Journal of Discrete Mathematical Sciences & Cryptography
ISSN 0972-0529 (Print), ISSN 2169-0065 (Online)
Vol. 29 (2026), No. 4, pp. 1629–1650
DOI : 10.47974/JDMSC-2346

The antiadjacency spectrum of some graph operations

Mohammad Farhan [§]
Peter John [†]
Denny Riama Silaban *
Department of Mathematics
Faculty of Mathematics and Natural Sciences
Universitas Indonesia
Depok 16424
Indonesia

Abstract

Let $\Gamma = (V, E)$ be a graph of order η with $V = \{1, 2, 3, \ldots, \eta\}$ and $E \subseteq [V]^2$. The antiadjacency matrix of Γ is an $\eta \times \eta$ matrix where the ij-entry is 1 if $ij \notin E$ and 0 if otherwise. An antiadjacency matrix can be seen as the opposite of the adjacency matrix of graphs. In this paper, we give the spectrum of the antiadjacency matrix of some graph operations, namely, disjoint union of circulant graphs, Cartesian product of cycle graph and complete graph of order two, disjoint union of complete graphs, complement of a graph, and join of graphs.

Subject Classification: *05C50.*

Keywords: *Antiadjacency matrix, Spectrum, Cartesian product, Join, Disjoint union.*

1. Introduction

Graphs can be represented using various mathematical tools, among which matrices are particularly useful. Matrix properties, such as eigenvalues, rank, and determinant, provide insights into the structure of the graph they represent. Extensive results on this topic can be found in [1-3], and references therein. Several types of matrices are used to represent graphs, including the adjacency matrix, Laplacian matrix, Seidel matrix, and distance matrix, each offering unique insights into different aspects of the graph. In this paper, we focus on the antiadjacency matrix, a relatively less explored graph matrix.

[§] *ORCID-ID:* **0009-0000-2381-7919**
* *ORCID-ID:* **0000-0003-1721-4438**

[§] *E-mail:* **mohammad.farhan81@alumni.ui.ac.id**
[†] *E-mail:* **peter.john@sci.ui.ac.id**
* *E-mail:* **denny@sci.ui.ac.id (Corresponding Author)**

We generally refer to [4, 5] for basic definitions in graph theory. We denote by K_η, $K_{\eta,\tau}$, and C_η the complete graph of η vertices, the complete bipartite graph consisting of two partitions with η and τ elements, and the cycle graph of η vertices, respectively. For any two integers α and β where $\alpha \leq \beta$, let $[\alpha, \beta] := \{x \in \mathbb{Z} : \alpha \leq x \leq \beta\}$. Let $\Gamma = (V, E)$ be a graph of order η with $V = [1, \eta]$. The *degree* of a vertex $v \in V$ in Γ is defined as the number of vertices in Γ adjacent to v. We call Γ a *regular* graph of degree ρ if all vertices in Γ has degree ρ. The *adjacency matrix* $\mathbf{A}(\Gamma) = [\alpha_{ij}]$ of Γ is a $\eta \times \eta$ matrix encoding adjacency relations, that is, $\alpha_{ij} = 1$ if $ij \in E$ and $\alpha_{ij} = 0$ if otherwise. On the other hand, *antiadjacency matrix* of Γ, denoted by $\mathbf{B}(\Gamma) = [\beta_{ij}]$, is a $\eta \times \eta$ matrix where $\beta_{ij} = 1$ if $ij \notin E$ and $\beta_{ij} = 0$ if otherwise. Therefore, $\mathbf{B}(\Gamma) = \mathbf{J} - \mathbf{A}(\Gamma)$ where $\mathbf{J}$ is the $\eta \times \eta$ all-ones matrix. While the adjacency matrix has been extensively studied, the antiadjacency matrix offers a complementary perspective by capturing the non-adjacency structure of a graph.

Let $\mathbf{A}$ be a real symmetric matrix. The *spectrum* of $\mathbf{A}$, denoted by $\mathrm{Spec}(\mathbf{A})$, is the multiset of eigenvalues of $\mathbf{A}$ together with their multiplicity. If λ_i and μ_i, $i \in [1, s]$, are the distinct eigenvalues of $\mathbf{A}$ and their respective multiplicities, then the spectrum of $\mathbf{A}$ may be written as

$$\mathrm{Spec}(\mathbf{A}) = \left\{\lambda_1^{(\mu_1)}, \lambda_2^{(\mu_2)}, \ldots, \lambda_s^{(\mu_s)}\right\}.$$

We may omit the superscript if the multiplicity of an eigenvalue is 1. For a graph Γ, we define $\mathrm{Spec}_{\mathbf{A}}(\Gamma) := \mathrm{Spec}(\mathbf{A}(\Gamma))$ and $\mathrm{Spec}_{\mathbf{B}}(\Gamma) := \mathrm{Spec}(\mathbf{B}(\Gamma))$, and we refer them as the *adjacency* and *antiadjacency spectrum* of Γ, respectively.

Although the adjacency matrix has been extensively studied, research on the antiadjacency matrix is relatively recent. The first known mention of this concept appeared in Stanley's work [6] on counting paths in directed acyclic graphs. Later, Skandera [7] formally introduced the term 'antiadjacency matrix' in the study of partially ordered sets. Since then, research has extended to other partially ordered sets [8-10], directed graphs [11-24] and undirected graphs [25-32]. Despite these contributions, many spectral properties of the antiadjacency matrix remain unexplored, particularly in the context of how graph operations affect its eigenvalues and determinant. This subtopic has been conducted recently for several graph operations [33-35].

A square matrix $\mathbf{\Omega}$ is called *circulant* if the ith row of $\mathbf{\Omega}$ is obtained by shifting the first row of $\mathbf{\Omega}$ to the right by $i - 1$ steps. Thus, $\mathbf{\Omega}$ is fully determined by its first row. A graph Γ is *circulant* if we can order its vertices such that its adjacency matrix $\mathbf{A}(\Gamma)$ is circulant [2]. Note that a circulant graph must be regular, and thus a circulant graph contains an isolated vertex, that is, a vertex of degree 0, if and only if it is isomorphic to $\overline{K_\eta}$.

Graph operations play a crucial role in understanding structural and spectral properties. One such fundamental operation is the Cartesian product, which combines two graphs while preserving their local structures. Let Γ_1 and Γ_2 be graphs. The *Cartesian product* $\Gamma_1 \times \Gamma_2$ has $V(\Gamma_1 \times \Gamma_2) = V(\Gamma_1) \times V(\Gamma_2)$, and two vertices (x_1, y_1) and (x_2, y_2) in $\Gamma_1 \times \Gamma_2$ are adjacent if (1) $x_1 = x_2$ and

$y_1y_2 \in E(\Gamma_2)$ or (2) $y_1 = y_2$ and $x_1x_2 \in E(\Gamma_1)$. Several recent studies have investigated structural aspects of Cartesian product graphs, including domination-related parameters [36, 37] and irregularity strength [38]. While these studies primarily focus on combinatorial properties, we explore the spectral characteristics of Cartesian products and other graph operations with respect to the antiadjacency matrix.

We denote by $\Gamma_1 \cup \Gamma_2$ the *disjoint union* of two disjoint graphs Γ_1 and Γ_2. In general, $\mu\Gamma$ denotes the graph consisting of μ disjoint copies of Γ. The *complement* $\overline{\Gamma}$ of a graph Γ is the graph where $V(\overline{\Gamma}) = V(\Gamma)$ and $E(\overline{\Gamma}) = \{xy: xy \notin E(\Gamma)\}$. The *join* $\Gamma_1 + \Gamma_2$ consists of $\Gamma_1 \cup \Gamma_2$ and edges joining each vertex of Γ_1 and each vertex of Γ_2.

Despite the growing interest in spectral graph theory, the spectral properties of the antiadjacency matrix for key graph operations remain largely unexamined. In this paper, we investigate the antiadjacency matrix spectrum for various graph families, including circulant graphs, cycle graphs, Cartesian products, disjoint unions, complements, and joins. In addition to spectral analysis, we also study the antiadjacency matrix determinant, which is a direct consequence of multiplying all its eigenvalues.

2. Preliminary Results

First, we give some results on matrix theory. For generalization, we assume all matrices are over a complex field $\mathbb{C}$. Nevertheless, since we consider only simple and undirected graphs, their adjacency and antiadjacency matrices are real and symmetric. For a matrix $\mathbf{\Omega}$, the notation $\mathbf{\Omega}^*$ denotes the conjugate transpose of $\mathbf{\Omega}$. In the real case, it is equivalent to the usual transpose of a matrix.

Theorem 2.1: ([39]). *If $\boldsymbol{\Omega}_{11}$ and $\boldsymbol{\Omega}_{22}$ are square matrices, and either $\boldsymbol{\Omega}_{12}$ or $\boldsymbol{\Omega}_{21}$ is a zero matrix, then*

$$\det\begin{bmatrix}\mathbf{\Omega}_{11} & \mathbf{\Omega}_{12}\\ \mathbf{\Omega}_{21} & \mathbf{\Omega}_{22}\end{bmatrix} = \det(\mathbf{\Omega}_{11})\det(\mathbf{\Omega}_{22}).$$

Theorem 2.2: ([39]). *If $\boldsymbol{\Omega}_{11}$ and $\boldsymbol{\Omega}_{22}$ are square matrices, then*

$$\det\begin{bmatrix}\mathbf{\Omega}_{11} & \mathbf{\Omega}_{12}\\ \mathbf{\Omega}_{21} & \mathbf{\Omega}_{22}\end{bmatrix} = \begin{cases}\det(\mathbf{\Omega}_{11})\det(\mathbf{\Omega}_{22} - \mathbf{\Omega}_{21}\mathbf{\Omega}_{11}^{-1}\mathbf{\Omega}_{12}), & \textit{when } \mathbf{\Omega}_{11}^{-1} \textit{ exists};\\ \det(\mathbf{\Omega}_{22})\det(\mathbf{\Omega}_{11} - \mathbf{\Omega}_{12}\mathbf{\Omega}_{22}^{-1}\mathbf{\Omega}_{21}), & \textit{when } \mathbf{\Omega}_{22}^{-1} \textit{ exists}.\end{cases}$$

Theorem 2.3: ([40]). *If $\boldsymbol{\Omega}_{11}, \boldsymbol{\Omega}_{12}, \boldsymbol{\Omega}_{21}, \boldsymbol{\Omega}_{22}$ are square matrices and $\boldsymbol{\Omega}_{21}\boldsymbol{\Omega}_{22} = \boldsymbol{\Omega}_{22}\boldsymbol{\Omega}_{21}$, then*

$$\det\begin{bmatrix}\mathbf{\Omega}_{11} & \mathbf{\Omega}_{12}\\ \mathbf{\Omega}_{21} & \mathbf{\Omega}_{22}\end{bmatrix} = \det(\mathbf{\Omega}_{11}\mathbf{\Omega}_{22} - \mathbf{\Omega}_{12}\mathbf{\Omega}_{21}).$$

Theorem 2.4: ([41]). *Let $\boldsymbol{\Omega}$ is a square matrix. Then λ is an eigenvalue of $\boldsymbol{\Omega}$ if and only if $det(\lambda \boldsymbol{I} - \boldsymbol{\Omega}) = 0$ which is called the characteristic equation of $\boldsymbol{\Omega}$.*

In some literature of graphs and matrices, the equation $\det(\lambda\mathbf{I} - \mathbf{\Omega}) = 0$ can also be written as $\det(\mathbf{\Omega} - \lambda\mathbf{I}) = 0$ since they are equivalent. So, for the following discussions, we will use $\det(\mathbf{\Omega} - \lambda\mathbf{I}) = 0$ as the characteristic equation of $\mathbf{\Omega}$.

Theorem 2.5: ([39]). *If $\mathbf{\Omega}$ is a $\eta \times \eta$ matrix with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_\eta$, then*

$$\det(\mathbf{\Omega}) = \prod_{i=1}^{\eta} \lambda_i.$$

We now mention some results on circulant matrices.

Proposition 2.6: ([42]). The eigenvalues of the circulant matrix with the first row $(\omega_0, \omega_1, \omega_2, \dots, \omega_{n-1})^*$ are

$$\chi_\alpha = \sum_{j=0}^{\eta-1} \omega_j \xi^{\alpha j},$$

where $\xi = \exp\left(\frac{2\pi i}{n}\right)$ and $\alpha \in [0, \eta - 1]$.

Throughout this paper, we will use Proposition 2.6 frequently to determine the eigenvalues of circulant graphs or matrices. Therefore, we will refer the eigenvalues χ_α of a circulant matrix in Proposition 2.6 as *circulant eigenvalues*.

Lemma 2.7: ([42]). *Let $\eta \geq 2$ and α be integers, and let $\xi = exp\left(\frac{2\pi i}{n}\right)$. Then*

$$\sum_{j=0}^{\eta-1} \xi^{\alpha j} = \begin{cases} 0, & if\ \eta\ does\ not\ \ divide\ \alpha; \\ \eta, & \text{otherwise} \end{cases}$$

Theorem 2.8: ([43]). *Let $\mathbf{\Omega}$ and $\mathbf{\Lambda}$ be two circulant matrices. Then*

1. $\mathbf{\Omega\Lambda} = \mathbf{\Lambda\Omega}$ *and* $\mathbf{\Omega\Lambda}$ *is also circulant, and*
2. $\mathbf{\Omega} + \mathbf{\Lambda}$ *is also circulant.*

Now we give the definition and properties of the Kronecker product of two matrices. Let $\mathbf{\Omega} = [\omega_{ij}]$ and $\mathbf{\Lambda}$ be $\alpha \times \beta$ and $\gamma \times \delta$ matrices, respectively. The *Kronecker product* $\mathbf{\Omega} \otimes \mathbf{\Lambda}$ of $\mathbf{\Omega}$ and $\mathbf{\Lambda}$ is defined as the partitioned matrix

$$\mathbf{\Omega} \otimes \mathbf{\Lambda} = \begin{bmatrix} \omega_{11}\mathbf{\Lambda} & \omega_{12}\mathbf{\Lambda} & \cdots & \omega_{1n}\mathbf{\Lambda} \\ \omega_{21}\mathbf{\Lambda} & \omega_{22}\mathbf{\Lambda} & \cdots & \omega_{2n}\mathbf{\Lambda} \\ \vdots & \vdots & & \vdots \\ \omega_{m1}\mathbf{\Lambda} & \omega_{m2}\mathbf{\Lambda} & \cdots & \omega_{mn}\mathbf{\Lambda} \end{bmatrix}.$$

The matrix $\mathbf{\Omega} \otimes \mathbf{\Lambda}$ has the size $\alpha\gamma \times \beta\delta$ that contains $\alpha\beta$ blocks with the (i, j)th block is the matrix $\omega_{ij}\mathbf{\Lambda}$ with the size $\gamma \times \delta$ [44].

Theorem 2.9: ([44]). *Let $\mathbf{\Omega}$ and $\mathbf{\Lambda}$ be two matrices with size $m \times m$ and $n \times n$, respectively. If λ_i and x_i, $i \in [1, m]$, are the eigenvalues and their corresponding eigenvector of $\mathbf{\Omega}$, and μ_j and y_j, $j \in [1, n]$, are the eigenvalues and their corresponding eigenvector of $\mathbf{\Lambda}$, then $\mathbf{\Omega} \otimes \mathbf{\Lambda}$ has eigenvalues $\lambda_i\mu_j$ with corresponding eigenvectors $x_i \otimes y_j$.*

Finally, we give some known results on the adjacency matrix of some graph classes.

Theorem 2.10: ([1]). *For every positive integers η and τ,*

1. $\mathrm{Spec}_{\mathbf{A}}(K_\eta) = \{\eta - 1, (-1)^{(\eta-1)}\}$, *and*
2. $\mathrm{Spec}_{\mathbf{A}}(K_{\eta,\tau}) = \{\pm\sqrt{\eta\tau}, 0^{(\eta+\tau-2)}\}$.

Theorem 2.11: ([45]). *For every integer $\eta \geq 3$,*

$$\det(\mathbf{A}(C_\eta)) = \begin{cases} 0, & \eta \equiv 0 \ (\mathrm{mod}\ 4); \\ 2, & \eta \equiv 1, 3 \ (\mathrm{mod}\ 4); \\ -4, & \eta \equiv 2 \ (\mathrm{mod}\ 4). \end{cases}$$

3. Antiadjacency Spectrum of Circulant Graphs and Their Disjoint Union

We now discuss the antiadjacency spectrum of a circulant graph and two copies of a circulant graph. First, we give a relation between the adjacency and antiadjacency spectrums of a circulant graph in the form of circulant eigenvalues.

Theorem 3.1: *Let Γ be a circulant graph of order η and degree ρ. If the circulant eigenvalues of $\boldsymbol{A}(\Gamma)$ are χ_α, $\alpha \in [0, \eta - 1]$, then the circulant eigenvalues of $\boldsymbol{B}(\Gamma)$ are*

$$\chi'_\alpha = \begin{cases} \eta - \rho, & \alpha = 0; \\ -\chi_\alpha, & \alpha \in [1, \eta - 1]. \end{cases}$$

Proof: Suppose that the first row of the circulant matrix $\mathbf{A}(\Gamma)$ is $(\omega_0, \omega_1, \dots, \omega_{\eta-1})^*$, so the first row of the circulant matrix $\mathbf{B}(\Gamma)$ is $(1 - \omega_0, 1 - \omega_1, \dots, 1 - \omega_{\eta-1})^*$. By Proposition 2.6, we have $\chi_\alpha = \sum_{j=0}^{\eta-1} \omega_j \xi^{\alpha j}$ and

$$\chi'_\alpha = \sum_{j=0}^{\eta-1} (1 - \omega_j)\xi^{\alpha j} = \sum_{j=0}^{\eta-1} \xi^{\alpha j} - \sum_{j=0}^{\eta-1} \omega_j \xi^{\alpha j} = \sum_{j=0}^{\eta-1} \xi^{\alpha j} - \chi_\alpha.$$

We know the values of $\sum_{j=0}^{\eta-1} \xi^{\alpha j}$ by Lemma 2.7, and $\chi_0 = \rho$, hence the result follows.

Corollary 3.2: *Let $\Gamma \neq \overline{K_\eta}$ be a circulant graph of order η and degree ρ. Then*

$$\det(\mathbf{B}(\Gamma)) = (-1)^{\eta-1} \frac{\eta - \rho}{\rho} \det(\mathbf{A}(\Gamma)).$$

Proof: Suppose that the circulant eigenvalues of $\mathbf{A}(\Gamma)$ are χ_α, $\alpha \in [0, \eta - 1]$. Note that $\rho > 0$ since $\Gamma \neq \overline{K_\eta}$. Theorems 3.1 and 3.2 imply

$$\begin{aligned} \det(\mathbf{B}(\Gamma)) &= \prod_{\alpha=0}^{\eta-1} \chi'_\alpha = (-1)^{\eta-1}(\eta - \rho) \prod_{\alpha=1}^{\eta-1} \chi_\alpha = (-1)^{\eta-1} \frac{(\eta-\rho)}{\chi_0} \prod_{\alpha=0}^{\eta-1} \chi_\alpha \\ &= (-1)^{\eta-1} \frac{\eta - \rho}{\rho} \det(\mathbf{A}(\Gamma)). \end{aligned}$$

Next, we discuss the antiadjacency spectrum of two copies of circulant graphs. For that, we need the following lemmas.

Lemma 3.3: *Let $\boldsymbol{J}_\eta$ be the $\eta \times \eta$ all-ones matrix. Then $Spec(\boldsymbol{J}_\eta) = \{\eta, 0^{(\eta-1)}\}$.*

Proof: Observe that $\mathbf{J}_\eta$ is circulant with the first row $(1,1,\dots,1)^*$. Then by Proposition 2.6, the eigenvalues of $\mathbf{J}_\eta$ are $\chi_\alpha = \sum_{j=0}^{\eta-1} \xi^{\alpha j}$ for $\alpha \in [0,\eta-1]$. Since η divides α when $\alpha = 0$ and η does not divide α when $\alpha \in [1,\eta-1]$, then by Lemma 2.7, the circulant eigenvalues of $\mathbf{J}_\eta$ are

$$\chi_\alpha = \sum_{j=0}^{\eta-1} \xi^{\alpha j} = \begin{cases} \eta, & \alpha = 0; \\ 0, & \alpha \in [1,\eta-1]. \end{cases}$$

Lemma 3.4: *If $\boldsymbol{\Omega}$ and $\boldsymbol{\Lambda}$ are two $\eta \times \eta$ circulant matrices with circulant eigenvalues λ_α and μ_α, $\alpha \in [0,\eta-1]$, respectively, then the eigenvalues of the matrix $\boldsymbol{\Pi}$ where*

$$\boldsymbol{\Pi} = \begin{bmatrix} \boldsymbol{\Omega} & \boldsymbol{\Lambda} \\ \boldsymbol{\Lambda} & \boldsymbol{\Omega} \end{bmatrix}$$

are $\lambda_\alpha \pm \mu_\alpha$ for all $\alpha \in [0,\eta-1]$.

Proof: Let λ be an eigenvalue of $\boldsymbol{\Pi}$, so $\det(\boldsymbol{\Pi} - \lambda \mathbf{I}_{2\eta}) = 0$ by Theorem 2.4. Observe that

$$\boldsymbol{\Pi} - \lambda \mathbf{I}_{2\eta} = \begin{bmatrix} \boldsymbol{\Omega} - \lambda \mathbf{I}_\eta & \boldsymbol{\Lambda} \\ \boldsymbol{\Lambda} & \boldsymbol{\Omega} - \lambda \mathbf{I}_\eta \end{bmatrix}.$$

Since $\boldsymbol{\Omega}$, $\boldsymbol{\Lambda}$, and $\mathbf{I}_\eta$ are circulant, the matrix $\boldsymbol{\Omega} - \lambda \mathbf{I}_\eta$ is also circulant by Theorem 2.8 (2), and $(\boldsymbol{\Omega} - \lambda \mathbf{I}_\eta)\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(\boldsymbol{\Omega} - \lambda \mathbf{I}_\eta)$ by Theorem 2.8 (1). Thus, by Theorem 2.3, we have

$$\begin{aligned} \det(\boldsymbol{\Pi} - \lambda \mathbf{I}_{2n}) &= \det((\boldsymbol{\Omega} - \lambda \mathbf{I}_\eta)^2 - \boldsymbol{\Lambda}^2) \\ &= \det((\boldsymbol{\Omega} - \lambda \mathbf{I}_\eta + \boldsymbol{\Lambda})(\boldsymbol{\Omega} - \lambda \mathbf{I}_\eta - \boldsymbol{\Lambda})) \\ &= \det(\boldsymbol{\Omega} + \boldsymbol{\Lambda} - \lambda \mathbf{I}_\eta)\det(\boldsymbol{\Omega} - \boldsymbol{\Lambda} - \lambda \mathbf{I}_\eta). \end{aligned}$$

Observe that $\det(\boldsymbol{\Pi} - \lambda \mathbf{I}_{2n}) = 0$ if and only if $\det(\boldsymbol{\Omega} + \boldsymbol{\Lambda} - \lambda \mathbf{I}_\eta) = 0$ or $\det(\boldsymbol{\Omega} - \boldsymbol{\Lambda} - \lambda \mathbf{I}_\eta) = 0$. This implies that $\mathrm{Spec}(\boldsymbol{\Pi}) = \mathrm{Spec}(\boldsymbol{\Omega} + \boldsymbol{\Lambda}) \cup \mathrm{Spec}(\boldsymbol{\Omega} - \boldsymbol{\Lambda})$.

Let the first rows of $\boldsymbol{\Omega}$ and $\boldsymbol{\Lambda}$ are $(\omega_0, \omega_1, \dots, \omega_{\eta-1})^*$ and $(\delta_0, \delta_1, \dots, \delta_{\eta-1})^*$, respectively. Then by Proposition 2.6, the circulant eigenvalues of $\boldsymbol{\Omega}$ and $\boldsymbol{\Lambda}$ are $\lambda_\alpha = \sum_{j=0}^{\eta-1} \omega_j \xi^{\alpha j}$ and $\mu_\alpha = \sum_{j=0}^{\eta-1} \delta_j \xi^{\alpha j}$, $\alpha \in [0,\eta-1]$, respectively. Since $\boldsymbol{\Omega}$ and $\boldsymbol{\Lambda}$ are circulant, by Theorem 2.8 (2), $\boldsymbol{\Omega} + \boldsymbol{\Lambda}$ and $\boldsymbol{\Omega} - \boldsymbol{\Lambda}$ are also circulant with the first rows $(\omega_0 + \delta_0, \dots, \omega_{n-1} + \delta_{\eta-1})^*$ and $(\omega_0 - \delta_0, \dots, \omega_{\eta-1} - \delta_{\eta-1})^*$, respectively. Let χ_α^+ and χ_α^-, $\alpha \in [0,\eta-1]$, be the circulant eigenvalues of $\boldsymbol{\Omega} + \boldsymbol{\Lambda}$ and $\boldsymbol{\Omega} - \boldsymbol{\Lambda}$, respectively. Then

$$\chi_\alpha^+ = \sum_{j=0}^{\eta-1} (\omega_j + \delta_j)\xi^{\alpha j} = \sum_{j=0}^{\eta-1} \omega_j \xi^{\alpha j} + \sum_{j=0}^{\eta-1} \delta_j \xi^{\alpha j} = \lambda_\alpha + \mu_\alpha$$

and

$$\chi_\alpha^- = \sum_{j=0}^{\eta-1} (\omega_j - \delta_j)\xi^{\alpha j} = \sum_{j=0}^{\eta-1} \omega_j \xi^{\alpha j} - \sum_{j=0}^{\eta-1} \delta_j \xi^{\alpha j} = \lambda_\alpha - \mu_\alpha.$$

Therefore, the eigenvalues of $\mathbf{\Pi}$ are the union of all χ_α^+ and χ_α^-, that is, $\lambda_\alpha \pm \mu_\alpha$ for all $\alpha \in [0, \eta - 1]$.

Theorem 3.5: *Let Γ be a circulant graph of order η and degree ρ. If the circulant eigenvalues of $\boldsymbol{A}(\Gamma)$ are χ_α, $\alpha \in [0, \eta - 1]$, then*

$$\mathrm{Spec}_{\mathbf{B}}(2\Gamma) = \{2\eta - \rho, -\rho\} \cup \{(-\chi_\alpha)^{(2)}: \alpha \in [1, \eta - 1]\}.$$

Proof: Let $\mathbf{J}_\eta$ be the $\eta \times \eta$ all-ones matrix. We know that $\mathbf{B}(\Gamma)$ and $\mathbf{J}_\eta$ are circulant matrices. Suppose that the circulant eigenvalues of $\mathbf{B}(\Gamma)$ are $\lambda_\alpha = \chi_{\alpha'}$ for $\alpha \in [0, \eta - 1]$ according to Theorem 3.1. By Lemma 3.3, the circulant eigenvalues of $\mathbf{J}_\eta$ are

$$\mu_\alpha = \begin{cases} \eta, & \alpha = 0; \\ 0, & \alpha \in [1, \eta - 1]. \end{cases}$$

Then, by Lemma 3.4, the eigenvalues of

$$\mathbf{B}(2\Gamma) = \begin{bmatrix} \mathbf{B}(\Gamma) & \mathbf{J}_\eta \\ \mathbf{J}_\eta & \mathbf{B}(\Gamma) \end{bmatrix}$$

are $\lambda_\alpha \pm \mu_\alpha$. For $\alpha = 0$, $\lambda_0 \pm \mu_0 = \chi_{0'} \pm \eta = (\eta - \rho) \pm \eta = 2\eta - \rho$ and $-\rho$ with multiplicity 1 each by Theorem 3.1, and for each $\alpha \in [1, \eta - 1]$, $\lambda_\alpha \pm \mu_\alpha = \chi_{\alpha'} \pm 0 = \chi_{\alpha'} = -\chi_\alpha$ with multiplicity 2 since $-\chi_\alpha$ holds for both $\lambda_\alpha + \mu_\alpha$ and $\lambda_\alpha - \mu_\alpha$

From Theorems 3.5 and 2.5, we obtain Corollary 3.6.

Corollary 3.6: *Let Γ be a circulant graph of order η and degree ρ. Then*

$$\det(\mathbf{B}(2\Gamma)) = \frac{(\eta-\rho)^2 - \eta^2}{(\eta-\rho)^2} (\det(\mathbf{B}(\Gamma)))^2.$$

4. Antiadjacency Spectrum of Cycle Graphs and Their Disjoint Union

We now discuss the antiadjacency spectrum of cycle graphs and their disjoint union using the fact that cycle graphs are circulant. The following theorem gives the antiadjacency spectrum of a cycle graph as a subclass of a circulant graph whose spectrum has been discussed in the previous section. Note that this is a general result of the spectrum of $\mathbf{B}(C_\eta)$ which was previously discussed by Alyani in [30]. She proved that the antiadjacency spectrum of the cycle graph C_η contains $\eta - 2$ for odd η and $\{2, \eta - 2\}$ for even η.

Lemma 4.1: *For every integer $\eta \geq 3$, the circulant eigenvalues of $\boldsymbol{A}(C_\eta)$ are $\chi_\alpha = 2cos(\frac{2\pi\alpha}{\eta})$ for $\alpha \in [0, \eta - 1]$.*

Proof: Since $\mathbf{A}(C_\eta)$ is circulant with first row $(\omega_0, \omega_1, \omega_2, \dots, \omega_{n-1})^* = (0,1,0,\dots,0,1)^*$, then the circulant eigenvalues of $\mathbf{A}(C_\eta)$ are $\chi_\alpha = \sum_{j=0}^{\eta-1} \omega_j \xi^{\alpha j} = \xi^\alpha + \xi^{(\eta-1)\alpha} = 2\cos\left(\frac{2\pi\alpha}{\eta}\right)$ for $\alpha \in [0, \eta - 1]$.

Theorem 4.2: *For every integer* $\eta \geq 3$*, the circulant eigenvalues of* $\boldsymbol{B}(C_\eta)$ *are*

$$\chi'_\alpha = \begin{cases} \eta - 2, & \alpha = 0; \\ -2\cos\left(\frac{2\pi\alpha}{\eta}\right), & \alpha \in [1, \eta - 1]. \end{cases}$$

Proof: The result follows from Lemma 4.1 and Theorem 3.1.

Corollary 4.3: *For every integer* $\eta \geq 3$*,*

$$\det(\mathbf{B}(C_\eta)) = \begin{cases} 0, & \eta \equiv 0 \pmod 4; \\ \eta - 2, & \eta \equiv 1, 3 \pmod 4; \\ 2(\eta - 2), & \eta \equiv 2 \pmod 4. \end{cases}$$

Next, we give an identity that is useful in the following sections. This identity can be proved by utilizing previous results and can also be found in [46, 47]. We also give another proof in Appendix A by using several calculus observations.

Corollary 4.4: *For every integer* $\eta \geq 2$*,*

$$\prod_{\alpha=1}^{\eta-1} \cos\left(\frac{2\pi\alpha}{\eta}\right) = \begin{cases} 0, & \eta \equiv 0 \pmod 4; \\ \frac{1}{2^{\eta-1}}, & \eta \equiv 1, 3 \pmod 4; \\ -\frac{1}{2^{\eta-2}}, & \eta \equiv 2 \pmod 4. \end{cases}$$

Proof: The formula is true for $\eta = 2$. Now assume that $\eta \geq 3$. By using Theorem 2.5 on the eigenvalues of $B(C_\eta)$ in Theorem 4.2, we have

$$\det(\mathbf{B}(C_\eta)) = \prod_{\alpha=0}^{\eta-1} \chi_{\alpha'} = (\eta - 2)(-2)^{\eta-1} \prod_{\alpha=1}^{\eta-1} \cos\left(\frac{2\pi\alpha}{\eta}\right),$$

and by matching this formula with Corollary 4.3, we obtain the result.

Next, we present the antiadjacency spectrum of two copies of C_η. From Theorems 3.5 and 4.2, we obtain Theorem 4.5.

Theorem 4.5: *For every integer* $\eta \geq 3$*,*

$$\mathrm{Spec}_{\mathbf{B}}(2C_\eta) = \{2\eta - 2, -2\} \cup \left\{\left(-2\cos(2\pi\alpha/\eta)\right)^{(2)} : \alpha \in [1, \eta - 1]\right\}.$$

From Theorem 2.11 and Corollary 3.6, we obtain Corollary 4.6

Corollary 4.6: *For every integer* $\eta \geq 3$*,*

$$\det(\mathbf{B}(2C_\eta)) = \begin{cases} 0, & n \equiv 0 \pmod 4; \\ (-4)(\eta - 1), & \eta \equiv 1, 3 \pmod 4; \\ (-16)(\eta - 1), & \eta \equiv 2 \pmod 4. \end{cases}$$

Next, as a generalization of Corollary 4.6, we discuss the determinant of the antiadjacency matrix of the disjoint union of two cycle graphs with different orders.

Theorem 4.7: *Let η and τ be positive integers with $\eta,\tau \geq 3$, and $\eta \not\equiv 0(mod4)$ or $\tau \not\equiv 0(mod4)$. Then*

$$\det(\mathbf{B}(C_\eta \cup C_\tau)) = \begin{cases} 0, & \eta \equiv 0(\text{mod } 4) \text{ or } \tau \equiv 0 \text{ (mod } 4); \\ (-2)(\eta+\tau-2), & \eta,\tau \equiv 1{,}3(\text{mod } 4); \\ (-4)(\eta+\tau-2), & \eta,\tau \not\equiv 0(\text{mod } 4) \text{ and } \eta \not\equiv \tau(\text{mod } 4); \\ (-8)(\eta+\tau-2), & \eta,\tau \equiv 2(\text{mod } 4). \end{cases}$$

Proof: We may write the antiadjacency matrix of $C_\eta \cup C_\tau$ as

$$\mathbf{B}(C_\eta \cup C_\tau) = \begin{bmatrix} \mathbf{B}(C_\eta) & \mathbf{J}_{\eta\times\tau} \\ \mathbf{J}_{\tau\times\eta} & \mathbf{B}(C_\tau) \end{bmatrix}.$$

To use Theorem 2.2, it must be the case that $B(C_\eta)^{-1}$ or $B(C_\tau)^{-1}$ exists, which Corollary 4.3 implies that $\eta \not\equiv 0(\text{mod}4)$ or $\tau \not\equiv 0(\text{mod}4)$. Suppose first that $\eta \not\equiv 0(\text{mod}4)$, so $\mathbf{B}(C_\eta)^{-1}$ exists by Corollary 4.3. From Theorem 4.2, it can be easily confirmed that $\eta-2$ is an eigenvalue of $\mathbf{B}(C_\eta)$ with eigenvector $\mathbf{e}_\eta$ where $\mathbf{e}_\eta$ is the $\eta\times 1$ all-ones vector. Then $\mathbf{B}(C_\eta)\mathbf{e}_\eta = (\eta-2)\mathbf{e}_\eta$ implies $\mathbf{B}(C_\eta)^{-1}\mathbf{e}_\eta = \frac{1}{\eta-2}\mathbf{e}_\eta$. Observe that

$$\begin{aligned} \mathbf{J}_{\tau\times\eta}\mathbf{B}(C_\eta)^{-1}\mathbf{J}_{\eta\times\tau} &= (\mathbf{J}_{\tau\times\eta})\mathbf{B}(C_\eta)^{-1}(\mathbf{e}_\eta\mathbf{e}_\tau^*) \\ &= (\mathbf{J}_{\tau\times\eta})(\mathbf{B}(C_\eta)^{-1}\mathbf{e}_\eta)\mathbf{e}_\tau^* \\ &= \mathbf{J}_{\tau\times\eta}\left(\frac{1}{\eta-2}\mathbf{e}_\eta\right)\mathbf{e}_\tau^* \\ &= \frac{1}{\eta-2}\mathbf{J}_{\tau\times\eta}(\mathbf{e}_\eta\mathbf{e}_\tau^*) \\ &= \frac{1}{\eta-2}\mathbf{J}_{\tau\times\eta}\mathbf{J}_{\eta\times\tau} = \frac{\eta}{\eta-2}\mathbf{J}_{\tau\times\tau}. \end{aligned}$$

Then, by Theorem 2.2 and the above calculation, we have

$$\begin{aligned} \det(\mathbf{B}(C_\eta \cup C_\tau)) &= \det(\mathbf{B}(C_\eta))\det(\mathbf{B}(C_\tau) - \mathbf{J}_{\tau\times\eta}\mathbf{B}(C_\eta)^{-1}\mathbf{J}_{\eta\times\tau}) \\ &= \det(\mathbf{B}(C_\eta))\det(\mathbf{B}(C_\tau) - \sigma\mathbf{J}_{\tau\times\tau}) \end{aligned}$$

where $\sigma = \eta/(\eta-2)$. Since $\mathbf{B}(C_\tau)$ is circulant with the first row $(1,0,1,1,\dots,1,0)^*$, then $\mathbf{B}(C_\tau) - \sigma\mathbf{J}_{\tau\times\tau}$ is also circulant with the first row $(\omega_0,\omega_1,\dots,\omega_{\tau-1})^* = (1-\sigma,-\sigma,1-\sigma,1-\sigma,\dots,1-\sigma,-\sigma)^*$. Proposition 2.6 implies that the circulant eigenvalues of $\mathbf{B}(C_\tau) - \sigma\mathbf{J}_{\tau\times\tau}$ are

$$\begin{aligned} \chi_\alpha &= \textstyle\sum_{j=0}^{\tau-1} \omega_j \xi^{\alpha j} = (1-\sigma) - \sigma\xi^\alpha + (1-\sigma)\sum_{j=2}^{\tau-2} \xi^{\alpha j} - \sigma\xi^{(\tau-1)\alpha} \\ &= (1-\sigma)\left(\textstyle\sum_{j=0}^{\tau-1} \xi^{\alpha j}\right) - (\xi^\alpha + \xi^{(\tau-1)\alpha}) \end{aligned}$$

for $\alpha \in [0,\tau-1]$. Since τ divides 0, then by Lemma 2.7, $\chi_0 = \tau(1-\sigma) - 2$, and for $\alpha \in [1,\tau-1]$, since τ does not divide α, then $\chi_\alpha = (1-\sigma)(0) - (\xi^\alpha + \xi^{-\alpha}) = -2\cos\left(\frac{2\pi\alpha}{\tau}\right)$. Then the circulant eigenvalues of $\mathbf{B}(C_\tau) - \sigma\mathbf{J}_{\tau\times\tau}$ are

$$\chi_\alpha = \begin{cases} \tau(1-\sigma)-2, & \alpha = 0; \\ -2\cos\left(\frac{2\pi\alpha}{\tau}\right), & \alpha \in [1, \tau-1]. \end{cases}$$

Theorem 2.5 implies

$$\begin{aligned} \det(\mathbf{B}(C_\tau) - \sigma \mathbf{J}_{\tau\times\tau}) &= \prod_{\alpha=0}^{\tau-1} \chi_\alpha = (\tau(1-\sigma)-2)(-2)^{\tau-1} \prod_{\alpha=1}^{\tau-1} \cos\left(\frac{2\pi\alpha}{\tau}\right) \\ &= (-2)^\tau \left(\frac{\eta+\tau-2}{\eta-2}\right) \prod_{\alpha=1}^{\tau-1} \cos\left(\frac{2\pi\alpha}{\tau}\right). \end{aligned}$$

By Corollary 4.3 and Corollary 4.4, there are six cases for η and τ.

Case 1: If $\eta \equiv 1,3 \pmod 4$ and $\tau \equiv 0 \pmod 4$, then

$$\det(\mathbf{B}(C_\eta \cup C_\tau)) = (\eta-2)(2^\tau)\left(\frac{\eta+\tau-2}{\eta-2}\right)(0) = 0.$$

Case 2: If $\eta \equiv 1,3 \pmod 4$ and $\tau \equiv 1,3 \pmod 4$, then

$$\det(\mathbf{B}(C_\eta \cup C_\tau)) = (\eta-2)(-2^\tau)\left(\frac{\eta+\tau-2}{\eta-2}\right)\left(\frac{1}{2^{\tau-1}}\right) = (-2)(\eta+\tau-2).$$

Case 3: If $\eta \equiv 1,3 \pmod 4$ and $\tau \equiv 2 \pmod 4$, then

$$\det(\mathbf{B}(C_\eta \cup C_\tau)) = (\eta-2)(2^\tau)\left(\frac{\eta+\tau-2}{\eta-2}\right)\left(-\frac{1}{2^{\tau-2}}\right) = (-4)(\eta+\tau-2).$$

Case 4: If $\eta \equiv 2 \pmod 4$ and $\tau \equiv 0 \pmod 4$, then

$$\det(\mathbf{B}(C_\eta \cup C_\tau)) = 2(\eta-2)(2^\tau)\left(\frac{\eta+\tau-2}{\eta-2}\right)(0) = 0.$$

Case 5: If $\eta \equiv 2 \pmod 4$ and $\tau \equiv 1,3 \pmod 4$, then

$$\det(\mathbf{B}(C_\eta \cup C_\tau)) = 2(\eta-2)(-2^\tau)\left(\frac{\eta+\tau-2}{\eta-2}\right)\left(\frac{1}{2^{\tau-1}}\right) = (-4)(\eta+\tau-2).$$

Case 6: If $\eta \equiv 2 \pmod 4$ and $\tau \equiv 2 \pmod 4$, then

$$\det(\mathbf{B}(C_\eta \cup C_\tau)) = 2(\eta-2)(2^\tau)\left(\frac{\eta+\tau-2}{\eta-2}\right)\left(-\frac{1}{2^{\tau-2}}\right) = (-8)(\eta+\tau-2).$$

Now, we consider the case where $\tau \not\equiv 0 \pmod 4$. With similar arguments from the case $\eta \not\equiv 0 \pmod 4$, we have

$$\det(\mathbf{B}(C_\eta \cup C_\tau)) = \det(B(C_\tau))(-2)^\eta \left(\frac{\eta+\tau-2}{\tau-2}\right) \prod_{\alpha=1}^{\eta-1} \cos\left(\frac{2\pi\alpha}{\eta}\right)$$

and we obtain two additional cases as follows.

Case 7: If $\eta \equiv 0 \pmod 4$ and $\tau \equiv 1,3 \pmod 4$, then

$$\det(\mathbf{B}(C_\eta \cup C_\tau)) = (\tau-2)(2^\eta)\left(\frac{\eta+\tau-2}{\tau-2}\right)(0) = 0.$$

Case 8: If $\eta \equiv 0 \pmod 4$ and $\tau \equiv 2 \pmod 4$, then

$$\det(\mathbf{B}(C_\eta \cup C_\tau)) = 2(\tau-2)(2^\eta)\left(\frac{\eta+\tau-2}{\tau-2}\right)(0) = 0.$$

The formula in Theorem 4.7 is summarized into Table 1. Note that we have not found the result for the case $\eta \equiv 0 (\mathrm{mod}\ 4)$ and $\tau \equiv 0 (\mathrm{mod}\ 4)$ since if these conditions hold, then $\mathbf{B}(C_\eta)^{-1}$ and $\mathbf{B}(C_\tau)^{-1}$ do not exist by Corollary 4.3, so Theorem 2.2 cannot be used to find $\det(\mathbf{B}(C_\eta \cup C_\tau))$.

Table 1

Determinant of $\mathbf{B}(\boldsymbol{C_\eta} \cup \boldsymbol{C_\tau})$

$\det(\mathbf{B}(\boldsymbol{C_\eta} \cup \boldsymbol{C_\tau}))$	$\boldsymbol{\tau \equiv 0}$ **(mod 4)**	$\boldsymbol{\tau \equiv 1,3}$ **(mod 4)**	$\boldsymbol{\tau \equiv 2}$ **(mod 4)**
$\eta \equiv 0$ (mod 4)	?	0	0
$\eta \equiv 1{,}3$ (mod 4)	0	$(-2)(\eta+\tau-2)$	$(-4)(\eta+\tau-2)$
$\eta \equiv 2$ (mod 4)	0	$(-4)(\eta+\tau-2)$	$(-8)(\eta+\tau-2)$

5. Antiadjacency Spectrum of $\boldsymbol{C_\eta \times K_2}$

We now discuss the antiadjacency spectrum of $C_\eta \times K_2$. For that, we need the following lemma.

Lemma 5.1: *For every integer $\eta \geq 1$, the circulant eigenvalues of $\boldsymbol{A}(K_\eta)$ are*

$$\chi_\alpha = \begin{cases} \eta - 1, & \alpha = 0; \\ -1, & \alpha \in [1, \eta - 1]. \end{cases}$$

Proof: Observe that $\mathbf{A}(K_\eta)$ is a $\eta \times \eta$ circulant matrix with the first row $(\omega_0, \omega_1, \dots, \omega_{\eta-1})^* = (0,1,1,\dots,1)^*$. Then, by Proposition 2.6, $\mathbf{A}(K_\eta)$ has eigenvalues

$$\chi_\alpha = \sum_{j=0}^{\eta-1} \omega_j \xi^{\alpha j} = \sum_{j=1}^{\eta-1} \xi^{\alpha j} = \left(\sum_{j=0}^{\eta-1} \xi^{\alpha j}\right) - 1$$

for $\alpha \in [0, \eta-1]$. By Lemma 2.7, $\chi_0 = \eta - 1$ since η divides 0, and for $\alpha \in [1, \eta - 1]$, η does not divide α, so $\chi_\alpha = 0 - 1 = -1$.

Theorem 5.2: *For every integer $\eta \geq 3$,*

$$\mathrm{Spec}_{\mathbf{B}}(C_\eta \times K_2) = \{2\eta - 3, -1\} \cup \left\{-2\cos\left(\frac{2\pi\alpha}{\eta}\right) \pm 1 : \alpha \in [1, \eta - 1]\right\}.$$

Proof: Observe that

$$\mathbf{B}(C_\eta \times K_2) = \begin{bmatrix} \mathbf{B}(C_\eta) & \mathbf{A}(K_\eta) \\ \mathbf{A}(K_\eta) & \mathbf{B}(C_\eta) \end{bmatrix}.$$

By Theorem 4.2, the circulant eigenvalues of $\mathbf{B}(C_\eta)$ are

$$\lambda_\alpha = \begin{cases} \eta - 2, & \alpha = 0; \\ -2\cos\left(\frac{2\pi\alpha}{\eta}\right), & \alpha \in [1, \eta - 1], \end{cases}$$

and by Lemma 5.1, the circulant eigenvalues of $\mathbf{A}(K_\eta)$ are

$$\mu_\alpha = \begin{cases} \eta - 1, & \alpha = 0; \\ -1, & \alpha \in [1, \eta - 1]. \end{cases}$$

Then, by Lemma 3.4, the eigenvalues of $\mathbf{B}(C_\eta \times K_2)$ are $\lambda_\alpha \pm \mu_\alpha$ for $\alpha \in [0, \eta - 1]$. For $\alpha = 0$, we have $\lambda_0 \pm \mu_0 = (\eta - 2) \pm (\eta - 1) = (2\eta - 3), -1$, and for $\alpha \in [1, \eta - 1]$, we have $\lambda_\alpha \pm \mu_\alpha = -2\cos\left(\frac{2\pi\alpha}{\eta}\right) \pm (-1) = -2\cos\left(\frac{2\pi\alpha}{\eta}\right) \mp 1$. This completes the proof.

Corollary 5.3: *For every integer* $\eta \geq 3$,

$$\det(\mathbf{B}(C_\eta \times K_2)) = (-1)(2\eta - 3) \prod_{\alpha=1}^{\eta-1} \left(4\cos^2\left(\frac{2\pi\alpha}{\eta}\right) - 1\right).$$

Proof: This follows directly from Theorem 5.2 and Theorem 2.5.

6. Antiadjacency Spectrum of Complete Graphs and Their Disjoint Union

We now discuss the antiadjacency spectrum of complete graphs and their disjoint union using the concept of Kronecker product and its eigenvalues. Alyani [30] found the antiadjacency spectrum of the complete graph K_η for $\eta \geq 4$ in Theorem 6.1, which follows immediately from the fact that $\mathbf{B}(K_\eta) = \mathbf{I}_\eta$.

Theorem 6.1: ([30]). *For every integer* $\eta \geq 1$, $Spec_{\boldsymbol{B}}(K_\eta) = \{1^{(\eta)}\}$.

It is a direct consequence, using Theorem 2.5, that $\det(\mathbf{B}(K_n)) = 1$, a result given by Irawan and Sugeng in [27]. Next, we give the antiadjacency spectrum of the disjoint union of complete graphs. For that, we need the following obvious Lemma.

Lemma 6.2: *Let* $\boldsymbol{A}$ *be a square matrix and* $\boldsymbol{I}$ *be the identity matrix of the same size as* $\boldsymbol{A}$. *Then* $Spec(\boldsymbol{A} + \boldsymbol{I}) = \{\lambda + 1 : \lambda \in Spec(\boldsymbol{A})\}$.

Proof: Note that $\det(\mathbf{A} - \lambda\mathbf{I}) = \det((\mathbf{A} + \mathbf{I}) - (\lambda + 1)\mathbf{I})$. It is clear that $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ if and only if $\det((\mathbf{A} + \mathbf{I}) - (\lambda + 1)\mathbf{I}) = 0$. Then, by Theorem 2.4, λ is an eigenvalue of $\mathbf{A}$ if and only if $\lambda + 1$ is an eigenvalue of $\mathbf{A} + \mathbf{I}$. Consequently, this preserves the multiplicities, and the assertion follows.

Theorem 6.3: *For every positive integers* η *and* μ,

$$\mathrm{Spec}_{\mathbf{B}}(\mu K_\eta) = \left\{\eta(\mu - 1) + 1, 1^{(\mu(\eta-1))}, (-(\eta - 1))^{(\mu-1)}\right\}.$$

Proof: Let $\mathbf{I}_\eta$ be the $\eta \times \eta$ identity matrix, and let $\mathbf{J}_\eta$ be the $\eta \times \eta$ all-ones matrix. Since $\mathbf{B}(K_\eta) = \mathbf{I}_\eta$, we may write the antiadjacency matrix of μK_η as

$$\mathbf{B}(\mu K_\eta) = \begin{bmatrix} \mathbf{I}_\eta & \mathbf{J}_\eta & \mathbf{J}_\eta & \cdots & \mathbf{J}_\eta \\ \mathbf{J}_\eta & \mathbf{I}_\eta & \mathbf{J}_\eta & \cdots & \mathbf{J}_\eta \\ \mathbf{J}_\eta & \mathbf{J}_\eta & \mathbf{I}_\eta & \cdots & \mathbf{J}_\eta \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{J}_\eta & \mathbf{J}_\eta & \mathbf{J}_\eta & \cdots & \mathbf{I}_\eta \end{bmatrix} = \mathbf{A}(K_\mu) \otimes \mathbf{J}_\eta + \mathbf{I}_{\mu\eta}.$$

Applying Theorem 2.9 on $\mathrm{Spec}_{\mathbf{A}}(K_\mu)$ in Theorem 2.10 and $\mathrm{Spec}(\mathbf{J}_\eta)$ in Lemma 3.3 gives

$$\mathrm{Spec}(\mathbf{A}(K_\mu)\otimes\mathbf{J}_\eta)=\{\eta(\mu-1),0^{(\mu(\eta-1))},(-\eta)^{(\mu-1)}\}.$$

and applying Lemma 6.2 gives

$$\mathrm{Spec}_{\mathbf{B}}(\mu K_\eta)=\mathrm{Spec}(\mathbf{A}(K_\mu)\otimes\mathbf{J}_\eta+\mathbf{I}_{\mu\eta})=$$
$$\{\eta(\mu-1)+1,1^{(\mu(\eta-1))},(-(\eta-1))^{(\mu-1)}\}.$$

Corollary 6.4: *For every positive integers η and μ,*

$$\det(\mathbf{B}(\mu K_\eta))=(-1)^{\mu-1}(\eta-1)^{\mu-1}(\eta(\mu-1)+1).$$

Finally, we give the antiadjacency spectrum of disjoint union of two complete graphs with different orders.

Theorem 6.5: *For every positive integers η and τ,*

$$\mathrm{Spec}_{\mathbf{B}}(K_\eta\cup K_\tau)=\{1\pm\sqrt{\eta\tau},1^{(\eta+\tau-2)}\}$$

Proof: Observe that $\mathbf{B}(K_\eta)=\mathbf{I}_\eta$ and $\mathbf{B}(K_\tau)=\mathbf{I}_\tau$. Then

$$\mathbf{B}(K_\eta\cup K_\tau)=\begin{bmatrix}\mathbf{B}(K_\eta) & \mathbf{J}_{\eta\times\tau}\\ \mathbf{J}_{\tau\times\eta} & \mathbf{B}(K_\tau)\end{bmatrix}=\begin{bmatrix}\mathbf{I}_\eta & \mathbf{J}_{\eta\times\tau}\\ \mathbf{J}_{\tau\times\eta} & \mathbf{I}_\tau\end{bmatrix}=\begin{bmatrix}\mathbf{O} & \mathbf{J}_{\eta\times\tau}\\ \mathbf{J}_{\tau\times\eta} & \mathbf{O}\end{bmatrix}+\begin{bmatrix}\mathbf{I}_\eta & \mathbf{O}\\ \mathbf{O} & \mathbf{I}_\tau\end{bmatrix}$$
$$=\mathbf{A}(K_{\eta,\tau})+\mathbf{I}_{\eta+\tau}$$

where $\mathbf{J}_{\eta\times\tau}$ is the $\eta\times\tau$ all-ones matrix and $\mathbf{O}$ is the zero matrix. Theorem 2.10 implies that $\mathrm{Spec}_{\mathbf{A}}(K_{\eta,\tau})=\{\pm\sqrt{\eta\tau},0^{(\eta+\tau-2)}\}$. The result follows by Lemma 6.2.

From Theorem 6.5, we obtain Corollary 6.6 which gives the determinant of $\mathbf{B}(K_\eta\cup K_\tau)$. This result was discussed by Edwina and Sugeng in [32] for $\eta,\tau>2$ by using Theorem 2.2. Here, we give another proof of this result by using Theorem 2.5 for $\eta,\tau\geq 1$.

Corollary 6.6: ([32]). *For every positive integers η and τ,*

$$\det(\mathbf{B}(K_\eta\cup K_\tau))=-(\eta\tau-1).$$

Proof: We have $\det(\mathbf{B}(K_\eta\cup K_\tau))=(1+\sqrt{\eta\tau})(1-\sqrt{\eta\tau})(1)^{\eta+\tau-2}=-(\eta\tau-1)$ by Theorem 6.5 and Theorem 2.5.

7. Antiadjacency Spectrum of the Complement and Join of Graphs

In this final section, we present direct results on the antiadjacency spectrum of two graph operations: complement and join. The first result gives the relation between the adjacency spectrum of a graph and the antiadjacency spectrum of its complement.

Theorem 7.1: *For every graph Γ,*

$$\mathrm{Spec}_{\mathbf{B}}(\overline{\Gamma})=\{\lambda+1:\lambda\in\mathrm{Spec}_{\mathbf{A}}(\Gamma)\}.$$

Proof: Since $\mathbf{B}(\overline{\Gamma}) = \mathbf{A}(\Gamma) + \mathbf{I}$, the result follows by Lemma 6.2.

The following theorem gives the determinant and spectrum of the antiadjacency matrix of graph joins. The determinant part was previously proved by Irawan and Sugeng in [27].

Theorem 7.2: *Let Γ_1 and Γ_2 be two disjoint graphs. Then*

$$\mathrm{Spec}_{\mathbf{B}}(\Gamma_1 + \Gamma_2) = \mathrm{Spec}_{\mathbf{B}}(\Gamma_1) \cup \mathrm{Spec}_{\mathbf{B}}(\Gamma_2) \text{ and } \det(\mathbf{B}(\Gamma_1 + \Gamma_2)) = \det(\mathbf{B}(\Gamma_1))\det(\mathbf{B}(\Gamma_2).$$

Proof: Suppose that $|V(\Gamma_1)| = \eta$ and $|V(\Gamma_2)| = \tau$. Let $\mathbf{I}_\tau$ be the $\tau \times \tau$ identity matrix. Observe that

$$\mathbf{B}(\Gamma_1 + \Gamma_2) = \begin{bmatrix} \mathbf{B}(\Gamma_1) & \mathbf{O} \\ \mathbf{O} & \mathbf{B}(\Gamma_2) \end{bmatrix} \Rightarrow \mathbf{B}(\Gamma_1 + \Gamma_2) - \lambda \mathbf{I}_{\eta+\tau} = \begin{bmatrix} \mathbf{B}(\Gamma_1) - \lambda \mathbf{I}_\eta & \mathbf{O} \\ \mathbf{O} & \mathbf{B}(\Gamma_2) - \lambda \mathbf{I}_\tau \end{bmatrix}.$$

By Theorem 2.1, we have $\det(\mathbf{B}(\Gamma_1 + \Gamma_2)) = \det(\mathbf{B}(\Gamma_1))\det(\mathbf{B}(\Gamma_2))$ and $\det(\mathbf{B}(\Gamma_1 + \Gamma_2) - \lambda \mathbf{I}_{\eta+\tau}) = \det(\mathbf{B}(\Gamma_1) - \lambda \mathbf{I}_\eta)\det(\mathbf{B}(\Gamma_2) - \lambda \mathbf{I}_\tau)$. It is clear that $\det(\mathbf{B}(\Gamma_1 + \Gamma_2) - \lambda \mathbf{I}_{\eta+\tau}) = 0$ if and only if $\det(\mathbf{B}(\Gamma_1) - \lambda \mathbf{I}_\eta)\det(\mathbf{B}(\Gamma_2) - \lambda \mathbf{I}_\tau) = 0$ if and only if $\det(\mathbf{B}(\Gamma_1) - \lambda \mathbf{I}_\eta) = 0$ or $\det(\mathbf{B}(\Gamma_2) - \lambda \mathbf{I}_\tau) = 0$. Therefore, by Theorem 2.4, the eigenvalues of $\mathbf{B}(\Gamma_1 + \Gamma_2)$ are the union of the eigenvalues of $\mathbf{B}(\Gamma_1)$ and $\mathbf{B}(\Gamma_2)$. This completes the proof.

Note that the wheel graph $W_\eta = C_{\eta-1} + K_1$, complete bipartite graph $K_{\mu,\eta} = \overline{K_\mu} + \overline{K_\eta}$, star graph $S_\eta = \overline{K_{\eta-1}} + K_1$, and friendship graph $F_\eta = \eta K_2 + K_1$ are some classes of graphs formed by the join of two disjoint graphs. The spectrum and determinant of these graphs can be obtained by using Theorem 7.2 and results from previous discussions. Some research has given the antiadjacency spectrum of the aforementioned graph classes. Alyani [30] found the spectrum of $\mathbf{B}(K_{\mu,\eta})$ and $\mathbf{B}(S_\eta)$, and Poniam [31] found the spectrum of $\mathbf{B}(F_\eta)$. Irawan and Sugeng [27] determined the characteristic polynomial, hence we can obtain the spectrum, of the antiadjacency matrix of the complete bipartite graph and friendship graph.

Acknowledgement

This publication is funded by the Faculty of Mathematics and Natural Sciences, Universitas Indonesia, under the Publication Grant scheme 2025.

Appendix A

In this appendix, we provide an alternative proof of Corollary 4.4 using calculus observations for the possibility of its application in subjects other than graph theory. We may now restate the corollary as a theorem.

Theorem: *For every integer* $\eta \geq 2$,

$$\prod_{\alpha=1}^{\eta-1} \cos\left(\frac{2\pi\alpha}{\eta}\right) = \begin{cases} 0, & \eta \equiv 0 \pmod 4; \\ \frac{1}{2^{\eta-1}}, & \eta \equiv 1,3 \pmod 4; \\ -\frac{1}{2^{\eta-2}}, & \eta \equiv 2 \pmod 4. \end{cases}$$

Proof: There are three cases for η.

Case 1: If $\eta \equiv 0 \pmod 4$, then there exists $\alpha = \frac{\eta}{4} \in [1, \eta-1]$ such that $\cos(\frac{2\pi\alpha}{\eta}) = 0$, and the result follows.

Case 2: Suppose that η is odd. Let

$$C := \prod_{\alpha=1}^{\eta-1} \cos\left(\frac{2\pi\alpha}{\eta}\right) \text{ and } S := \prod_{\alpha=1}^{\eta-1} \sin\left(\frac{2\pi\alpha}{\eta}\right). \tag{1}$$

Observe that,

$$\begin{aligned} CS &= \left(\prod_{\alpha=1}^{\eta-1} \cos\left(\frac{2\pi\alpha}{\eta}\right)\right)\left(\prod_{\alpha=1}^{\eta-1} \sin\left(\frac{2\pi\alpha}{\eta}\right)\right) \\ &= \prod_{\alpha=1}^{\eta-1} \sin\left(\frac{2\pi\alpha}{\eta}\right)\cos\left(\frac{2\pi\alpha}{\eta}\right) \\ &= \frac{1}{2^{\eta-1}}\prod_{\alpha=1}^{\eta-1} 2\sin\left(\frac{2\pi\alpha}{\eta}\right)\cos\left(\frac{2\pi\alpha}{\eta}\right) \\ &= \frac{1}{2^{\eta-1}}\prod_{\alpha=1}^{\eta-1} \sin\left(\frac{4\pi\alpha}{\eta}\right). \end{aligned} \tag{2}$$

Consider $\prod_{\alpha=1}^{\eta-1} \sin\left(\frac{4\pi\alpha}{\eta}\right)$. Since η is odd and $\eta-1$ is even, then $\prod_{\alpha=1}^{\eta-1} \sin\left(\frac{4\pi\alpha}{\eta}\right)$ may be written as a product of two factors as follows:

$$\prod_{\alpha=1}^{\eta-1} \sin\left(\frac{4\pi\alpha}{\eta}\right) = \left(\prod_{p=1}^{\frac{\eta-1}{2}} \sin\left(\frac{4\pi p}{\eta}\right)\right)\left(\prod_{q=\frac{\eta+1}{2}}^{\eta-1} \sin\left(\frac{4\pi q}{\eta}\right)\right). \tag{3}$$

Consider the first factor on the right hand side of Equation 3. We define a new index $\alpha := 2p$. Then the indices $p = 1,2,3,\ldots,\frac{\eta-1}{2}$ are changed into $\alpha = 2,4,6,\ldots,\eta-1$, and hence we can write this factor as

$$\prod_{p=1}^{\frac{\eta-1}{2}} \sin\left(\frac{4\pi p}{\eta}\right) = \prod_{p=1}^{\frac{\eta-1}{2}} \sin\left(\frac{2\pi(2p)}{\eta}\right) = \prod_{\substack{1\le\alpha\le\eta-1 \\ \alpha \text{ is even}}} \sin\left(\frac{2\pi\alpha}{\eta}\right). \tag{4}$$

Now consider the second factor on the right hand side of Equation 3. We define a new index $\alpha := 2q - \eta q = (\alpha+\eta)/2$. Recall that for every $x \in \mathbb{R}$, $\sin(2\pi + x) = \sin(x)$. Then

$$\sin\left(\frac{4\pi q}{\eta}\right) = \sin\left(\frac{4\pi((\alpha+\eta)/2)}{\eta}\right) = \sin\left(\frac{2\pi(\alpha+\eta)}{\eta}\right) = \sin\left(2\pi + \frac{2\pi\alpha}{\eta}\right) = \sin\left(\frac{2\pi\alpha}{\eta}\right)$$

and the indices $q = \frac{\eta+1}{2}, \frac{\eta+1}{2}+1, \frac{\eta+1}{2}+2, \ldots, \eta-1$ are changed into $\alpha = 1,3,5,\ldots,\eta-2$, and hence we can write this factor as

$$\prod_{q=\frac{\eta+1}{2}}^{\eta-1} \sin\left(\frac{4\pi q}{\eta}\right) = \prod_{\substack{1\le\alpha\le\eta-1 \\ \alpha \text{ is odd}}} \sin\left(\frac{2\pi\alpha}{\eta}\right). \tag{5}$$

By substituting Equations 4 and 5 to Equation 3, we have

$$\prod_{\alpha=1}^{\eta-1} \sin\left(\frac{4\pi\alpha}{\eta}\right) = \left(\prod_{\substack{1\le\alpha\le\eta-1 \\ \alpha \text{ iseven}}} \sin\left(\frac{2\pi\alpha}{\eta}\right)\right)\left(\prod_{\substack{1\le\alpha\le\eta-1 \\ \alpha \text{ isodd}}} \sin\left(\frac{2\pi\alpha}{\eta}\right)\right)$$
$$= \prod_{\alpha=1}^{\eta-1} \sin\left(\frac{2\pi\alpha}{\eta}\right)$$

and hence

$$\prod_{\alpha=1}^{\eta-1} \sin\left(\frac{4\pi\alpha}{\eta}\right) = S. \tag{6}$$

Substituting Equation 6 to Equation 2 gives

$$CS = \frac{1}{2^{\eta-1}} S. \tag{7}$$

Consider $S := \prod_{\alpha=1}^{\eta-1} \sin\left(\frac{2\pi\alpha}{\eta}\right)$. Observe that the only number $\alpha \in \mathbb{R}$, $1 \le \alpha \le \eta - 1$, satisfying $\sin\left(\frac{2\pi\alpha}{\eta}\right) = 0$ is $\alpha = \frac{\eta}{2}$. However, since η is odd, then $\alpha = \frac{\eta}{2} \notin \mathbb{Z}$, hence no index $\alpha \in \mathbb{Z}$, $1 \le \alpha \le \eta - 1$, exists such that $\sin\left(\frac{2\pi\alpha}{\eta}\right) = 0$. So $S \ne 0$, and from Equation 7, we have

$$C = \frac{1}{2^{\eta-1}}.$$

Therefore, we have proved that if $\eta \equiv 1 \pmod 4$ or $\eta \equiv 3 \pmod 4$, then

$$\prod_{\alpha=1}^{\eta-1} \cos\left(\frac{2\pi\alpha}{\eta}\right) = \frac{1}{2^{\eta-1}}.$$

Case 3: Suppose that $\eta \equiv 2 \pmod 4$. Then $\eta = 4k + 2$ for some $k \in \mathbb{Z}$. Note that $\eta = 4k + 2 \frac{\eta}{2} = 2k + 1$. It means that $\frac{\eta}{2}$ is odd and $\frac{\eta}{2} - 1$ is even. First, for $\eta = 2$, we have

$$\prod_{\alpha=1}^{\eta-1} \cos\left(\frac{2\pi\alpha}{\eta}\right) = \cos(\pi) = -1 = -\frac{1}{2^{\eta-2}}.$$

Now let us assume that $\eta > 2$. Observe that

$$\prod_{\alpha=1}^{\eta-1} \cos\left(\frac{2\pi\alpha}{\eta}\right) = (-1)\left(\prod_{\alpha=1}^{\frac{\eta}{2}-1} \cos\left(\frac{2\pi\alpha}{\eta}\right)\right)\left(\prod_{\alpha=\frac{\eta}{2}+1}^{\eta-1} \cos\left(\frac{2\pi\alpha}{\eta}\right)\right).$$

Consider the last factor, which is $\prod_{\alpha=\frac{\eta}{2}+1}^{\eta-1} \cos\left(\frac{2\pi\alpha}{\eta}\right)$. We define a new index $\beta := \alpha - \frac{\eta}{2} \alpha = \beta + \frac{\eta}{2}$. Then the indices $\alpha = \frac{\eta}{2} + 1, \frac{\eta}{2} + 2, \frac{\eta}{2} + 3, \dots, \eta - 1$ are changed into $\beta = 1,2,3,\dots,\frac{\eta}{2} - 1$. Since $\cos(x + \pi) = -\cos(x)$ for all $x \in \mathbb{R}$, we have

$$\cos\left(\frac{2\pi\alpha}{\eta}\right) = \cos\left(\frac{2\pi(\beta+\eta/2)}{\eta}\right) = \cos\left(\pi + \frac{2\pi\beta}{\eta}\right) = -\cos\left(\frac{2\pi\beta}{\eta}\right).$$

Then we have

$$\begin{aligned}\prod_{\alpha=1}^{\eta-1}\cos\left(\frac{2\pi\alpha}{\eta}\right) &= (-1)\left(\prod_{\alpha=1}^{\frac{\eta}{2}-1}\cos\left(\frac{2\pi\alpha}{\eta}\right)\right)\left(\prod_{\beta=1}^{\frac{\eta}{2}-1} -\cos\left(\frac{2\pi\beta}{\eta}\right)\right)\\ &= (-1)\left(\prod_{\alpha=1}^{\frac{\eta}{2}-1}\cos\left(\frac{2\pi\alpha}{\eta}\right)\right)\left((-1)^{\frac{\eta}{2}-1}\prod_{\beta=1}^{\frac{\eta}{2}-1}\cos\left(\frac{2\pi\beta}{\eta}\right)\right)\\ &= (-1)^{\frac{\eta}{2}}\left(\prod_{\alpha=1}^{\frac{\eta}{2}-1}\cos\left(\frac{2\pi\alpha}{\eta}\right)\right)^2\\ &= (-1)\left(\prod_{\alpha=1}^{\frac{\eta}{2}-1}\cos\left(\frac{2\pi\alpha}{\eta}\right)\right)^2.\end{aligned}$$

Now let

$$C := \prod_{\alpha=1}^{\eta-1}\cos\left(\frac{2\pi\alpha}{\eta}\right) \text{ and } S := \prod_{\alpha=1}^{\frac{\eta}{2}-1}\sin\left(\frac{2\pi\alpha}{\eta}\right).$$

From previous results, we have

$$\begin{aligned} C &= (-1)\left(\prod_{\alpha=1}^{\frac{\eta}{2}-1}\cos\left(\frac{2\pi\alpha}{\eta}\right)\right)^2\\ CS^2 &= (-1)\left(\prod_{\alpha=1}^{\frac{\eta}{2}-1}\cos\left(\frac{2\pi\alpha}{\eta}\right)\right)^2\left(\prod_{\alpha=1}^{\frac{\eta}{2}-1}\sin\left(\frac{2\pi\alpha}{\eta}\right)\right)^2\\ &= (-1)\left(\prod_{\alpha=1}^{\frac{\eta}{2}-1}\sin\left(\frac{2\pi\alpha}{\eta}\right)\cos\left(\frac{2\pi\alpha}{\eta}\right)\right)^2\\ &= (-1)\left(\frac{1}{2^{\frac{\eta}{2}-1}}\prod_{\alpha=1}^{\frac{\eta}{2}-1}2\sin\left(\frac{2\pi\alpha}{\eta}\right)\cos\left(\frac{2\pi\alpha}{\eta}\right)\right)^2\\ &= -\frac{1}{2^{\eta-2}}\left(\prod_{\alpha=1}^{\frac{\eta}{2}-1}\sin\left(\frac{4\pi\alpha}{\eta}\right)\right)^2.\end{aligned}$$

Now we will prove that

$$\prod_{\alpha=1}^{\frac{\eta}{2}-1}\sin\left(\frac{4\pi\alpha}{\eta}\right) = (-1)^{\frac{\eta-2}{4}}S. \tag{8}$$

Observe that

$$\prod_{\alpha=1}^{\frac{\eta}{2}-1}\sin\left(\frac{4\pi\alpha}{\eta}\right) = \left(\prod_{p=1}^{\frac{\eta/2-1}{2}}\sin\left(\frac{4\pi p}{\eta}\right)\right)\left(\prod_{q=\frac{\eta/2-1}{2}+1}^{\frac{\eta}{2}-1}\sin\left(\frac{4\pi q}{\eta}\right)\right) \tag{9}$$

Consider the first factor $\prod_{p=1}^{\frac{\eta-2}{4}}\sin\left(\frac{4\pi p}{\eta}\right)$. We define a new index $\alpha := 2p$. Then the indices $p = 1, 2, 3, \ldots, \frac{\eta-2}{4}$ are changed into $\alpha = 2, 4, 6, \ldots, \frac{\eta}{2}-1$ and

$$\prod_{p=1}^{\frac{\eta-2}{4}}\sin\left(\frac{4\pi p}{\eta}\right) = \prod_{p=1}^{\frac{\eta-2}{4}}\sin\left(\frac{2\pi(2p)}{\eta}\right) = \prod_{\substack{1\le\alpha\le\frac{\eta}{2}-1\\ \alpha \text{ is even}}}\sin\left(\frac{2\pi\alpha}{\eta}\right). \tag{10}$$

Now consider the second factor $\prod_{q=\frac{\eta+2}{4}}^{\frac{\eta}{2}-1} \sin\left(\frac{4\pi q}{\eta}\right)$. Again we let $\alpha := 2q - \frac{\eta}{2}$, and $q = \frac{2\alpha+\eta}{4}$. Then the indices $q = \frac{\eta+2}{4}, \frac{\eta+2}{4}+1, \frac{\eta+2}{4}+2, \dots, \frac{\eta}{2}-1$ are changed into $\alpha = 1,3,5,\dots,\frac{\eta}{2}-2$. Note that there are $\frac{\eta-2}{4}$ indices. Since $\sin(x+\pi) = -\sin(x)$ for all $x \in \mathbb{R}$, we have

$$\begin{aligned}
\prod_{q=\frac{\eta+2}{4}}^{\frac{\eta}{2}-1} \sin\left(\frac{4\pi q}{\eta}\right) &= \prod_{\substack{1\le\alpha\le\frac{\eta}{2}-1 \\ \alpha \text{ is odd}}} \sin\left(\frac{4\pi\left(\frac{2\alpha+\eta}{4}\right)}{\eta}\right) \\
&= \prod_{\substack{1\le\alpha\le\frac{\eta}{2}-1 \\ \alpha \text{ is odd}}} \sin\left(\frac{2\pi\alpha+\eta\pi}{\eta}\right) \\
&= \prod_{\substack{1\le\alpha\le\frac{\eta}{2}-1 \\ \alpha \text{ is odd}}} \sin\left(\frac{2\pi\alpha}{\eta}+\pi\right) \\
&= \prod_{\substack{1\le\alpha\le\frac{\eta}{2}-1 \\ \alpha \text{ is odd}}} -\sin\left(\frac{2\pi\alpha}{\eta}\right) \\
&= (-1)^{\frac{\eta-2}{4}} \prod_{\substack{1\le\alpha\le\frac{\eta}{2}-1 \\ \alpha \text{ is odd}}} \sin\left(\frac{2\pi\alpha}{\eta}\right).
\end{aligned}$$

Then we have

$$\prod_{q=\frac{\eta+2}{4}}^{\frac{\eta}{2}-1} \sin\left(\frac{4\pi q}{\eta}\right) = (-1)^{\frac{\eta-2}{4}} \prod_{\substack{1\le\alpha\le\frac{\eta}{2}-1 \\ \alpha \text{ is odd}}} \sin\left(\frac{2\pi\alpha}{\eta}\right). \tag{11}$$

Substituting Equations 10 and 11 to Equation 9 implies

$$\begin{aligned}
\prod_{\alpha=1}^{\frac{\eta}{2}-1} \sin\left(\frac{4\pi\alpha}{\eta}\right) &= \left(\prod_{p=1}^{\frac{\eta/2-1}{2}} \sin\left(\frac{4\pi p}{\eta}\right)\right)\left(\prod_{q=\frac{\eta/2-1}{2}+1}^{\frac{\eta}{2}-1} \sin\left(\frac{4\pi q}{\eta}\right)\right) \\
&= \left(\prod_{\substack{1\le\alpha\le\frac{\eta}{2}-1 \\ \alpha \text{ is even}}} \sin\left(\frac{2\pi\alpha}{\alpha}\right)\right)\left((-1)^{\frac{\eta-2}{4}} \prod_{\substack{1\le\alpha\le\frac{\eta}{2}-1 \\ \alpha \text{ is odd}}} \sin\left(\frac{2\pi\alpha}{\eta}\right)\right) \\
&= (-1)^{\frac{\eta-2}{4}}\left(\prod_{\substack{1\le\alpha\le\frac{\eta}{2}-1 \\ \alpha \text{ is even}}} \sin\left(\frac{2\pi\alpha}{\eta}\right)\right)\left(\prod_{\substack{1\le\alpha\le\frac{\eta}{2}-1 \\ \alpha \text{ is odd}}} \sin\left(\frac{2\pi\alpha}{\eta}\right)\right) \\
&= (-1)^{\frac{\eta-2}{4}} \prod_{\alpha=1}^{\frac{\eta}{2}-1} \sin\left(\frac{2\pi\alpha}{\eta}\right) \\
&= (-1)^{\frac{\eta-2}{4}} S.
\end{aligned}$$

So we have proved that Equation 8 is true, and hence

$$\begin{aligned} CS^2 &= -\frac{1}{2^{\eta-2}}\left(\prod_{\alpha=1}^{\frac{\eta}{2}-1} \sin\left(\frac{4\pi\alpha}{\eta}\right)\right)^2 \\ &= -\frac{1}{2^{\eta-2}}((-1)^{\frac{\eta-2}{4}}S)^2 \\ &= -\frac{1}{2^{\eta-2}}(-1)^{\frac{\eta}{2}-1}S^2 \\ &= -\frac{1}{2^{\eta-2}}S^2. \end{aligned}$$

Observe that $S \neq 0$ since none of the indices $\alpha = 1,2,3,\dots,\frac{\eta}{2}-1$ implies $\sin\left(\frac{2\pi\alpha}{\eta}\right) = 0$ Then, we have

$$C = -\frac{1}{2^{\eta-2}}.$$

Therefore, we have proved that if $\eta \equiv 2 \pmod 4$, then

$$\prod_{\alpha=1}^{\eta-1} \cos\left(\frac{2\pi\alpha}{\eta}\right) = -\frac{1}{2^{\eta-2}}.$$

References

[1] R. B. Bapat, Graphs and Matrices, Springer (2010).

[2] N. Biggs, Algebraic Graph Theory, 2nd ed., Cambridge University Press (1993).

[3] D. Cvetković, P. Rowlinson and S. Simić, An Introduction to the Theory of Graph Spectra, Cambridge University Press (2010).

[4] G. Chartrand and P. Zhang, A First Course in Graph Theory, Dover Publications (2013).

[5] D. B. West, Introduction to Graph Theory, 2nd ed., Prentice Hall Upper Saddle River (2001).

[6] R. P. Stanley, "A matrix for counting paths in acyclic digraphs," *Journal of Combinatorial Theory, Series A* vol. 74, pp. 169-172, (1996).

[7] M. Skandera, "A characterization of -free posets," *Journal of Combinatorial Theory, Series A*, vol. 93, pp. 231-241 (2001).

[8] M. Skandera and B. Reed, "Total nonnegativity and -free posets," *Journal of Combinatorial Theory, Series A*, vol. 103, pp. 237-256 (2003).

[9] B. Shelton and M. Skandera, "Hecke algebra and quantum chromatic symmetric functions," *Discrete Mathematics & Theoretical Computer Science* Proceeding (2012).

[10] A. Chavez and F. Gotti, "Dyck paths and positroids from unit interval orders," *Journal of Combinatorial Theory, Series A*, vol. 154, pp. 507-532 (2018).

[11] K. A. Sugeng, F. Firmansyah, B. D. Wildan, N. H. Handari and M. Imran, "Several properties of antiadjacency matrices of directed graphs," *AIMS Mathematics*, vol. 9, no. 10, pp. 27834-27847 (2024).

[12] L. Widiastuti, S. Utama and S. Aminah, "Characteristic polynomial of antiadjacency matrix of directed cyclic wheel graph," *Journal of Physics: Conference Series*, vol. 1108, pp. 012009 (2018).

[13] B. H. Aji, K. A. Sugeng and S. Aminah, "Characteristic polynomial and eigenvalues of antiadjacency matrix of directed unicyclic flower vase graph," *Journal of Physics: Conference Series*, vol. 1722, pp. 012055 (2021).

[14] N. Anzana, S. Aminah, S. Utama and D. R. Silaban, "Characteristic polynomial of anti-adjacency matrix of directed cyclic friendship graph," *Journal of Physics: Conference Series*, vol. 1538, pp. 012007 (2020).

[15] M. I. A. Prayitno, S. Utama and S. Aminah, "Properties of anti-adjacency matrix of directed cyclic sun graph," *IOP Conference Series: Materials Science and Engineering*, vol. 567, pp. 012020 (2019).

[16] M. I. A. Prayitno and K. A. Sugeng, "On characteristic polynomial of antiadjacency matrix of a line digraph," *Jurnal Matematika UNAND*, vol. 11, no. 1, pp. 74-81 (2022).

[17] M. I. A. Prayitno and K. A. Sugeng, "On antiadjacency matrix of a digraph with directed digon(s)," *BAREKENG: Jurnal Ilmu Matematika dan Terapan*, vol. 16, no. 2, pp. 497-506 (2022).

[18] R. Stin, S. Aminah and S. Utama, "Characteristic polynomial and eigenvalues of the anti-adjacency matrix of cyclic directed prism graph," *AIP Conference Proceedings*, vol. 2168, pp. 020052 (2019).

[19] S. Budiyanto, S. Utama and S. Aminah, "Eigenvalues of antiadjacency matrix of directed cyclic dumbbell graph," *Journal of Physics: Conference Series*, vol. 1108, pp. 012015 (2018).

[20] D. Diwyacitta, A. P. Putra, K. A. Sugeng and S. Utama, "The determinant of an antiadjacency matrix of a directed cycle graph with chords," *AIP Conference Proceedings*, vol. 1862, pp. 030127 (2017).

[21] R. P. Okfradifa, S. Aminah and K. A. Sugeng, "Properties of characteristic polynomial and eigenvalues of antiadjacency matrix of directed unicyclic helm graph," *Journal of Physics: Conference Series*, vol. 1722, pp. 012056 (2021).

[22] Qomaruzzaman, S. Aminah, K. A. Sugeng, "Properties of characteristic polynomial and eigenvalues of antiadjacency matrix of directed unicyclic tadpole graph," *Journal of Physics: Conference Series*, vol. 1836, pp. 012002 (2021).

[23] N. Hasyyati, K. A. Sugeng and S. Aminah, "Characteristic polynomial and eigenvalues of anti-adjacency matrix of directed unicyclic corona graph," *Journal of Physics: Conference Series*, vol. 1836, pp. 012001 (2021).

[24] M. S. R. Solihin, S. Aminah and S. Utama, "Properties of anti-adjacency matrix of cyclic directed windmill graph ," *Advances in Intelligent Systems Research*, vol. 157, pp. 9-12 (2018).

[25] D. F. Jain and S. Naduvath, "A study on anti-adjacency spectra of graphs," *Journal of Interconnection Networks* (2025).

[26] M. F. Camilla, M. Rensiana and D. R. Silaban, "Spectral characteristics of the antiadjacency matrix of kite graph," *Jurnal Matematika, Statistika dan Komputasi*, vol. 21, no. 2, pp. 345-353 (2025).

[27] W. Irawan and K. A. Sugeng, "Characteristic antiadjacency matrix of graph join," *BAREKENG: Jurnal Ilmu Matematika dan Terapan*, vol. 16, no. 1, pp. 041-046 (2022).

[28] Murni, A. E. Hadi, I. Febry and Abdussakir, "Anti-adjacency and Laplacian spectra of inverse graph of group of integers modulo ," *IOP Conference Series: Materials Science and Engineering*, vol. 807, pp. 012033 (2020).

[29] J. Daniel, K. A. Sugeng and N. Hariadi, "Eigenvalues of antiadjacency matrix of Cayley graph of ," *Indonesian Journal of Combinatorics*, vol. 6, no. 1, pp. 66-76 (2022).

[30] F. Alyani, "*Spektrum matriks antiadjacency dari beberapa kelas graf tak berarah*," M.S. thesis, Universitas Indonesia, Depok, Indonesia (2014).

[31] B. Poniam, "Polinomial karakteristik dan spektrum matriks adjacency dan anti-adjacency dari graf friendship tak berarah dan berarah," M.S. thesis, Universitas Indonesia, Depok, Indonesia (2016).

[32] M. Edwina and K. A. Sugeng, "Determinant of antiadjacency matrix of union and join operation from two disjoint of several classes of graphs," *AIP Conference Proceedings*, vol. 1862, pp. 030158 (2017).

[33] A. Sutjijana and D. A. Azka, "Antiadjacency matrices for some strong products of graphs," *Journal of Fundamental Mathematics and Applications*, vol. 6, no. 1, pp. 27-36 (2023).

[34] A. A. Ataupah and G. L. Putra, "Characteristic polynomial of antiadjacency matrix of several classes of graph join," *Indonesian Journal of Combinatorics*, vol. 8, no. 2, pp. 76-82 (2024).

[35] G. L. Putra, "Characteristic polynomial and eigenvalues of anti-adjacency matrix for graph $K_m \odot K_1$ and $H_m \odot K_1$," *SITEKIN: Jurnal Sains, Teknologi dan Industri*, vol. 21, no. 2, pp. 357-361 (2024).

[36] A. D. Akwu, O. Oyewumi and D. O. A. Ajayi, "The outer-connected vertex edge domination number in Cartesian product graphs," *Journal of Discrete Mathematical Sciences and Cryptography*, vol. 25, no. 5, pp. 1275-1287 (2022).

[37] S. Arulanand, R. S. Rajan, S. Prabhu and S. Stephen, "Certain domination numbers for Cartesian product of graphs," *Journal of Discrete Mathematical Sciences and Cryptography*, vol. 27, no. 3, pp. 1045-1058 (2024).

[38] A. Ahmad, M. Baa and A. Semaniovčová–Feňovčíková, "Computation of edge -irregularity strength of Cartesian product of graphs," *Journal of Discrete Mathematical Sciences and Cryptography*, vol. 26, no. 2, pp. 403-410 (2023).

[39] C. D. Meyer, Matrix Analysis and Applied Linear Algebra, SIAM (2000).

[40] J. R. Silvester, "Determinants of block matrices," *The Mathematical Gazette*, vol. 84, no. 501, pp. 460-467 (2000).

[41] H. Anton and C. Rorres, Elementary Linear Algebra: Applications Version, John Wiley & Sons (2014).

[42] M. Krebs and A. Shaheen, Expander Families and Cayley Graphs: A Beginner's Guide, Oxford University Press (2011).

[43] R. M. Gray, Toeplitz and Circulant Matrices: A Review, Now Publishers Inc (2006).

[44] A. Graham, Kronecker Products and Matrix Calculus with Applications, Courier Dover Publications (2018).

[45] A. Abdollahi, "Determinants of adjacency matrices of graphs," *Transactions on Combinatorics*, vol. 1, no. 4, pp. 9-16 (2012).

[46] I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series, and Products, 7th ed., Academic Press (2007).

[47] L. B. W. Jolley, Summation of Series, Chapman & Hall (1925).

Received November, 2024