

2-purely extending modules

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Abstract

In this paper, we give a new generalization for important class in the category of modules. There are 2-purely extending and the direct sum of 2-purely extending R-modules. In this work, study the relationships of 2-purely extending with extending module and purely extending. Are given some properties and results of these modules and other related concepts. We give a condition under which a 2-pure submodule of a 2-purely extending module is 2-purely extending. If M is 2-purely extending, then it is not necessary that every essential submodule of M is 2-purely extending, and provide examples that delineate our results. Also show that a 2-purely extending module is a generalized of purely extending. Moreover, the relationships with other related notes are discussed.

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1. Introduction

In this paper, all rings are presumed to be commutative with unity, and all R -modules M are unitary left R -modules. Let M be an R -module; N a non-zero submodule of M is called an LM, since $N \cap (N \cap I) = N \cap I$. For all submodules L, M it is found $L \cap M = 0$ (see [1-3]). Definition 2.4: Let M be a R -module and $K \subseteq M$ be a submodule of M , then K is said to be M pure if $K \cap IM = I \cdot K$ holds for every ideal I of R . For example, if R is cyclic or real rings (the case 3 denoted by Known as Theorem a Authors [4-5]): (1) For any ideal $I = 0R, X, W: (W \in R, r \in R) (Y \in R) = W \cdot r \cdot Y$. R . Multiplication Modules. $\cdot$ -multiplicative (resp.) It is well-known that if $M \subseteq R$ be a finitely generated module, then there exists an ideal $I \supseteq R$ such that $IN = M$, cf.

The authors Pure Submodule The familiar concept of pure submodule was first introduced by the authors in [2] (also see [4]). It is defined by Al {the authors [6] that a submodule N of M is called 2-pure if for every ideal I of R , we have $N \cap I^2 = I^2 N$. A sub module N of M is said to be closed in case it does not have proper essentials extension in M i.e; If there exists an essential $K \leq M$ then ($K=N$ or $K=M$ [3], [7-8]. In case, if this is the case, we say that M is an extendable or CS-module, if every submodule of M has essential socle inside a direct summand of M , alternatively it also follows from previous observation that every closed submodule of M has factorizable closure.

Multiple authors in recent years have looked at extensions of modules among them in [9].

The authors [5] has given the definition for purely extending, Let M be an R -module, then M is said to be a pure extending if any submodule of M_0 is actually essential in a pure-submodule of M . (equivalently by [10]: A module M is a purely extending if and only if every closed-in- M module $\text{res } \text{cd}(M)$, is pure.

This paper aims to generalize the concept of an extending (purely extending) module into 2-purely extending. This imply that the strictly extending lies between extending and 2 -strictly extending module, i.e. extensions module $\rightarrow$ purely extension some $\rightarrow$ 2-purely extensional this work arranged as follows:

To define the concept of 2-purely extendible modules as well as some of their properties.

In Section 3 we summarize some basic properties of flat modules and pure submodules. A module M Over R is called flat if for every short exact sequence of R -modules

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

The sequence:

$$0 \rightarrow A \otimes M \rightarrow B \otimes M \rightarrow C \otimes M \rightarrow 0$$

is also exact, we study the direct sum of modules and the 2-purely extending property.

2. Purely Extending Modules

In section is studying the properties of 2-purely extending R-modules and study the relationships of 2-purely extending with extending module and purely extending.

Definition 2.1: An R-module M , is called 2-purely extending R-module if every submodule of M is essential in a 2- pure submodule of M .

Lemma 2.2: *An R-module M . if A is a submodule of M , then there exist a closed submodule H in M such that A is essential in H [10].*

The following theorem gives a characterization of 2-purely extending modules.

Theorem 2.3: *Let M be an R-module, then M is 2-purely extending if and only if every closed submodule in M is 2-pure in M .*

Proof: Let M be a 2-purely extending and let a submodule K of M is closed in M . Since M is 2-purely extending R-module, then there exist a 2-pure submodule B of M such that K is essential in B , but a submodule K is closed, then $K=B$. Hence K is 2-puer in M . Conversely, a submodule A of M by lemma (2.2) there exist a submodule H is closed such that A is essential in H . since H is closed then H is 2-pure by assumption then A is essential in H and H is 2-pure in M , so M is 2-purely extending.

Remarks and Examples 2.4:

1. Every extending R-module is purely extending R-module hence is 2-puerly extending. To show that, a submodule A of M , by assumption A is essential in a direct summand of M [9]. Since every direct summand is pure hence is 2-pure [11], then A is essential in 2-pure of M . Hence M is 2-purely extending.

The convers in general is not true for example: Z_4 as Z -module is 2-purely extending but is not purely extending.

2. Every direct summand of 2-purely extending R-modules is 2-purely extending.

Proof: An R-module M be a 2-purely extending R-module, assume that $M=A\oplus B$. To show that A is 2-purely extending module. Let a submodule K is closed in A , since A is a direct summand of M , then a submodule A is closed in M . Hence K is closed in M [1]. But M is 2-purely extending therefor K is 2-pure in M , since $K\subseteq A$ then by [11] K is 2- pure in A and hence A is 2-purely extending module.

3. Every uniform R-module is 2-purely extending module, where an R-module M is called uniform if every submodule of M is essential in M [1].
4. Proof: Since M is uniform R-module, then every submodule is essential in M [1]. Hence M is 2-purely extending R-module.

5. Every 2-regular R-module is 2-purely extending, whereas R-module M is called 2 regulars if every submodule of M is 2-pure [11].
6. Every semisimple R-module is 2-purely extending. Since every direct summand is pure [12] hence is a 2-pure submodule, so by definition of 2-purely extending the result follows.
7. Consider the module $M = \mathbb{Z}_8 \oplus \mathbb{Z}_2$ as a Z-module, $A = ((2,1))$ be the submodule generated by (2,1). It can be easily checked that A is closed in M but not 2-pure, thus M is not 2-purely extending.

Proposition 2.5: Let M be an R-module. If M is 2-purely extending R-module, then every closed submodule of M is 2-purely extending R-module.

Proof: A submodule N is closed of M and a submodule A is closed of N, then by [1] A is closed in M. But M is 2-purely extending R-module therefor, A is 2-pure in M since $A \subseteq N$ then by [6] A is 2-pure in N.

Proposition 2.6: An R-module M is 2-purely extending R-module and a submodule N is a closed of M, then $\frac{M}{N}$ is 2-purly extending.

Proof: Let a submodule $\frac{K}{N}$ of $\frac{M}{N}$, since M is 2-purely extending R-module, then there exist a 2-pure submodule A of M such that K is essential in A since $N \subseteq K$ and a submodule N is closed in M, then by [10] $\frac{K}{N}$ is essential in $\frac{M}{N}$. But A is 2-pure in M, so by [8] $\frac{A}{N}$ is 2-pure in $\frac{M}{N}$.

We do not know whether a 2-pure submodule of a 2-purely extending module is 2-purely extending or not.

Recall that, An R-module M has the 2-pure intersection property (briefly) if the intersection of any two 2-pure submodule is again 2-pure.

Now, we give a condition under which a 2-pure submodule of 2-purely extending module is 2-purely extending.

Proposition 2.7: An R-module M be a 2-purely extending R-module and let a submodule N be a 2-pure of M. if M has the 2-PIP, then N is 2-purely extending R-module.

Proof: Let a submodule A be a closed in N by [1], there exist a submodule B is closed in M such that A is essential in B. Since N is essential in N, then A is essential in B and N is essential in N hence $A \cap N$ is essential in $B \cap N$ and $A = A \cap N$ is essential in $B \cap N \subseteq N$. But a submodule A is closed in N, there for $A = B \cap N$, since M is 2-puerly extending R-module and a submodule B is closed in M, then by theorem (2.3) B is 2-pure in M and M has 2-PIP so $A = B \cap N$ is 2-pure in M then $A = B \cap N$ is 2-pure in M and hence A is 2-pure in N. Hence N is 2-purely extending R-module.

Corollary 2.8: *Let M be a multiplication R -module and let N be a 2-pure submodule of M . If M 2-pure extending then N is 2-purely extending.*

Remark 2.9: An R -module M . If M is 2-purely extending, then it is not necessary that every essential submodule of M is 2-purely extending.

Proof: Suppose that M is not 2-purely extending and let $E(M)$ be the injective hull of M and M is essential of $E(M)$. Since $E(M)$ is injective hence by [12] $E(M)$ is extending, hence is 2-purely extending.

Theorem 2.10: *An R -module M , the statements are equivalent:*

1. M is 2-purely extending R -module.
2. Each closed submodule of M is 2-pure in M .
3. If a submodule A is summand of the injective hull of M , then $A \cap M$ is a 2-pure submodule of M .

Proof: 1 $\rightarrow$ 2 By theorem (2.3)

2 $\rightarrow$ 3 suppose that $E(M)=A \oplus B$, B is a submodule of the injective hull $E(M)$ of M . To prove that $A \cap M$ is closed in R -module M . Let $A \cap M$ is essential in H where H is a submodule of R -module M . Let $h \in H$, so $h \in M$, $h=a+b$, $a \in A$ and $b \in B$. suppose that $h \in A$, thus $0 \neq b \in B$. But M is essential in $E(M)$, so by [1] there exist $r \in R$ such that $0 \neq rb \in M$. Now, $rb=ra+rb$ and hence $ra=rb-rb \in A \cap M \subseteq H$ thus $rb=rh-ra \in B \cap H$. since $A \cap M$ is essential in H and B is essential in B then $0=(A \cap M) \cap B$ is essential in $H \cap B$ [1] and hence $H \cap B=0$, thus $rb=0$ this is contradiction. Thus $h \in A$ thus, $A \cap M$ is closed in M so by (2) $A \cap M$ is 2-pure in M .

3 $\rightarrow$ 1 Let a submodule A of M and let a submodule B is a relative complement of A , then by [1] $A \oplus B$ is essential in M . But M is essential in $E(M)$ there for, $A \oplus B$ is essential in $E(M)$, thus $E(A) \oplus E(B) = E(A \oplus B) = E(M)$, since $E(M)$ is a summand of $E(M)$, then $E(A) \cap M$ is 2-pure submodule of R -module M , but A is essential in $E(A)$ and M is essential in M so by [1] $A=A \cap M$ is essential in $E(A) \cap M$. So, A is essential in $E(A) \cap M$ and it is 2-pure in R -module M , hence M is 2-purely extending R -module.

Recall that, a ring R is called Bezout ring if every finitely generated ideal is principal see [12].

An R -module M is a Prime R -module, if $\text{ann}(a) = \text{ann}(b)$, for every non zero element $a, b \in M$.

Let R be an integral domain, recall that an R -module M is called a torsion free if $\text{ann}(x) = 0$ for every non-zero element x in M see [12].

Before we give our result, we need the following lemma.

Lemma 2.11: *An R -module M be a prime and let a submodule A be a 2-pure of a module E if M is essential in E , then $A \cap M$ is closed in M .*

Proof: Assume that $A \cap M$ is essential in K , where A is a submodule of M . To show that $A \cap M = K$. Assume not and let $0 \neq k \in K$ such that $k \notin A$. Since $A \cap M$ is essential in K there exist $r \in R$ such that $0 \neq rk \in A \cap M$. Hence $2rk \in 2rE \cap A$. Since A is 2-pure in E then $2rE \cap A = 2rA$ and hence $2rk \in 2rA$. Thus $2rk = 2ra$ where $a \in A$ and hence $2r(k-a) = 0$ thus $2r \in \text{ann}(k-a)$. Since M is prime then $2r \in \text{ann}(k)$, which is a contradiction.

Theorem 2.12: Let an R -module M be a prime. The following are equivalent:

1. M is 2-purely extending.
2. Every closed submodule in M is 2-pure.
3. If A is a 2-pure submodule of the injective hull $E(M)$ of M , then $A \cap M$ is 2-pure in M .

Proof: $1 \rightarrow 2$ Is clear by theorem (2.9)

$2 \rightarrow 3$ Since A is 2-pure in $E(M)$ and M is essential in $E(M)$ by lemma (2.10) $A \cap M$ is closed in M and hence by assumption $A \cap M$ is a 2-pure in M .

$3 \rightarrow 1$ Let a submodule A of M and let a submodule B is a relative complement of A in M and hence by [10], $A \oplus B$ is essential in M but M is essential in $E(M)$ there for $A \oplus B$ is essential in $E(M)$ and $E(A) \oplus E(B) = E(M)$. Whereas a submodule $E(A)$ is a 2-pure of $E(M)$ and hence $E(A) \cap M$ is 2-pure in M . Where as A is essential in $E(A)$, then $A = A \cap M$ is essential in $E(A) \cap M$ and it is 2-pure in M . Hence R -module M is 2-purely extending.

The following corollary follows immediately from theorem (2.12).

Corollary 2.13: Let R be an integral domain and let M be a torsion free R -module, then the following are equivalence:

1. M is 2-purely extending.
2. Every closed submodule in M is 2-pure.
3. If A is a 2-pure submodule of the injective hull $E(M)$ of M , then $A \cap M$ is 2-pure in M .

3. The Direct Sum of 2-Purely Extending Modules

In section, we give a condition under which $M \oplus N$ is 2-purely extending R -module where M and N are 2-purely extending R -module and Every torsion free R -module over a Bezout domain is 2-purely extending.

We start this section by the following proposition.

Proposition 3.1: Let R be Bezout domain and let M be an R -module, then M is a flat if and if M is torsion free see [12].

Proposition 3.2: Let M be an R -module, for any 2- pure submodule A of the injective hull $E(M)$ of M , $A+M$ is flat R -module, then M is 2-purely extending.

Proof: In the following short exact sequences:

$$0 \rightarrow A \cap M \xrightarrow{i_1} M \xrightarrow{f_1} \frac{M}{A \cap M} \rightarrow 0 \dots\dots\dots (1)$$

$$0 \rightarrow A + M \xrightarrow{i_2} M \xrightarrow{f_2} \frac{A + M}{M} \rightarrow 0 \dots\dots\dots (2)$$

Where i_1, i_2 are the inclusion maps and f_1, f_2 are the natural epimorphism. By the second isomorphism theorem $\frac{M}{A \cap M} \cong \frac{A + M}{M}$. Whereas A is a 2-pure of $E(M)$ and $A \subseteq A + M$ then A is 2- pure of $A + M$ [6] and hence the second sequences splits. But $A + M$ is a flat R -module, then $\frac{M}{A \cap M} \cong \frac{A + M}{M}$ is a flat R -module then $A \cap M$ is pure in M and hence $A \cap M$ is 2- pure in M .

Remark 3.3: The converse of proposition (3.1) is not true in general as the following example shows. For example: let $M = Z_{p^\infty} \oplus Z_{p^\infty}$ as a Z -module. It is clear that M is injective. Let $A = Z_{p^\infty} \oplus 0$. A is a summand of M , so A is a pure submodule of M and hence it is a 2- pure submodule of M , but $A + M = Z_{p^\infty} \oplus Z_{p^\infty}$ is not torsion free and hence is not flat R -module.

Corollary 3.4: Every torsion free R -module over a Bezout domain is 2-purely extending.

Proof: Where as M is torsion free, then it can be easily checed that the injective hull of M is torsion free. Now let A be a summand of $E(M)$. Where as $A + M$ is a submodule of $E(M)$, then $A + M$ is torsion free. But R is a a Bezout domain, so $A + M$ is a flat R -module and hence by prop. (3.2) M is 2-purely extending.

Remark 3.5: A homomorphic image of a 2-purely extending may not be 2-purely extending. For example, consider the following short sequence:

$$0 \rightarrow 8Z \oplus 2Z \xrightarrow{i_1} Z \oplus Z \xrightarrow{f_1} \frac{Z \oplus Z}{8Z \oplus 2Z} \rightarrow 0$$

Where i_1 is the inclusion map and f_1 is the natural epimorphism Where as $8Z \oplus 2Z$ and $Z \oplus Z$ are torsion free over Bezout domain, then by coro. (3.4) $8Z \oplus 2Z$ and $Z \oplus Z$ are 2-purely extending on the other hand $\frac{Z \oplus Z}{8Z \oplus 2Z} \approx Z_8 \oplus Z_2$ is not 2-purely extending. See (2.4,5).

Proposition 3.6: Let $M = \bigoplus_{i \in I} M_i$ be an R -module. where each M_i is a submodule of M . If M is 2-purely extending, then each M_i is 2-purely extending. The converse is true if each closed submodule of M is fully invariant.

Proof: Assume M is 2-purely extending. Whereas M_i is a summand of M , then by Remarks and Examples (2.4, 2), M_i is 2-purely extending.

Remark 3.7: The converse, let A be a closed submodule of M and let $p_i: M \rightarrow M_i$ be the natural projection on M_i for each $i \in I$. Let $x \in A$, then $x = \sum_{i \in I} m_i$, where $m_i \in M_i$. $p_i(x) = m_i$. But A is closed in M , then (by our assumption) A is fully invariant and hence $p_i(A) \subseteq A \cap M_i$ for each i . So $p_i(x) = m_i \in A \cap M_i$. Thus $x \in \bigoplus_{i \in I} (A \cap M_i)$ and hence $A \subseteq \bigoplus_{i \in I} (A \cap M_i)$. Where as $\bigoplus_{i \in I} (A \cap M_i)$

$\subseteq A$ then $\bigoplus_{i \in I} (A \cap M_i) = A$. Whereas $A \cap M_i$ is a summand of A then, $A \cap M_i$ is closed in A but A is closed in M therefor $A \cap M_i$ is closed in M . Whereas $A \cap M_i \subseteq M_i$ then $A \cap M_i$ is closed in M_i for each i . But is 2-purely extending. So, $A \cap M_i$ is 2-pure in M_i . By [6], $\bigoplus_{i \in I} (A \cap M_i) = A$ is 2-pure in $M = \bigoplus_{i \in I} M_i$.

Proposition 3.8: R -modules M and N be purely extending, such that $\text{ann}(M) + \text{ann}(N) = R$, then $M \oplus N$ is purely extending [10].

Proposition 3.9: R -modules M and N be 2-purely extending, such that $\text{ann}(M) + \text{ann}(N) = R$, then $M \oplus N$ is 2-purely extending.

5. Conclusions

In this work, 2-purely Extending R -modules is details of Extending R -modules. We presented many supporting lemmas that contributed to proving this theory. We also show some of the following results:

- If M is 2-purely extending R -module, then every closed submodule of M is purely extending R -module.
- If M is 2-purely extending R -module and a submodule N is a closed of M , then $\frac{M}{N}$ is 2-purly extending.
- An R -module M for any 2-pure submodule A of the injective hull $E(M)$ of M , $A + M$ is flat R -module, then M is 2-purely extending.
- Every torsion free R -module over a Bezout domain is 2-purely extending.
- If M be a 2-purely extending R -module and let a submodule N be a 2-pure of M . if M has the 2-PIP, then N is 2-purely extending R -module.
- If M and N be 2-purely extending, such that $\text{ann}(M) + \text{ann}(N) = R$, then $M \oplus N$ is 2-purely extending.

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