules, Annihilator essential submodule, Essential submodule.*

1. Introduction

In [1], introduces the concept of essential submodule: a submodule $0 \neq T \leq M$ is essential, if $T \cap V \neq 0$ for all $0 \neq V \leq M$ (denoted by $T \leq_e M$. A module is called uniform if all non-zero submodule is an essential. Furthermore, a module M is said to be non-singular if $Z(M) = 0$ and singular if $Z(M) = M$, where $Z(M) = \{a \in M | aJ = 0$ for some J is essential ideal of $R\}$. The set $ann(J) = \{a \in R: aJ = 0\}$ is the annihilator of ideal J of a ring R.

In [2], introduces the concept of annihilator essential submodule (denoted by $\leq_{a.e}$ or ANN-essential), a submodule $0 \neq T \leq M$ is ANN-essential if $T \cap V = 0$ then $ann(V) \leq_e R$ for all $V \leq M$. In equivalent terms, a submodule

* *E-mail:* **zhmaster93@uomustansiriyah.edu.iq (Corresponding Author)**
† *E-mail:* **mahdisaleh773@uomustansiriyah.edu.iq**

$0 \neq T \leq M$ is ANN-essential if $T \cap V \neq 0$ such as $ann(V) \nleq_e R$ for any $V \leq M$. Where $0 \neq T$ has no proper ANN-essential extension, then is called annihilator closed submodule (denoted by ANN-closed or $\leq_{a.c}$).

In [3], introduces the concept of injective modules. Let T, V and Y are an R-modules, if for any monomorphism θ from T to V and any homomorphism $\beta$from T to Y, there is a homeomorphism α from V to Y such that $\alpha \circ \theta = \beta$. Many researchers have extensively explored various generalizations of injective modules. For further details, see [4-5].

In [6], the author introduces two concepts of injective modules: the first concept nearly injective. A module T is nearly V-injective, if all homomorphism $\theta: D \to T$ where $D \leq V$ and $\ker\theta \neq 0$ can be extended to a homomorphism $\alpha: V \to T$. the second concept is essentially injective, the module T is essentially V-injective if $\ker\theta \leq_e D$. On related and generalized structures within module theory, see [7-9].

This research introduces and characterizes a generalization of injective and essentially injective modules using ANN-essential submodules. We explore the relationships between this new class and existing concepts, supported by illustrative examples. Finally, key theorems are established to characterize the properties of these modules.

2. ANN-Essentially Injective Modules

Definition 2.1: Let M_1 and M_2 be modules. The module M_2 is annihilator essentially M_1-injective (denoted by ANN-essentially M_1-injective) if every homomorphism $\alpha: A \to M_2$, where A is a submodule of $M_1$and ($\ker\alpha \leq_{a.e} A$), can be extended to a homomorphism $\beta: M_1 \to M_2$.

Remarks and examples 2.2

1) Every injective module is ANN-essentially injective module. This relationship does not imply the reverse; for example: Z/pZ is ANN-essentially Z/p^2Z-injective module but it is not injective (by [8]).
2) Every ANN-essentially injective module is essentially injective module.
3) Every nearly injective module is ANN-essentially injective module. This relationship does not imply the reverse; for example:Z is ANN-essentially ($Z \oplus Z$)-injective (by proposition 2.9), but Z is not nearly ($Z \oplus Z$)-injective (by [8]).
4) For every M, K modules. Then M is ANN-essentially K-injective, if K is simple module. This relationship does not imply the reverse; for example: Z is ANN-essentially Z-injective.

The following diagram summarizes the relationships between the key concepts discussed in the previous remarks.

Injective$\Longrightarrow$ Nearly injective $\Longrightarrow$ ANN-essentially injective $\Longrightarrow$ Essentially injective.

In the following results, we will examine the necessary conditions for the converse of the previous relations to hold:

Proposition 2.3: Let S a non-singular module. Then T is essentially S-injective if and only if T is ANN-essentially S-injective.

Proof: Assume T is essentially S-injective module and let $K \leq M$, and $\alpha: K \to N$ be any homomorphism with $ker\alpha \leq_{a.e} K$. Since S is non-singular module then $Z(S) = 0$. It follows that $Z(K) = K \cap Z(S) = 0$. Consequently K is non-singular, then $\ker \alpha \leq_e K$ [10]. So, by hypothesis there exist a homomorphism $\epsilon: S \to N$ that extend α. Therefore T is ANN-essentially S-injective module. The other direction, it is clear by remark 1.2 (2).

Proposition 2.4: The following statements are equivalent for a uniform module S:

1) T is nearly S-injective module.
2) T is ANN-essentially S-injective module.
3) T is essentially S-injective module.

Proof: (1)⟹(2)⟹(3) it is clear by remark 1.2. (3) ⟹(1) it is clear by[9].

The following lemma provides several equivalent characterizations of ANN-essentially injective.

Lemma 2.5: *Let S_1 and S_2 be modules and let $S = S_1 \oplus S_2$ then the following conditions are equivalent:*

(1) S_2 is ANN-essentially S_1-injective module.
(2) S_2 is $({}^{S_1}/_{C})$ -injective, for every $C \leq_{a.e} S_1$.
(3) For every $C \leq S$ such that $C \cap S_1 \leq_{a.e} S_1$ and $C \cap S_2 = 0$, there is $C' \leq S$ and $C \leq C'$, $S = C' \oplus S_2$.
(4) For every $C \leq_{a.c} S$ such that $C \cap S_1 \leq_{a.e} S_1$ and $C \cap S_2 = 0$, $S = C \oplus S_2$.
(5) For every (ANN-closed) $C \leq M$ such that $C \cap S_1 \leq_{a.e} S_1$, there is $H' \leq S$ and $C \leq H'$, $S = H' \oplus S_2$.

Proof: (1)⟹(2), Suppose that S_2 is ANN-essentially S_1-injective module, and let: $C \leq_{a.e} S_1$,so $K/C \leq S_1/C$ where $K \leq S_1$ and $f: K/C \to S_2$ be any homeomorphism. Let $\pi: S_1 \to S_1/C$ is natural R-homomorphism and $\pi' = \pi|_K$, then $f \circ \pi': K \to S_2$ is homomorphism. Since $C \leq_{a.e} S_1$ and $C \leq K \leq S_1$, then $C \leq_{a.e} K$ by ([8], (2.1.11)). Now $f \circ \pi'(C) = f(\pi'(C)) = f(0) = 0$ hence $C \leq \ker(f \circ \pi') \leq K$. This implies that $\ker(f \circ \pi') \leq_{a.e} K$ by ([8], (2.1.11)). Thus, by hypothesis there exist a homomorphism $\epsilon: S_1 \to S_2$ such that $\epsilon \circ i_k = f \circ \pi'$ where i_k is inclusion map from K into S_1. Now $\epsilon(C) = f \circ \pi'(C) = f(0) = 0$, then $\ker(\pi) \leq \ker(\epsilon)$ and consequently there exist $\rho: S_1/C \to S_2$ such that $\epsilon = \rho \circ \pi$. Then for every $k \in K$ $\rho(k + C) = \rho(\pi(k)) = \epsilon(k) = f(\pi'(k)) = f = (k + C)$. So, we have ρ extends f and therefore S_2 is (S_1/S) -injective. (2)⟹(1), Suppose that S_2 is (S_1/C)-injective, for every ANN-essentially C of S_1. Let $K \leq S_1$and $\gamma: K \to S_2$ be a homomorphism such that $ker\gamma \leq_{a.e} K$. consider the homomrphism $\gamma': K/\ker\gamma \to S_2$ such that $\gamma'(k +$

$ker\gamma) = \gamma(k)$ form $k \in K$. Now let T be a complement of K in S_1, then $C = \ker\gamma \oplus T \leq_{a.e} S_1$. consider the homomrfism $\vartheta: (K/\ker\gamma) \to S_1/C$ such that $\vartheta(k + ker\gamma) = k + C$, because $k \cap C = \ker\gamma \oplus (K \cap T) = ker\,\gamma$ so ϑ is monomorphism. Now by hypothesis S_2 is (S_1/C)-injective, then there exist a map $\beta: S_1/C \to S_2$ such that $\gamma'(k + ker\gamma) = \beta\vartheta(k + ker\gamma) = \beta(k + C)$, for $k \in K$. Let $\pi: S_1 \to S_2$, so $\beta(k + C) = \pi(k)$ then $\pi(k) = \gamma(\mathrm{k})$, for every $k \in K$. (5) $\Longrightarrow$(4), Suppose that condition (5) holds and let $C \leq_{a.c} S$ such that $C \cap S_1 \leq_{a.e} S_1$ and $C \cap S_2 = 0$. Let us remark that$C \cap S_1 \leq_{a.e} C$, because $(C \cap S_1) \oplus S_2 \leq_{a.e} S$ and $C \cap S_1 = C \cap [(C \cap S_1) \oplus S_2$. Then, by hypothesis, there exists a submodule C' of S such that $C < C'$ and $S = C' \oplus S_2$. We have $C \leq_{a.e} C'$, because $C \oplus S_2 \leq_{a.e} S$ and $C = C' \cap (C \oplus S_2)$. Since $C \leq_{a.c} S$ we can conclude that $C = C'$ and $S = C \oplus S_2$. (4) $\Longrightarrow$(3), It is clear by ([11], (2.1.1)). (3) $\Longrightarrow$(5), Suppose that condition (3) holds and let $C \leq M$ such that $\cap S_2 \leq_{a.e} C$. Let A be a complement of $C \cap S_1$ in S_1 . Then, $(C \oplus A) \cap S_1 = (C \cap S_1) \oplus A$. Also, $(C \cap S_1) \cap [C \cap (A \oplus S_2)] = C \cap [A \oplus (S_1 \cap S_2)] = C \cap A = 0$. As $\cap S_2 \leq_{a.e} C$, $C \cap (A \oplus S_2) = 0$ and, consequently, $(C \oplus A) \cap S_2 = 0$. By hypothesis, there is $H' \leq S$ such that $C \cap A \leq H'$ and $S = H' \oplus S_2$.

Before presenting the next proposition, we need to establish the following lemma:

Lemma 2.6: *Let M be an R-module and $A \leq N \leq M$. If $N/A \leq_{a.e} M/A$, then $N \leq_{a.e} M$.*

Proof: Let $0 \neq V \leq M$ such that $N \cap V = 0$, so $A \cap V = 0$. Now we have $A \leq A + V$, then $[(A + V)/A] \leq M/A$ but $V \neq 0$ and $A \cap V = 0$ therefor $A + V/W \neq 0$. Since $N/A \leq_{a.e} M/A$ then we get $[(A + V)/A] \cap N/A = 0$ and $ann(A + V/A) \leq_e R$. And by $W + V/A \cong V/V \cap A$, then $ann(V) = ann(A + VA) \leq_e R$. Then $N \leq_{a.e} M$.

Proposition 2.7: Let S_1 and S_2 be modules. If S_2 is ANN-essentially S_1-injective, then for every $L \leq S_1$, S_2 is ANN-essentially L-injective, and (S_1/L)-injective.

Proof: Let $F \leq L$ and $\alpha: F \to S_2$ be any R-homomorphism with $\ker\alpha \leq_{a.e} F$. Now since S_2 is ANN-essentially S_1-injective, then there exists a homomorphism $f: S_1 \to S_2$ such that $f \circ \sigma_L \circ \sigma_F = \alpha$, where $\sigma_F: F \to S_1$ and $\sigma_L: L \to S_1$ are inclusion maps. Let $\mu: L \to S_2$ is homomorphism, then $\mu = f \circ \sigma_L$ and hence μ is extend α. Then we have S_2 is ANN-essentially L-injective. Let $T \leq S_1$ be such that $L \leq T$ and $T/L \leq_{a.e} S_1/L$, then by lemma 1.6 $T \leq_{a.e} S_1$. Thus, by assumption and lemma 1.5 we get S_2 is ANN-essentially (S_1/T)-injective. Therefor it is also $[(S_1/T)/(T/L)]$-injective since these tow modules are isomorphic, then S_2 is ANN-essentially (S_1/L)-injective.

In what following, we shall shift our focus to the study of direct products and direct sums of ANN-essentially injective.

Proposition 2.8: Let S and $\{ S_i | i \in I \}$ be modules. Then $\prod_{i\in I} S_i$ is ANN-essentially S-injective if and only if S_i is ANN-essentially S-injective for eac $i \in I$.

Proof: Suppose $\prod_{i\in I} S_i$ is ANN-essentially S-injective. Let $T \leq S$ and $\sigma: T \to S_i$ be any homeomorphism for each $i \in I$ such that $\operatorname{Ker}\sigma \leq_{a.e} T$. Let $\theta: T \to \prod_{i\in I} S_i$ such that $\theta = \rho_i \circ \sigma$ for all $i \in I$, when $\rho_i: S_i \to \prod_{i\in I} S_i$ is injection mapping. Consequently θ is homomorphism, $\ker\theta = \ker(\rho_i \circ \sigma)$,and by ([8],((2.1.11)) we have $\ker\theta \leq_{a.e} T$. since $\prod_{i\in I} S_i$ is ANN-essentially S-injective, then there is a homomorphism $\beta: S \to \prod_{i\in I} S_i$ such that $\beta|_T = \theta$. Let $\beta': S \to S_i$ such that $\beta'(j) = \pi_i \circ \beta(j)$ for all $j \in S$, when $\pi_i: \prod_{i\in I} S_i \to S_i$ is a projection mapping. So β' is homomorphism and for any $\in T$ $\beta'(j) = \sigma(t)$, therefore β' is extend of σ, and so S_i is ANN-essentially S-injective for each $i \in I$. The other direction, suppose that S_i is ANN-essentially S-injective for any $i \in I$. Let $T \leq S$ and $\sigma: T \to \prod_{i\in I} S_i$ be any homomorphism with $\ker\sigma \leq_{a.e} T$. Set $g: T \to S_i$ such that $g = \pi_i \circ \sigma$, whenever π_i is projection mapping from $\prod_{i\in I} S_i$ to S_i so g is homomorphism and $\ker g \leq_{a.e} T$. Thus, by hypothesis, there exist θ be a homeomorphism such that $\theta \circ i_T = g$, where i_T is inclusion map. Now let $\theta': S \to \prod_{i\in I} S_i$ such that for $j \in S$, $\theta'(j) = \alpha_i \circ \ \theta(j)$, where α_i is injection mapping from S_i to $\prod_{i\in I} S_i$, for any $i \in I$. Hence θ' is a homomorphism and extend σ. Then $\prod_{i\in I} S_i$ is ANN-essentially S-injective.

In [12], the following was clarified $\prod_{i\in I} S_i = \oplus_{i\in I} S_i$, if I is finite index set. Based on this, we get the following example and result every direct sum of ANN-essentially injective is ANN-essentially injective. For example, Q is ANN-essentially Z-injective (since is injective), then $(Q \oplus Q)$ is ANN-essentially Z-injective.

Proposition 2.9: Let V and $\{ V_i | i \in I \}$ be modules. Then V is ANN-essentially V_i-injective for each $i \in I$ if and only if V is ANN-essentially $\oplus_{i\in I} V_i$-injective.

Proof: Assume that V is ANN-essentially V_i-injective for any $i \in I$, and let $T \leq_{a.e} \oplus_{i\in I} V_i$. So $T \cap V_i \leq_{a.e} V_i$ and by lemma 1.5 also with hypothesis we get V is $[V_i / T \cap V_i]$-injective. Then by ([12], [1.5]), we have V is $\{\oplus_{i\in I}[V_i / T \cap V_i]\}$ - injective. Therefore V is $\{[\oplus_{i\in I} V_i]/[\oplus_{i\in I}(T \cap V_i)]\}$-injective, and by ([12], (1.3)) V is $[\oplus_{i\in I} V_i / T]$-injective. Also, by lemma 1.5 we get V is ANN-essentially $\oplus_{i\in I} V_i$-injective. The other direction, it is clear by proposition 1.7.

If V_a is ANN-essentially V_b-injective, for all $a, b \in \{1,2\}$ and $a \neq b$, then the modules V_1 and V_2 are relatively ANN-essentially injective. The following result gives an equivalent condition for relatively ANN-essentially injective.

Proposition 2.10: Let S_1, and S_2 be modules and let $S = S_1 \oplus S_2$.Then S_1, and S_2 are relatively ANN-essentially injective if and only if, for all (ANN-closed) submodules B and T of S so that $S_1 \cap B \leq_{a.e} B$ and $T \cap S_2 \leq_{a.e} T$, there exist $B \leq B' \leq S$ and $T \leq T' \leq S$ and $S = B' \oplus T'$.

Proof: Suppose that S_1, and S_2 are relatively ANN-essentially injective and let B and T be (ANN-essentially) submodules of S such that $B \cap S_1 \leq_{a.e} B$ and $T \cap S_2 \leq_{a.e} T$. Then by lemma 1.5 that there exist $B \leq B'$ and $S = B' \oplus S_2$. So S_1 and B' are isomorphic and we get B' ANN-essentially S_2-injective. Also, by lemma 1.5, and since $T \cap S_2 \leq_{a.e} T$, then there exists a submodule $T \leq T'$ and $S = B' \oplus T'$. The other direction, let $B \leq S$ such that $B \cap S_1 \leq_{a.e}$H and let $T = S_2$. By assumption, there exist $B \leq B' \leq S$ and $T \leq T' \leq S$ and $S = B' \oplus T'$. So $T' = (S_1 \oplus S_2) \cap T' = (S_1 \cap T') \oplus S_2$ and $S = B' \oplus (S_1 \cap T') \oplus S_2$. Therefore, by lemma 1.5 we get S_2 is ANN-essentially S_1-injective. In the same way, we get S_1 is ANN-essentially S_2-injective.

5. Discussion and Conclusion

This study concludes that every injective module is ANN-essentially injective, which in turn is essentially injective. We proved that ANN-essential injectivity is a hereditary property and is closed under direct products. These results establish relative ANN-essential injectivity as an effective criterion for classifying module structures and ensuring their decomposition.

References

[1] K. R. Goodearl, “Ring Theory: Nonsingular Rings and Modules,” *Pure and Applied Mathematics,* vol. 33, p. 206 (1976).

[2] Y. A. Qasim and S. M. Yaseen, “Annihilator essential submodules,” *Journal of Physics: Conference Series*, vol. 1818, no. 1, art. no. 012213, pp. 1–5 (2021), doi: 10.1088/1742-6596/1818/1/012213.

[3] C. Faith, *Lectures on Injective Modules and Quotient Rings*, Lecture Notes in Mathematics, vol. 49. Berlin, Heidelberg, New York: Springer-Verlag (1967), doi: 10.1007/BFb0074319.

[4] Z. A. Fadel and M. S. Nayef, “On WS-rationally extending modules,” *AIP Conference Proceedings*, vol. 2834, pp. 080028-1–080028-8 (2023), doi: 10.1063/5.0161534.

[5] W. K. Nicholson, J. K. Park, and M. Yousif, “Principally quasi-injective modules,” *Commun. Algebra*, vol. 27, no. 4, pp. 1683–1693 (Jan. 1999), doi: 10.1080/00927879908826521.

[6] N. V. Dung, D. V. Huynh, P. F. Smith, and R. Wisbauer, *Extending Modules,* Pitman Research Notes in Mathematics Series, vol. 313. Harlow, UK/New York, USA: Longman Scientific & Technical, (1994), doi: 10.1201/9780203756331.

[7] M. Saleh, Z. M. A. Al-Majeed, Z. R. Shaker, and A. S. Dawood, "Supplement t-extending modules," *Journal of Discrete Mathematical Sciences and Cryptography,* vol. 27, no. 5, pp. 1591–1598 (2024).

[8] J. Chen, N. Ding, Y. Li, and Y. Zhou, "ON (m, n)-INJECTIVITY OF MODULES," *Commun. Algebra,* vol. 29, no. 12, pp. 5589–5603 (2001), doi: 10.1081/AGB-100107948.

[9] K. I. Beidar and F. Kasch, "Good conditions for the total," in *International Symposium on Ring Theory,* G. F. Birkenmeier, J. K. Park, and Y. S. Park, Eds., Mathematics Trends, Basel, Switzerland: Birkhäuser Boston, pp. 43–65 (2001), doi: 10.1007/978-1-4612-0181-6_3.

[10] C. A. de Santa Clara Gomes, "Some generalizations of injectivity," Ph.D. dissertation, University of Glasgow, Glasgow, U.K. (1998).

[11] F. Kasch, *Modules and Rings* (LMS Monogr. 17), D. A. R. Wallace, Trans. London, U.K.: Academic Press (1982).

[12] G. F. Birkenmeier, J. K. Park, and S. T. Rizvi, "Injectivity and Some of Its Generalizations," in *Extensions of Rings and Modules*, New York, USA: Springer, pp. 19–60 (2013), doi:10.1007/978-0-387-92716-9_2.

Received January, 2026