

A note on t-pure subhypermodules

Aseel H. Ali *

Mayssam Fadel Abood [†]

Muna Jasim Mohammed [§]

Department of Mathematics

College of Science for Women

University of Baghdad

Baghdad 10071

Iraq

Abstract

The theoretical construct of a T-pure sub hyper module also T-pure sub hyper module relative to a sub hyper module are introduced. Varies attributes of these constructs are currently under investigation.

Subject Classification: 16Y99, 20N20, 16D90.

Keywords: Hyper module, Sub hyper module, T-pure sub hyper module, T-pure sub hyper module relative to sub hyper module.

1. Introduction

Hyperalgebraic structures represent a natural extension of algebraic structures. In algebra, when we take two elements, their composition yields a single element, whereas in hyperalgebraic structures, it yields a set of elements [1-2]. These hyperstructures find broad applications across various domains of applied mathematics, including cryptography, geometry, automata theory, lattices, graph and hypergraph theory, fuzzy sets, probability theory, and rough set theory, among others. The concept of a hypergroup was first introduced in 1934 [3]. Later, hyperrings were introduced, where addition is defined as a hyperoperation while multiplication remains a classical operation. Furthermore, hypermodules over hyperrings generalize the classical notion of modules over rings. Let $H \neq \emptyset$, and denote $P^*(H)$ the set of all nonempty subsets of H . If there exists $+$: $H \times H \rightarrow P^*(H)$ satisfying the following conditions, then a canonical

* E-mail: aseel.h@csw.uobaghdad.edu.iq (Corresponding Author)

[†] E-mail: maissam.fadel11203a@csw.uobaghdad.edu.iq

[§] E-mail: munajm_math@csw.uobaghdad.edu.iq

hypergroup is $(H, +)$. Associativity: For all $a, b, c \in H$, $a, + (b + c) = (a + b) + c$. Commutativity: For all $a, b \in H$, $a + b = b + a$. Identity element: $\exists 0 \in H \exists 0 + a = \{a\}$ for all $a \in H$. Inverse element: For every $a \in H$, there exists a unique $a' \in H$ (denoted $-a$) such that $0 \in z + z'$. Reversibility: For all $z, t, c \in H$, if $c \in z + t$, then $t \in -z + c$ and $a \in c - b$. A subhypergroup is $A \subseteq H$ and, if $0 \in A$, $(A, +)$ also forms a hypergroup. A hyperalgebraic structure $(R, +, \cdot)$ is called a Krasner hyperring if: $(R, +)$ is a canonical hypergroup $(R, \cdot)$ is a semigroup with zero as a bilaterally absorbing element (i.e., $a \cdot 0 = 0 \cdot a = 0$). Distributivity: For all $a, b, c \in R$, multiplication distributes over hyperaddition: $a \cdot b + a \cdot c = a \cdot (b + c)$ and $a \cdot c + b \cdot c = (a + b) \cdot c$ [4-6].

2. Main Results

Definition 2.1: A sub hyper module X of Q is defined as T-pure (TPSHM) sub hyper module if $z^2 X = X \cap z^2 Q$ for each individual $z \in R$.

Remark 2.2: Put Q hyper module, X is a TPSHM in Q .

- 1- P is TPSHM in Q , if P is TPSHM in X .
- 2- X is a TPSHM in W , if W is a sub hyper module of Q containing X .
- 3- $\frac{X}{P}$ is a TPSHM in $\frac{Q}{P}$. If P is a TPSHM in X .
- 4- Put X, P sub hyper modules in Q , If X is TPSHM in P also $\frac{P}{X}$ is TPSHM in $\frac{Q}{X}$, then P is TPSHM of Q .

Proof:

- 1) Inasmuch as P is a TPSHM in X also X is TPSHM in Q , hence $P \cap z^2 X = z^2 P$ and $X \cap z^2 Q = z^2 X$, but P is a subhypermodule of X , therefore $P \cap z^2 Q \subseteq X \cap z^2 Q = z^2 X$. Since $P \cap z^2 Q \subseteq P \cap z^2 X = z^2 P$, therefore $P \cap z^2 Q = z^2 P$, given any $z \in R$.
- 2) Given that X is TPSHM in Q , therefore $X \cap z^2 Q = z^2 X$. Because a sub hyper module in Q , $X \cap z^2 W \subseteq X \cap z^2 Q = z^2 X$. In view of the fact that $z^2 X \subseteq X \cap z^2 W$, hence $X \cap z^2 W = z^2 X$, given any $z \in R$.
- 3) IT is establish that P is TPSHM in X , then $P \cap z^2 X = z^2 P$. Also $\frac{X}{P} \cap z^2 \left(\frac{Q}{P}\right) = \frac{X}{P} \cap \frac{z^2 Q + P}{P} = \frac{X \cap (z^2 Q + P)}{P} = \frac{X \cap z^2 Q + P}{P} = \left(\frac{z^2 X}{P}\right) = z^2 \left(\frac{X}{P}\right)$, therefore, $\frac{X}{P} \cap z^2 \left(\frac{Q}{P}\right) = z^2 \left(\frac{X}{P}\right)$.

Definition 2.3: Put Q a hypermodule, Q have the T-pure sub yper module intersection property (T-PIPSH|M), if we have the intersection of two TPSHM is again TPSHM.

Proposition 2.4: Put Q a hyper module.

1. Put X TPSHM in Q . Q has T-PIPSHM, if and only is, $\frac{Q}{X}$ has T-PIPSHM.
2. If Q owns the T-PIPSHM, then each TPSHM in Q has the T-PIPSHM.

Proof:

1. Suppose that $\frac{L}{X}, X$ are TPSHM in $\frac{Q}{X}$, where $z \in R$. Aim to demonstrate that $\left(\frac{L}{X} \cap \frac{F}{X}\right) \cap z^2\left(\frac{Q}{X}\right) = z^2\left(\frac{L}{X} \cap \frac{F}{X}\right)$. We claim that each one is TPSHM. To substantiate this claim, let $c \in L \cap z^2Q$. Because $\frac{L}{X}$ is TPSHM in $\frac{Q}{X}$, thus $\frac{L}{X} \cap z^2\left(\frac{Q}{X}\right) = z^2\left(\frac{L}{X}\right)$, therefore $\frac{L}{X} \cap \frac{z^2Q+X}{X} = \frac{z^2L+X}{X}$, then $\frac{L \cap (z^2Q+X)}{X} = \frac{z^2L+X}{X} = \frac{(L \cap z^2Q)+X}{X} = \frac{z^2L+X}{X}$, thus $(L \cap z^2Q) + X = z^2L + X$, and therefore $(L \cap z^2Q) = z^2L$. Since $c \in L \cap z^2Q \subseteq (L \cap z^2Q) + X$, thus $c \in z^2L + X$. Let $c = t + s$, t belongs to z^2L , where $s \in X$. Consider for instance $s = c - t \in X \cap z^2Q = z^2X \subseteq z^2X$ also thus L is TPSHM. But Q has the T-PIPSHM, hence $L \cap F$ is TPSHM in Q . Therefore $(L \cap F) \cap z^2Q = z^2(L \cap F)$. Now, let $c \in \left(\frac{L}{X} \cap \frac{F}{X}\right) \cap z^2\frac{Q}{X}$, therefore $t + X = c$, $t \in z^2Q$ and $c = x + X = y + X$, when $x \in L, y \in F$. Thus $c - x \in X \subseteq F, c - y \in X \subseteq F$ and therefore $t \in L \cap F$. Thus $t \in (L \cap F) \cap z^2Q = z^2(L \cap F)$. Therefore $c = t + X \in z^2\left(\frac{L \cap F}{X}\right) = z^2\left(\frac{L}{X} \cap \frac{F}{X}\right)$. From another perspective, put P, W is TPSHM, assume that X is a sub hyper module of P and X is a sub hyper module in W , thus $\frac{P}{X}, \frac{W}{X}$ are TPSHM of $\frac{Q}{X}$. But $\frac{Q}{X}$ owns the T-pure intersection property, then $\frac{P}{X} \cap \frac{W}{X} = \frac{P \cap W}{X}$ is TPSHM. Thus, $P \cap W$ is TPSHM in hyper module Q .

Theorem 2.5: A hyper module Q possesses the TPIPSHM if and only if $(z^2L \cap z^2F) = z^2(L \cap F)$ for all $z \in R$ also for all TPSHM F, L in Q .

Proof: Imposing Q possesses the T-PIPSHM therefore, $F \cap L$ is TPSHM. $z \in R$, thus $(L \cap F) \cap z^2Q = z^2(L \cap F)$. Clearly that $z^2(L \cap F) \subseteq (z^2L \cap z^2F)$. But $z^2L \cap z^2F \subseteq L \cap (F \cap z^2Q) = (L \cap F) \cap z^2Q = z^2(L \cap F)$. Then $z^2L \cap z^2F = z^2(L \cap F)$. From another perspective, $z \in R$, and N, M is TPSHM in Q , thus $(N \cap M) \cap z^2Q = N \cap (M \cap z^2Q) = N \cap z^2M$. Similarly $(N \cap M) \cap z^2Q = M \cap z^2N$. But N, M are T-pure subhypermodules in Q . Then $(N \cap M) \cap z^2Q \subseteq z^2N \cap z^2M = z^2(N \cap M)$.

Theorem 2.6: Put Q hypermodule. Q possesses the T-PIPSHM, if and only if, for each TPSHM, P and X sub hyper modules of a hyper module Q with for each homomorphism $\Theta : (P \cap X) \rightarrow Q$ when $(\text{Im } \Theta \cap P) + ((\text{Im } \Theta + P) \cap z^2Q) = \{0\}$ and $P + \text{Im } \Theta$ is TPSHM of Q and $\ker \Theta$ is TPSHM in Q .

Proof: Let Q be possesses the T-PIPSHM. Put L, D TPSHM in Q with $\Theta : L \cap D \rightarrow Q$ is homomorphism where $L \cap \text{Im } \Theta = 0$ also $\text{Im } \Theta + L$ which is TPSHM in Q . Put $W = \{g(c) + c \mid c \in L \cap D\}$. W is a sub hyper module. To demonstrate that W is TPSHM in Q . $z \in R$ and $t = \sum_{i=1}^n r_i m_i \in W \cap z^2Q$, $r_i \in R, m_i \in Q$. Hence $t = \sum_{i=1}^n r_i m_i = m + \Theta(m)$, $m \in (L \cap D)$ for some m . So $t = \sum_{i=0}^n r_i m_i = m + \Theta(m) \in L \cap (\text{Im } \Theta + D) \subseteq \text{Im } \Theta + L$ and $\text{Im } \Theta + L$ is

TPSHM in Q . Thus $c = \sum_{i=0}^n r_i m_i \in (\text{Im}\Theta + L) \cap z^2 Q = z^2 (\text{Im}\Theta + L) \cap z^2 Q$. Therefore $\sum_{i=0}^n r_i m_i = \sum_{i=0}^n r_i (z_i + k_i) + s$, $z_i \in L$, $k_i \in \text{Im}\Theta$, for each $i = 1, \dots, n$, where $s \in ((\text{Im}\Theta + L) \cap z^2 Q)$. Therefore $t = \sum_{i=0}^n r_i m_i = \sum_{i=0}^n r_i z_i + \sum_{i=0}^n r_i k_i + s$, hence $c - \sum_{i=0}^n r_i m_i = \sum_{i=0}^n r_i k_i - \Theta(c) + k \in (\text{Im}\Theta \cap L) + ((\text{Im}\Theta + L) \cap z^2 Q) = 0$. Thus $c \in (L \cap D) \cap z^2 L$. While $(L \cap D)$ is TPSHM in hyper module Q , thus L is TPSHM in Q . Therefore $(L \cap D) \cap z^2 L = z^2 (L \cap D)$ by theorem (2.5). Hence $m \in z^2 (L \cap D)$. Let $c = r_i \sum_{i=0}^n w_i + h$, $w_i \in (L \cap D)$, $h \in ((L \cap D) \cap z^2 D)$. Then $\Theta(c) = \sum_{i=1}^n r_i \Theta(w_i) + \Theta(h)$. Now $r = c + \Theta(c) = \sum_{i=0}^n r_i w_i + \sum_{i=0}^n r_i \Theta(w_i) + \Theta(h) = \sum_{i=0}^n r_i (w_i + \Theta(w_i)) + \Theta(h) \in z^2 W$. Thus $W \cap z^2 Q = z^2 W$ and W is TPSHM in Q . Now we want to prove, $\ker\Theta = (L \cap D) \cap W$. Presume that $c \in \ker\Theta$, thus $c \in (L \cap D)$ also $\Theta(c) = 0$. Therefore $c \in W$, Now let $c \in (L \cap D) \cap W$, thus $c = t + g(t)$, $t \in (L \cap D)$, then $c - t = g(t) \in L \cap \text{Im}\Theta \subseteq (L \cap \text{Im}\Theta) + (L + \text{Im}\Theta \cap z^2 Q) = 0$. Therefore $\Theta(c) = \Theta(t) = 0$ so $c \in \ker\Theta$. Since Q has the T-PIP, then $(L \cap D) \cap W = \ker\Theta$ is T-pure in Q . Reciprocally, let L, D be two TPSHM in Q . We define $g : (L \cap D) \rightarrow Q$ by $g(c) = 0$, $\forall c \in (L \cap D)$. So $(L \cap \text{Im}g) + (\text{Im}g + L) \cap z^2 Q = 0$ and $\text{Im}g + L = L$ is TPSHM in Q , thus $\ker g = (L \cap D)$ is TPSHM in Q .

Theorem 2.7: *A hyper module Q possesses the T-PIPSHM, if and only if, for each TPSHM W , a sub hyper module X in hyper module Q , for all homomorphisms $g : (W \cap X) \rightarrow D$, where D is a sub hyper module in Q , for this reason, $(W \cap D) + (W + D \cap z^2 Q) = 0$ and $W + D$ are TPSHM, and D is a TPSHM.*

3. Discussion and Conclusion

Various properties have been formally derived and verified within this framework. Put Q hyper module also X is TPSHM of Q if P is T-pure in X , we get P is TPSHM in Q also, if W in Q containing X , then X is a TPSHM in W , and also prove that if P is TPSHM in X , then $\frac{X}{P}$ is TPSHM in $\frac{Q}{P}$.

References

- [1] P. Ghiasvand, S. Mirvakili, and B. Davvaz, "New fundamental relations in hyperrings and the corresponding quotient structures," *J. Algebraic Syst.*, vol. 12, no. 1, pp. 21–41 (2024).
- [2] B. Davvaz and V. Leoreanu-Fotea, *Hyperring Theory and Applications*. International Academic Press (2007).
- [3] F. Marty, "On a generalization of the notion of group," in *Proc. 8th Congr. Scand. Math.*, Stockholm, Sweden, pp. 45–49 (1934).
- [4] F. Farzalipour and P. Ghiasvand, "On graded hyperrings and graded hypermodules," *Algebraic Struct. Appl.*, vol. 7, no. 2, pp. 15–28 (2020).

- [5] F. Farzalipour and P. Ghasvand, "On secondary subhypermodules," *J. Algebra Relat. Top.*, vol. 9, no. 1, pp. 143–158 (2021).
- [6] S. A. Al-Ameedee, H. A. Ahmed, and A. J. Obaid, "Differential subordinations of meromorphic multivalent harmonic functions," *J. Interdiscip. Math.*, vol. 27, no. 4, pp. 947–952 (2024), doi: 10.47974/JIM-1915.

Received May, 2025