Journal of Discrete Mathematical Sciences & Cryptography
ISSN 0972-0529 (Print), ISSN 2169-0065 (Online)
Vol. 29 (2026), No. 4, pp. 1823–1832
DOI : 10.47974/JDMSC-2599

On the rainbow vertex connection number of general unicyclic graphs

Ridho Alfarisi *
Department of Mathematics
University of Jember
Jember 68121
Indoensia

Arika Indah Kristiana †
Department of Mathematics Education
University of Jember
Jember 68121
Indoensia

Sharifah Kartini Said Husain §
Department of Mathematics and Statistics
Universiti Putra Malaysia
Serdang Selangor 43400
Malaysia

and

Institute for Mathematical Research
Universiti Putra Malaysia
Serdang Selangor 43400
Malaysia

Abstract

In a graph G, the distance between any two vertices is defined as the length of the shortest path connecting them. A path in G is termed *rainbow vertex-connected* if all internal vertices along the path have distinct colors. For every pair of vertices u and v in G, if there exists such a colored u-v path, the graph is considered rainbow vertex-connected. The minimum number of colors required to ensure that a connected graph G satisfies this condition is called

* *E-mail:* **alfarisi.fkip@unej.ac.id (Corresponding Author)**
† *E-mail:* **arika.fkip@unej.ac.id**
§ *E-mail:* **kartini@upm.edu.my**

the rainbow vertex-connection number, denoted by *rvc*(*G*). This study focuses on analyzing the rainbow vertex-connection number specifically for unicyclic graphs.

Subject Classification: *05C12.*

Keywords: *Rainbow vertex connected, Rainbow vertex connection number, Unicyclic graph.*

1. Introduction

All graph in this paper are simple, finite, and connected graph. For all the notation and terminology graph, can be seen in Hartsfield and Ringle [1]. Graph coloring and connectivity represent two of the most extensively studied areas within graph algorithm research. Over the years, various forms of color assignments and connectivity metrics have been explored.

As fundamental concepts in combinatorics, rainbow connection and rainbow vertex connection have garnered considerable interest among graph theorists. Beyond their theoretical significance, these notions have practical relevance, particularly in network design and communication systems. The motivation for these concepts originates from secure information transfer scenarios, such as inter-agency communication within governmental networks. Consider a scenario in a cellular network where each communication link along a route between two nodes must utilize a unique frequency channel. The fewest number of distinct channels required corresponds precisely to the rainbow connection number of the network's associated graph. For an in-depth discussion of various forms of rainbow connectivity, readers may consult the monograph in [2].

The notion of rainbow coloring was first introduced by Chartrand et al. [3]. A path in an edge-colored graph is said to be *rainbow connected* if every edge along the path has a different color. For a connected graph G, the *rainbow connection number*, denoted by $rc(G)$, is defined as the minimum number of colors needed to ensure that any pair of vertices in G is connected via such a rainbow path. There are some basic results of rainbow connection number of G, observe that $diam(G) \leq rc(G) \leq n-1$. It is easy to verify that $rc(G) = 1$ if and only if G is a complete graph, and $rc(G) = n-1$ if and only if G is a tree.

New variant of rainbow connection with vertex parameter, is called rainbow vertex connection. This concept was introduced by Krivelevich and Yuster [4]. An vertex-colored path is said rainbow vertex connected, if for every two vertices u and v in G, then there is an $u-v$ path with all internal vertices have different color. The *rainbow vertex-connection number* of a connected graph G, symbolized as $rvc(G)$, refers to the least number of colors required to assign to the vertices of G such that any two vertices are linked by a path whose internal vertices all receive distinct colors. It is straightforward to observe that for a graph G of order n, the bound $rvc(G) \leq n-2$ holds. Moreover, $rvc(G) =$ precisely when G is a complete graph. Additionally, one has the inequality $rvc(G) \geq diam(G) - 1$, where equality occurs in particular when the diameter of G is either 1 or 2. Figure 1 is an example of rainbow vertex connection.

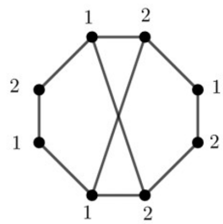

Figure 1
Example of rainbow vertex connected with 2-coloring

There are many results about rainbow vertex connection number. Some of them are stated by Li and Liu [5], which have determined the rainbow vertex connection number of cycle graph. There are some results of the rainbow connection number of pencil graphs [6], star fan graphs [7], bipartite and chordal graphs [8]. It was also shown that computing the rainbow vertex connection number of an arbitrary graph is NP-hard problem [9]. In addition, there is research related to distance in graphs, for example, the Sombor Index in a unicyclic graph [10, 11].

Definition 1.1: The unicyclic graph is a graph that has only one cycle.

Definition 1.2: The bicyclic graph is a graph which has only two cycles.

Based on Definition 1.1, a unicyclic graph can be formed by taking a tree structure and adding a single edge that joins two of its vertices, thereby creating exactly one cycle. Illustrations of a unicyclic graph (on the left) and a bicyclic graph (on the right) are presented in Figure 2.

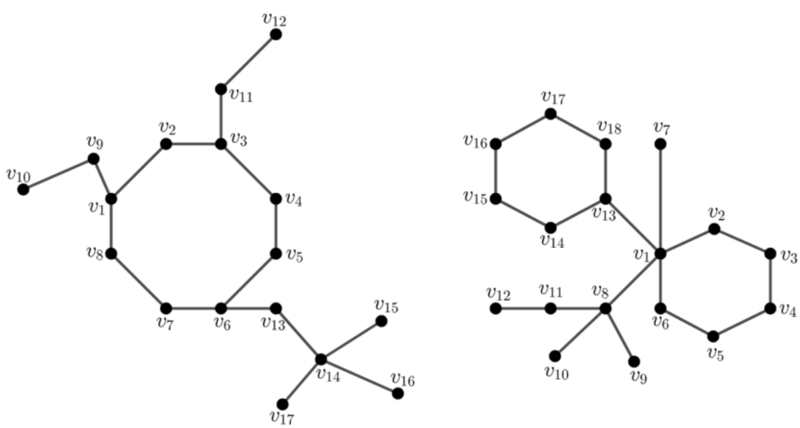

Figure 2
Unicyclic (left) and bicyclic (right) graphs

2. Results

In this paper, we have investigated the bounds and exact value of the rainbow vertex connection number of unicyclic graphs as follows.

Unicyclic graph, denoted by UC_n have only one cycle C_p. The number of leaf vertex in UC_n is l. This graph can be divide two cases, namely UC_n with properties a). $\exists v \in V(C_p)$ with $d(v) = 2$ and b). $\forall v \in V(C_p)$ with $d(v) > 2$.

Let UC_n be an unicyclic graph with $n \geq 3$ and $l \geq 1$. There are some condition for $u - v$ path in UC_n as follows:

(a) Every two vertices w' and w'' in T_i have unique path, such that internal vertices in $w' - w''$ have different color.

(b) Every two vertices $w_i \in V(T_i)$ and $w_j \in V(T_j)$ have unique path.

(c) All vertices in cycle C_p are internal vertices of path $w_i - w_j$ with $w_i \in V(T_i)$ and $w_j \in V(T_j)$.

Proof: Unicyclic graph contain cycle subgraph C_p. Suppose $V(C_p) = \{v_1, v_2, \dots, v_p\}$ and $E(C_p) = \{v_1v_p, v_iv_{i+1}; 1 \leq i \leq p-1\}$. We define $U = \{u; d(u) = 1\}$ and $W = \{w; w \notin V(C_p)$ and $d(w) \neq 1\}$. Let $G' = UC_n - E(C_p)$. Then G' is a disconnected graph that contains p components where each component is a tree. For $i \in \{1,2,3, \dots, p\}$, we define T_i as the component of G' containing vertex $v_i \in V(C_p)$. Suppose v is a vertex in UC_n, then there are $i \in \{1,2,3, \dots, p\}$ such that $v \in V(T_i)$. Suppose $x \in V(T_k)$ and $y \in V(T_l)$, then $d(x,y) = d(x, v_k) + d(v_k, v_l) + d(v_l, y)$. Since T_k and T_l are a tree and every two distinct vertices in the tree have a unique path between them. Thus, we have some condition of $u - v$ path in UC_n as follows.

(a) Every two vertices w' and w'' in T_i have unique path, such that internal vertices in $w' - w''$ have different color.

(b) Every two vertices $w_i \in V(T_i)$ and $w_j \in V(T_j)$ have unique path.

(c) All vertices in cycle C_p are internal vertices of path $w_i - w_j$ with $w_i \in V(T_i)$ and $w_j \in V(T_j)$.

Lemma 2.1: *Let UC_n be an unicyclic graph with $n \geq 3$ and $l \geq 1$. For all $v \in V(C_p)$ with $d(v) > 2$, then $rvc(UC_n) \leq n - l$.*

Proof: Unicyclic graph contain cycle subgraph C_p. Suppose $V(C_p) = \{v_1, v_2, \dots, v_p\}$ and $E(C_p) = \{v_1v_p, v_iv_{i+1}; 1 \leq i \leq p-1\}$. We define $U = \{u; d(u) = 1\}$ and $W = \{w; w \notin V(C_p)$ and $d(w) \neq 1\}$.

Now, let c be a $n-l$-rainbow vertex coloring of UC_n. We show $rvc(UC_n) \leq n - l$ by defining a $n-l$-rainbow vertex coloring $c: V(UC_n) \rightarrow \{1,2,3 \dots, n-l\}$ as follows

$$c(u) = 1$$

$$c(u_i) = i; 1 \leq i \leq p$$
$$c(w) = j; p + 1 \leq j \leq n - l$$

Observe that the coloring c above maintains the position of colors in UC_n and assigns distinct colors in $V(UC_n) - U$.

(a) For vertices $u \in U$ does not internal vertices, such that the $c(u) = c(w) = c(v_i)$.

(b) Every two vertices w' and w'' in T_i have unique path, such that internal vertices in $w' - w''$ have different color.

(c) Every two vertices $w_i \in V(T_i)$ and $w_j \in V(T_j)$ have unique path, such that internal vertices in $w_i - w_j$ have different color.

(d) All vertices in cycle C_p are internal vertices of path $w_i - w_j$ with $w_i \in V(T_i)$ and $w_j \in V(T_j)$, such that $c(v_k) \neq c(v_l)$.

We are able to find a rainbow path for every pair vertices x and y in $V(UC_n)$.

Table 1
Rainbow Vertex Path

x	y	Condition	Rainbow vertex path
v_k	v_l	1 k,l p	$v_k - v_{k+1} - v_l$
u_k	u_l	1 k,l p	$u_k - v_k - v_l - u_l$
w_k	u_l	1 k,l p	$w_k - v_k - v_l - u_l$
u_k	w_l	1 k,l p	$u_k - v_k - v_l - w_l$

Based on table above that every two vertices in UC_n is a rainbow path. Thus, $rvc(UC_n) \leq n - l$.

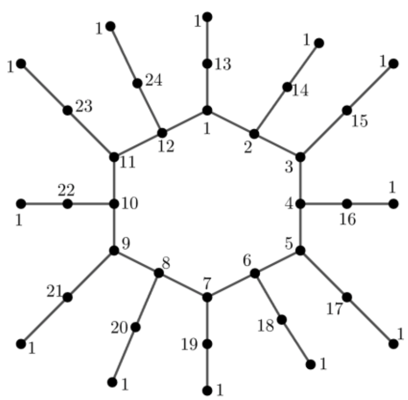

Figure 3
24-rainbow vertex coloring of unicyclic graphs

Theorem 2.1: *Let UC_n be an unicyclic graph with $n \geq 3$ and $l \geq 1$. For all $v \in V(C_p)$ with $d(v) > 2$, then $rvc(UC_n) = n - l$.*

Proof: Based on Lemma 2.1 that $rvc(UC_n) \leq n - l$. Furthermore, we show that $rvc(UC_n) \geq n - l$. Assume $rvc(UC_n) > n - l$, let c' be a $n - l - 1$-rainbow vertex coloring of UC_n by defining $n - l - 1$-rainbow vertex coloring with $c'\colon V(UC_n) \rightarrow \{1,2,\dots,n-l-1\}$ as follows:

(a) Every pendant vertex is given color 1, such that color in pendant vertex have same color with vertices $x \in V(UC_n)$ with $d(x) \neq 1$;

(b) Every vertex $x \in V(UC_n)$ with $d(x) \neq 1$ is given $n - l - 1$ color, such that there are at least two vertices $x, y \in V(UC_n)$ have same color. There are three condition namely: a) $x \in V(T_i)$ and $y \in V(T_j)$; b) $x \in V(T_i)$ and $y \in V(C_p)$; and c) $x, y \in V(C_p)$.

   (a) For $x \in V(T_i)$ and $y \in V(T_j)$ and $c(x) = c(y)$. Based Observation 2 that $x' - x - y - y'$ have unique path, such that there is at least two vertices have same color in $x' - y'$ path. It is contradiction.

   (b) for $x \in V(T_i)$ and $y \in V(C_p)$ and $c(x) = c(y)$. Based Observation 2 that $x' - x - y - y'$ have unique path and every vertices in cycle is internal vertices, such that there is at least two vertices have same color in $x' - y'$ path. It is contradiction.

   (c) for $x, y \in V(C_p)$ and $c(x) = c(y)$. We know that x and y are internal vertices of path between two vertices in different subgraph tree, then there is at least two vertices have same color in $x' - y'$ path. It is contradiction.

Based on 1 and 2 that every vertices in $V(UC_n) - U$ have different color such that we have $n - l$-rainbow vertex coloring in UC_n. Thus, $rvc(UC_n) = n - l$. Figure 3 is an example of rainbow vertex connection of unicyclic graph.

Furthermore, for this case specially $\exists v \in V(C_p)$ with $d(v) = 2$. Unicyclic graph contain cycle subgraph C_p. Suppose $V(C_p) = \{v_1, v_2, \dots, v_p\}$ and $E(C_p) = \{v_1 v_p, v_i v_{i+1}; 1 \leq i \leq p - 1\}$. We define $U = \{u\colon d(u) = 1\}$, $W = \{w\colon w \notin V(C_p) \text{ and } d(w) \neq 1\}$, and $S = \{s\colon d(s) > 2\}$ where $s \in V(C_p)$. Thus, we got the bounds of the rainbow vertex connection of UC_n as follows.

Theorem 2.2: *Let UC_n be an unicyclic graph with $n \geq 3$ and $l \geq 1$. $\exists v \in V(C_p)$ with $d(v) > 2$ and $|S| \geq diam(C_p)$, then $|S| + |W| \leq rvc(UC_n) \leq n - l$.*

Proof: We show that $rvc(UC_n) \geq |S| + |W|$. Assume $rvc(UC_n) > |S| + |W|$, let c' be a $|S| + |W| - 1$-rainbow vertex coloring of UC_n by defining $|S| + |W| - 1$-rainbow vertex coloring with $c'\colon V(UC_n) \rightarrow \{1,2,\dots,|S| + |W| - 1\}$ as follows:

(a) Every pendant vertex is given color 1, such that color in pendant vertex have same color with vertices $r \in V(UC_n)$ with $d(r) \neq 1$;

(b) Every vertex $w \in V(UC_n)$ with $d(w) \neq 1$ is given $|S| + |W| - 1$ color, such that there are at least two vertices $w, y \in V(UC_n)$ have same color. There are three condition namely: a) $x \in V(T_i)$ and $y \in V(T_j)$; b) $x \in V(T_i)$ and $y \in V(C_p)$; and c) $x, y \in V(C_p)$.

  (a) For $x \in V(T_i)$ and $y \in V(T_j)$ and $c(x) = c(y)$. Based Observation 2 that $x' - x - y - y'$ have unique path, such that there is at least two vertices have same color in $x' - y'$ path. It is contradiction.

  (b) for $x \in V(T_i)$ and $y \in V(C_p)$ and $c(x) = c(y)$. Based Observation 2 that $x' - x - y - y'$ have unique path and every vertices in cycle is internal vertices, such that there is at least two vertices have same color in $x' - y'$ path. It is contradiction.

  (c) for $x, y \in V(C_p)$ and $c(x) = c(y)$. We know that x and y are internal vertices of path between two vertices in different subgraph tree, then there is at least two vertices have same color in $x' - y'$ path. It is contradiction.

Based on 1 and 2 that every vertices in $V(UC_n) - U$ have different color such that we have $n - l$-rainbow vertex coloring in UC_n. Thus, $rvc(UC_n) \geq |S| + |W|$.

Now, let c be a $n - l$-rainbow vertex coloring of UC_n. We show $rvc(UC_n) \leq n - l$ by defining a $n - l$-rainbow vertex coloring $c: V(UC_n) \rightarrow \{1,2,3 \ldots, n - l\}$ as follows

$$c(u) = 1$$
$$c(u_i) = i; 1 \leq i \leq p$$
$$c(w) = j; p + 1 \leq j \leq n - l$$

Observe that the coloring c above maintains the position of colors in UC_n and assigns distinct colors in $V(UC_n) - U$.

(a) For vertices $u \in U$ does not internal vertices, such that the $c(u) = c(w) = c(v_i)$.

(b) Every two vertices w' and w'' in T_i have unique path, such that internal vertices in $w' - w''$ have different color.

(c) Every two vertices $w_i \in V(T_i)$ and $w_j \in V(T_j)$ have unique path, such that internal vertices in $w_i - w_j$ have different color.

(d) All vertices in cycle C_p are internal vertices of path $w_i - w_j$ with $w_i \in V(T_i)$ and $w_j \in V(T_j)$, such that $c(v_k) \neq c(v_l)$.

We are able to find a rainbow path for every pair vertices x and y in $V(UC_n)$. Thus, $rvc(UC_n) \leq n - l$. Figure 4 is an example of rainbow vertex connection of unicyclic graph.

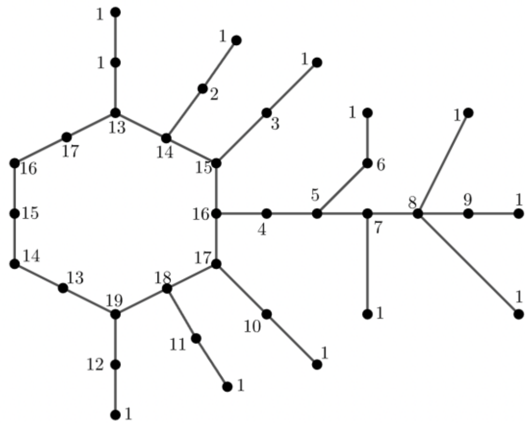

Figure 4

19-rainbow vertex coloring of unicyclic graphs

Theorem 2.3: *Let UC_n be an unicyclic graph with $n \geq 3$ and $l \geq 1$. $\exists v \in V(C_p)$ with $d(v) \neq 2$ and $|S| < diam(C_p)$, then $rc(C_p) + |W| \leq rvc(UC_n) \leq n - l$.*

Proof: Let c'' is k-rainbow vertex coloring in C_p). We define $c'': V(C_p) \rightarrow \{1,2,3, \ldots, k\}$, k is a minimal color in subgraph C_p. The color in subgraph C_p should be different with the color in W.

We show that $rvc(UC_n) \geq k + |W|$. Assume $rvc(UC_n) > k + |W|$, let c' be a $k + |W| - 1$-rainbow vertex coloring of UC_n by defining $k + |W| - 1$-rainbow vertex coloring with $c': V(UC_n) \rightarrow \{1,2, \ldots, k + |W| - 1\}$ as follows:

(a) Every pendant vertex is given color 1, such that color in pendant vertex have same color with vertices $r \in V(UC_n)$ with $d(r) \neq 1$;

(b) Every vertex $w \in V(UC_n)$ with $d(w) \neq 1$ is given $k + |W| - 1$ color, such that there are at least two vertices $w, y \in V(UC_n)$ have same color. There are three condition namely: a) $x \in V(T_i)$ and $y \in V(T_j)$; b) $x \in V(T_i)$ and $y \in V(C_p)$; and c) $x, y \in V(C_p)$.

   (a) For $x \in V(T_i)$ and $y \in V(T_j)$ and $c(x) = c(y)$. Based Observation 2 that $x' - x - y - y'$ have unique path, such that there is at least two vertices have same color in $x' - y'$ path. It is contradiction.

   (b) for $x \in V(T_i)$ and $y \in V(C_p)$ and $c(x) = c(y)$. Based Observation 2 that $x' - x - y - y'$ have unique path and every vertices in cycle is internal vertices, such that there is at least two vertices have same color in $x' - y'$ path. It is contradiction.

   (c) for $x, y \in V(C_p)$ and $c(x) = c(y)$. We know that x and y are internal vertices of path between two vertices in different subgraph tree, then there is at least two vertices have same color in $x' - y'$ path. It is contradiction.

Based on 1 and 2 that every vertices in $V(UC_n) - U$ have different color such that we have $n - l$-rainbow vertex coloring in UC_n. Thus, $rvc(UC_n) \geq rc(C_p) + |W|$.

Now, let c be a $n - l$-rainbow vertex coloring of UC_n. We show $rvc(UC_n) \leq n - l$ by defining a $n - l$-rainbow vertex coloring $c: V(UC_n) \rightarrow \{1,2,3 \dots, n - l\}$ as follows

$$c(u) = 1$$
$$c(u_i) = i; 1 \leq i \leq p$$
$$c(w) = j; p + 1 \leq j \leq n - l$$

Observe that the coloring c above maintains the position of colors in UC_n and assigns distinct colors in $V(UC_n) - U$.

(a) For vertices $u \in U$ does not internal vertices, such that the $c(u) = c(w) = c(v_i)$.
(b) Every two vertices w' and w'' in T_i have unique path, such that internal vertices in $w' - w''$ have different color.
(c) Every two vertices $w_i \in V(T_i)$ and $w_j \in V(T_j)$ have unique path, such that internal vertices in $w_i - w_j$ have different color.
(d) All vertices in cycle C_p are internal vertices of path $w_i - w_j$ with $w_i \in V(T_i)$ and $w_j \in V(T_j)$, such that $c(v_k) \neq c(v_l)$.

We are able to find a rainbow path for every pair vertices x and y in $V(UC_n)$. Thus, $rc(C_p) + |W| \leq rvc(UC_n) \leq n - l$.

3. Conclusion

We have bounds and exact value of rainbow conenction number of unicyclic graphs. Bounds of rainbow connection number is affected by rainbow connection of subgraph cycle and the number of pendant vertex.

Acknowledgement

We gratefully acknowledge the funding support by University of Jember of year 2025.

References

[1] N. Hartsfield and G. Ringel, Pearls in Graph Theory: A Comprehensive Introduction. Mineola, NY, USA: Courier Corporation (2013).

[2] X. Li and Y. Sun, Rainbow Connections of Graphs. New York, NY, USA: Springer (2012).

[3] G. Chartrand, L. Johns, K. A. McKeon, and P. Zhang, "Rainbow connection in graphs," *Math. Bohem.*, vol. 133, pp. 85–98 (2008).

[4] M. Krivelevich and R. Yuster, "The rainbow connection of a graph is (at most) reciprocal to its minimum degree," *J. Graph Theory*, vol. 63, no. 3, pp. 185–191 (2009).

[5] X. Li and S. Liu, "Rainbow vertex-connection number of 2-connected graphs," arXiv preprint, arXiv:1110.5770 [math.CO] (2011).

[6] D. N. Simamora and A. N. M. Salman, "The rainbow (vertex) connection number of pencil graphs," *Procedia Comput. Sci.*, vol. 74, pp. 138–142 (2015).

[7] A. W. Bustan and A. N. M. Salman, "The rainbow vertex-connection number of star fan graphs," *CAUCHY: Jurnal Matematika Murni dan Aplikasi*, vol. 5, no. 3, pp. 112–116 (2018).

[8] P. Heggernes, D. Issac, J. Lauri, P. T. Lima, and E. J. van Leeuwen, "Rainbow vertex coloring of bipartite graphs and chordal graphs," in Proc. 43rd Int. Symp. Mathematical Foundations of Computer Science (MFCS) (2018).

[9] L. Chen, X. Li, and Y. Shi, "The complexity of determining the rainbow vertex-connection of a graph," *Theor. Comput. Sci.*, vol. 412, pp. 4531–4535 (2011).

[10] T. Zhou, T. Lin, Z. Miao, and L. Lianying, "The extremal Sombor index of trees and unicyclic graphs with given matching number," *J. Discrete Math. Sci. Cryptography*, vol. 26, pp. 1–12 (2022).

[11] S. Javed and A. S. M. Javaid, "Extremal unicyclic graphs with fixed leaves via Sombor index," *J. Discrete Math. Sci. Cryptography,* vol. 28, no. 3, pp. 783–795 (2025).

Received April, 2025