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On the b-chromatic number of 3-petal graph families

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Abstract

The definition of b-coloring of a graph leads to the construction of what we call as Color Class Complete Graph (CCCG). We find the b-chromatic number of the p-petal graph, when p = 3, using the CCC Graph. We also find the b-chromatic number of the line graph, middle graph and the total graph of the p-petal graph.

Subject Classification: *05C15.*

Keywords: *b-coloring, b-chromatic number, p-petal graph, Line graph, Complement graph.*

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1. Introduction

In a b-coloring of G, in every color class C_i of $C = \{C_i : i = 1\ to\ k\}$ there exists at least one vertex, called a b-vertex that is adjacent to at least one vertex in every other color class. The largest possible integer k is called the b-chromatic number of G and is denoted by $\varphi(G)$ [1].

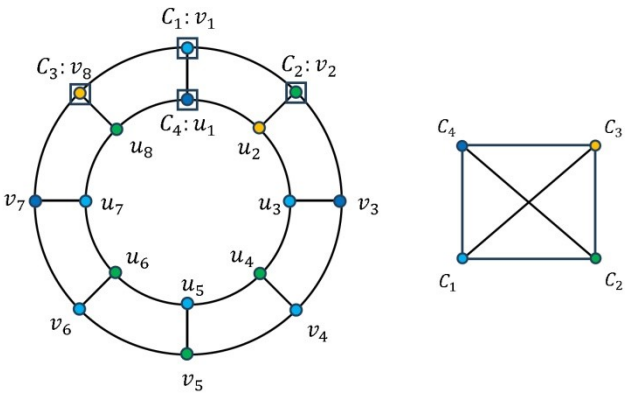

Figure 1
The b-chromatic number of the Prism graph Y_8 is Four

Refer to Figure 1. The b-vertices are enclosed by a square. The b-chromatic number of the Prism graph Y_n is Four [2]. For an even number n, we construct the four vertex color class complete graph K_4 with color classes

1. $C_1 = A_1 \cup B_1 = \{v_1\} \cup \{v_4, v_6, \dots, v_{n-4}, v_{n-2}, u_3, u_5, u_7, \dots, u_{n-3}, u_{n-1}\}$; The b - vertex is v_1
2. $C_2 = A_2 \cup B_2 = \{v_2, u_n\} \cup \{v_5, v_7, \dots, v_{n-3}, u_4, u_6, u_8, \dots, u_{n-4}, u_{n-2}\}$ where $B_2 = \emptyset$; The b - vertex is v_2
3. $C_3 = A_3 \cup B_3 = \{v_n, u_2\}$ where $B_3 = \emptyset$; The b - vertex is v_n
4. $C_4 = A_4 \cup B_4 = \{v_3, v_{n-1}, u_1\}$ where $B_4 = \emptyset$; The b - vertex is u_1

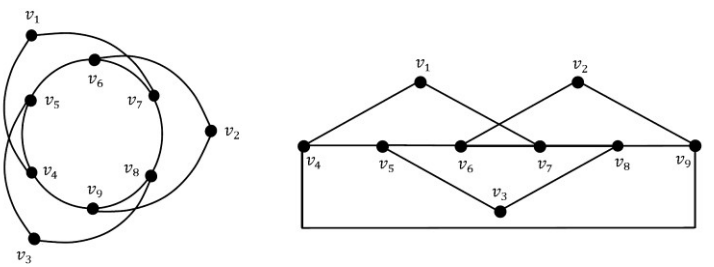

Figure 2
Two different versions of the 3-petal graph with $a = 3$ petals

A petal graph G of n vertices and a petals with petal sequence $\{P_j\}$ is said to be a p-petal graph [3], denoted $G = P_{n,p}$, if every petal in G is of size p and $l(P_i, P_{i+1}) = 2$, $i = 0,1,2,\ldots,a-1$with $P_{a+1} = P_0$. In a p-petal graph the petal size p is always odd.

In this paper, we consider 3-petal graphs with $a \geq 3$ petals. Figure 2 shows two different versions of the 3-petal graph with the same number of petals.

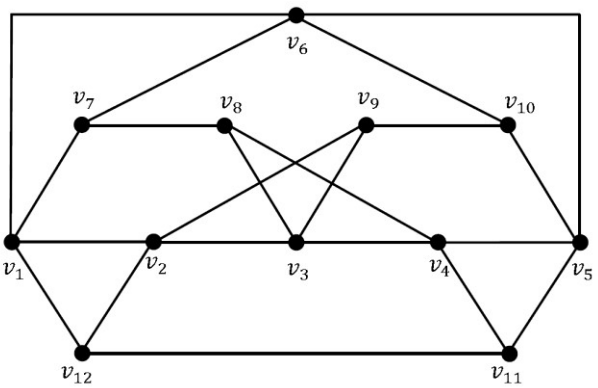

Figure 3
Line graph of the 3-petal graph with $a = 3$ petals

In the works of Whitney [4] originated the concept of line graph. Figure 3 shows the line graph of the 3-petal graph with 3 petals.

Hamada and Yoshimura [5] introduced the middle graph of a graph G, denoted $M(G)$. Figure 4 shows the middle graph of the 3-petal graph with 3 petals.

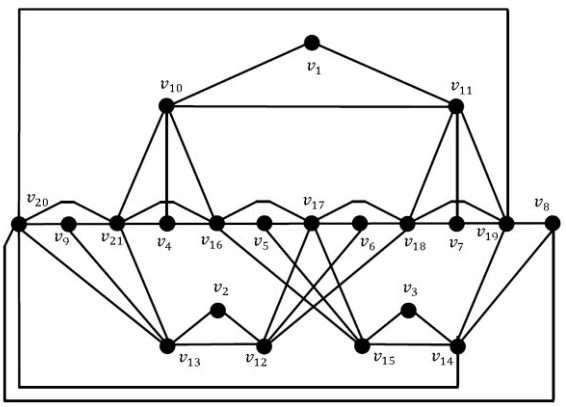

Figure 4
Middle graph of the 3-petal graph with $a = 3$ petals

Behzad and Chartrand [6] defined the total graph of a graph. Figure 5 shows the total graph of the 3-petal graph with 3 petals.

Irving and Manlove provided an upper bound for the b-chromatic number of a graph G. They proved that $\varphi(G) \leq m(G)$, where $m(G)$ is the m-degree of the graph. They also proved that the b-chromatic number is NP-complete.

Kouider and Maheo [7] proved that $\varphi(G) \leq \frac{1}{2} + \sqrt{2m + \frac{1}{4}}$, for a graph G of size m. Vijayalakshmi et al [8] gave the b-chromatic number of the middle graph of some graphs.

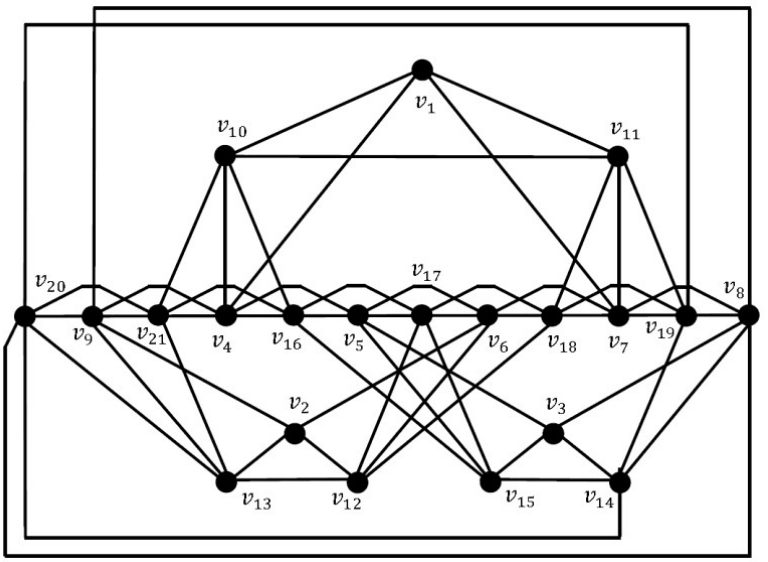

Figure 5
Total graph of the 3-petal graph with $a = 3$ petals

2. Main Results

We prove the following theorems:

Theorem 2.1: *The b-chromatic number of the* 3*-petal graph* $G = P_{n,3}$ *is given by* $\varphi(G) = \begin{cases} 4, & a > 3 \\ 3, & a = 3 \end{cases}$

Proof: The chromatic number of the 3-petal graph G is three. The maximum degree of G is three and there are at least $2a = 6$ vertices of degree three. Hence the largest possible induced complete subgraph is K_4 and therefore, the b-chromatic number $\varphi(G) \leq 4$. The graph G has $2a$ vertices on its core each of degree three. Hence, the maximum possible complete graph is K_4.

Case 1: $a > 3$

Let us consider the vertices v_{a+3}, v_{a+4}, v_{a+5} and v_{a+2} of degree three. We assign a proper coloring to the vertex v_{a+3} as shown in Figure 6.

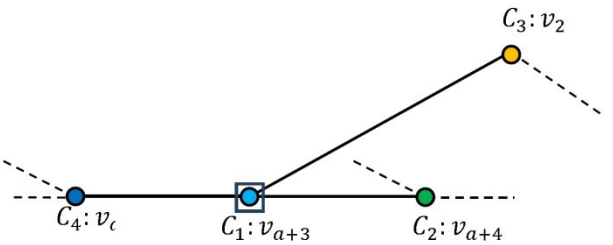

Figure 6

v_{a+3} is the b-vertex of the color class C_1

Similarly, b-coloring is done for each of the remaining three vertices v_{a+4}, v_{a+5} and v_{a+2} together with its neighbours using these four colors. These four vertices form the b-vertex set of the b-coloring of the graph G. The b-coloring of these four vertices is shown in Figure 7.

The petal graph has minimum 8 vertices on the core of the graph. We include these vertices in the color classes A_i, $i = 1$ to 4. These sets of vertices will have the b-vertices and their neighbours only. The graph also has more vertices which are not elements of A_i. We include these vertices in separate color classes denoted B_i. The union C_i of the color classes A_i and B_i will have all the vertices of the graph and form the color class complete graph.

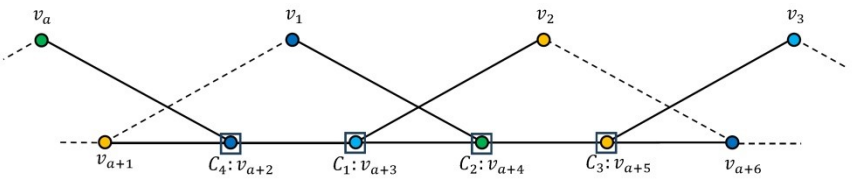

Figure 7

The b-vertex of each color class from C_1 to C_4

The sets A_i will be the same for graphs with number of petals $a \geq 4$. The sets B_i will be distinct for graphs with respect to the number of petals.

Let us consider the vertices v_{a+3}, v_{a+4}, v_{a+5} and v_{a+2} of degree thee. We assign a proper coloring with four colors to the vertex v_{a+3} and its neighbours. The same is done for all the remaining vertices mentioned above using the same four colors.

This proper coloring creates the set of color classes A_i as follows.

1. $A_1 = \{v_3, v_{a+3}\}$; The b - vertex is v_{a+3}

2. $A_2 = \{v_a, v_{a+4}\}$; The b - vertex is v_{a+4}
3. $A_3 = \{v_2, v_{a+1}, v_{a+5}\}$; The b - vertex is v_{a+5}
4. $A_4 = \{v_1, v_{a+2}, v_{a+6}\}$; The b - vertex is v_{a+2}

A proper coloring is done for the remaining vertices using the same colors. When the number of petals are increasing, the color classes B_i will be distinct.

1. $B_1 = \{v_{a+7}, v_{a+9}, \dots, v_{3a-1}\}$
2. $B_2 = \{v_{a+8}, v_{a+10}, \dots, v_{3a}\}$
3. $B_3 = \{v_4, v_5, \dots, v_{a-1}\}$
4. $B_4 = \{\ \}$

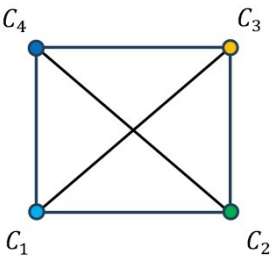

Figure 8
The Color Class Complete Graph K_4 proves that $\varphi(G) = 4$, when $a > 3$

Refer Figure 8.
Thus, when $a > 3$, $\varphi(G) = 4$

Case 2: $a = 3$

When we try to apply case 1, the vertex $v_3 = v_a$ gets two colors C_1 and C_2 when we do a proper coloring of the b - vertices that creates the set of color classes A_i as follows.

1. $A_1 = \{\boldsymbol{v_3}, v_{a+3} = v_6\}$; The b - vertex is $v_{a+3} = v_6$
2. $A_2 = \{v_a = \boldsymbol{v_3}, v_{a+4} = v_7\}$; The b - vertex is $v_{a+4} = v_7$

Thus, $\varphi(G) \neq 4$.

We try to reorganise the color classes as follows:

The even suffixed vertices in the core cycle of G are assigned color c_1, the odd suffixed vertices are assigned color c_2, and the central vertices in the petals of G are assigned color c_3.

The petal graph has 6 vertices on the core of the graph. We include the vertices v_6, v_7 and v_8 in the color classes A_i, $i = 1$ to 3. These sets of vertices will have the b-vertices and their neighbours only. The graph also has more vertices which are not elements of A_i. We include these vertices in separate color classes denoted B_i. The union C_i of the color classes A_i and B_i will have all the vertices of the graph and form the color class complete graph.

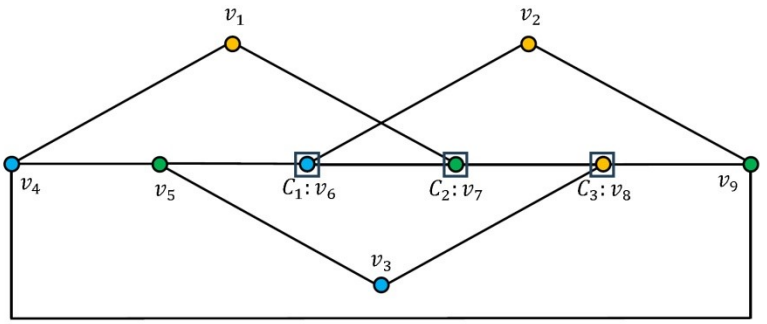

Figure 9
The b-vertex of each color class from C_1 to C_3

We assign a proper coloring with three colors to the vertex v_6 and its neighbours. Refer Figure 9. The same is done for all the remaining vertices mentioned above using the same three colors.

This proper coloring creates the set of color classes A_i as follows.

1. $A_1 = \{v_3, v_6\}$; The b - vertex is v_6
2. $A_2 = \{v_7\}$; The b - vertex is v_7
3. $A_3 = \{v_2, v_8\}$; The b - vertex is v_8

A proper coloring is done for the remaining vertices using the same colors.

1. $B_1 = \{v_4\}$
2. $B_2 = \{v_5, v_9\}$
3. $B_3 = \{v_1\}$

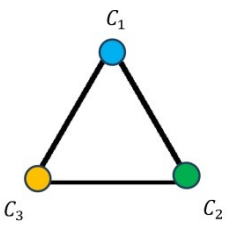

Figure 10
The Color Class Complete Graph K_3 proves that $\varphi(G) = 3$ when $a = 3$

Refer Figure 10.
Thus, $\varphi(G) = 3$.

Theorem 2.2: *The b-chromatic number of the line graph of the 3-petal graph G is given by* $\varphi(L(G)) = 5$

Proof: The maximum degree of $L(G)$ is four and there are at least 5 vertices of degree four. Hence the largest possible color class complete graph is K_5 and therefore, $\varphi(L(G)) \leq 5$.

Here, L(G) has $2a$ vertices each of degree four. When $a = 3$, the core graph has at least five vertices of degree four. Hence, the maximum possible complete graph is K_5. Moreover, there are six vertices of degree four in the core graph of L(G).

Let us consider the vertices v_1, v_2, v_3, v_4 and v_5 of degree four. We assign a proper coloring to the vertex v_1 as shown in Figure 11.

Similarly, b-coloring is done for each of the remaining four vertices v_2, v_3, v_4 and v_5 together with its neighbours using these five colors. These five vertices form the b-vertex set of the b-coloring of the line graph L(G).

The b-coloring of these five vertices is shown in Figure 12.

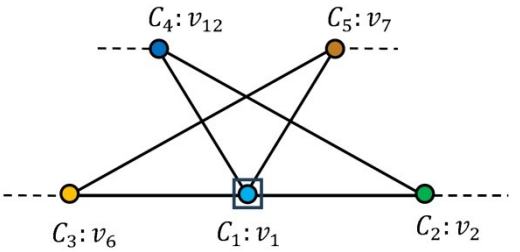

Figure 11
v_1 is the b-vertex of the color class C_1

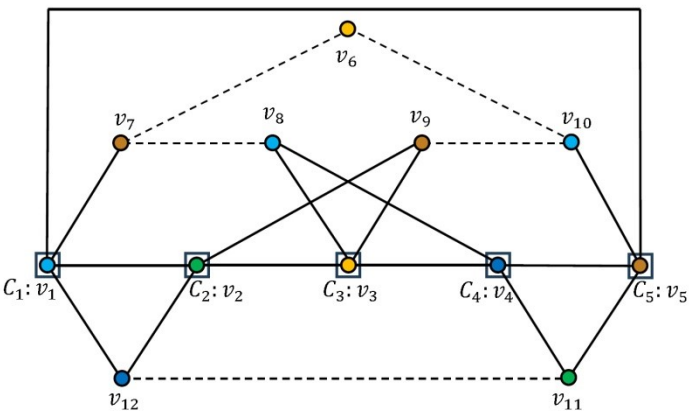

Figure 12
The b-vertex of each color class from C_1 to C_5

Consider the complete graph K_5 with color classes C_1, C_2, C_3, C_4 and C_5.

Case 1: $a = 3$

The $4a = 12$ vertices of $L(G)$ are included in the color classes as follows:

1. $C_1 = \{v_1, v_8, v_{10}\}$; The b-vertex is v_1
2. $C_2 = \{v_2, v_{11}\}$; The b-vertex is v_2
3. $C_3 = \{v_3, v_6\}$; The b-vertex is v_3
4. $C_4 = \{v_4, v_{12}\}$; The b-vertex is v_4
5. $C_5 = \{v_5, v_7, v_9\}$; The b-vertex is v_5

Thus, $\varphi(L(G)) = 5$

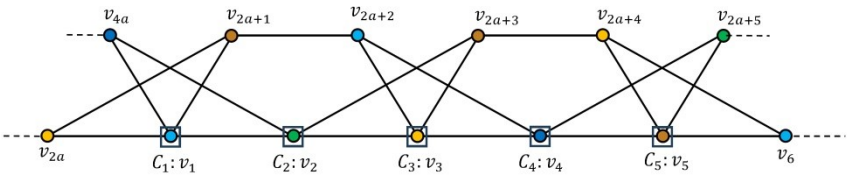

Figure 13

The b-vertex of each color class from C_1 to C_5

Case 2: $a \geq 4$

The line graph has minimum 8 vertices on the core of the graph. We include these vertices in the color classes A_i, $i = 1$ to 5. These sets of vertices will have the b-vertices and their neighbours only. The graph also has more vertices which are not elements of A_i. We include these vertices in separate color classes denoted B_i. The union C_i of the color classes A_i and B_i will have all the vertices of the graph and form the color class complete graph.

The sets A_i will be the same for graphs with number of petals $a \geq 4$. The sets B_i will be distinct for graphs with respect to the number of petals.

Let us consider the vertices v_1, v_2, v_3, v_4 and v_5 of degree four. We assign a proper coloring with five colors to the vertex v_1 and its neighbours. Refer Figure 13. The same is done for all the remaining vertices mentioned above using the same five colors.

This proper coloring creates the set of color classes A_i as follows.

1. $A_1 = \{v_1, v_6, v_{2a+2}\}$; The b - vertex is v_1
2. $A_2 = \{v_2, v_{2a+5}\}$; The b - vertex is v_2
3. $A_3 = \{v_3, v_{2a}, v_{2a+4}\}$; The b - vertex is v_3
4. $A_4 = \{v_4, v_{4a}\}$; The b - vertex is v_4
5. $A_5 = \{v_5, v_{2a+1}, v_{2a+3}\}$; The b - vertex is v_5

A proper coloring is done for the remaining vertices using the same colors. When the number of petals are increasing, the color classes B_i will be distinct.

We have the following cases:

Case 2a: $a = 4$

1. $B_1 = \{v_{14}\}$
2. $B_2 = \{v_7\}$
3. $B_3 = \{v_{15}\}$
4. $B_4 = \{\ \}$
5. $B_5 = \{\ \}$

Case 2b: $a = 5$

1. $B_1 = \{v_{16}, v_{18}, v_{19}\}$
2. $B_2 = \{v_7, v_9\}$
3. $B_3 = \{v_8, v_{17}\}$
4. $B_4 = \{\ \}$
5. $B_5 = \{\ \}$

Case 2c: $a > 5$, a is even

1. $B_1 = \{v_{11}, v_{15}, v_{19}, \dots, v_{2a-1}, v_{2a+6}, v_{2a+8}, v_{2a+9}, v_{2a+12}, v_{2a+13}, v_{2a+16}, \dots, v_{4a-4}, v_{4a-3}\}$
2. $B_2 = \{v_7, v_9, v_{13}, v_{17}, v_{21}, \dots, v_{2a-3}, v_{2a+10}, v_{2a+11}, v_{2a+14}, v_{2a+15}, v_{2a+18}, \dots, v_{4a-2}, v_{4a-1}\}$
3. $B_3 = \{v_8, v_{10}, v_{12}, \dots, v_{2a-2}, v_{2a+7}\}$
4. $B_4 = \{\ \}$
5. $B_5 = \{\ \}$

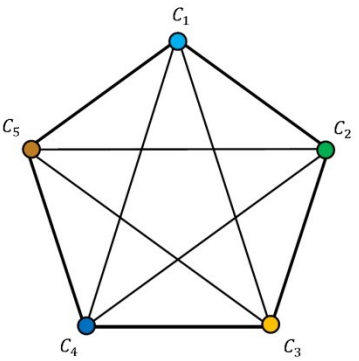

Figure 14
The Color Class Complete Graph K_5 proves that $\varphi(L(G)) = 5$

Case 2d: $a > 5$, a is odd

1. $B_1 = \{v_{11}, v_{15}, v_{19}, \dots, v_{2a-3}, v_{2a+6}, v_{2a+8}, v_{2a+9}, v_{2a+12}, v_{2a+13}, v_{2a+16}, \dots, v_{4a-2}, v_{4a-1}\}$
2. $B_2 = \{v_7, v_9, v_{13}, v_{17}, v_{21}, \dots, v_{2a-1}, v_{2a+10}, v_{2a+11}, v_{2a+14}, v_{2a+15}, v_{2a+18}, \dots, v_{4a-4}, v_{4a-3}\}$
3. $B_3 = \{v_8, v_{10}, v_{12}, \dots, v_{2a-2}, v_{2a+7}\}$
4. $B_4 = \{\ \}$
5. $B_5 = \{\ \}$

Refer Figure 14.

Thus, $\varphi(L(G)) = 5$

Theorem 2.3: *The b-chromatic number of the middle graph of the* 3*-petal graph* $G = P_{n,3}$ *is given by* $\varphi(M(G)) = \begin{cases} 7, & a > 3 \\ 6, & a = 3 \end{cases}$

Proof: Consider G with a petals. Then M(G) has $2a$ vertices each of degree six. When $a = 3$, there are six vertices of degree six. Hence, the maximum possible complete graph is K_6. When $a > 3$, there are at least seven vertices of degree six. Hence, the maximum possible complete graph is K_7.

Therefore two cases arise.

Case 1: $a = 3$

Let us consider the vertices v_{17}, v_{18}, v_{19}, v_{20}, v_{21}, and v_{16} of degree six. We assign a proper coloring to the vertex v_{17} as shown in Figure 15. The sixth neighbouring vertex v_{12} is assigned color C_6 that is already assigned to the vertex v_{16}, because, it does not have an excess color to be assigned.

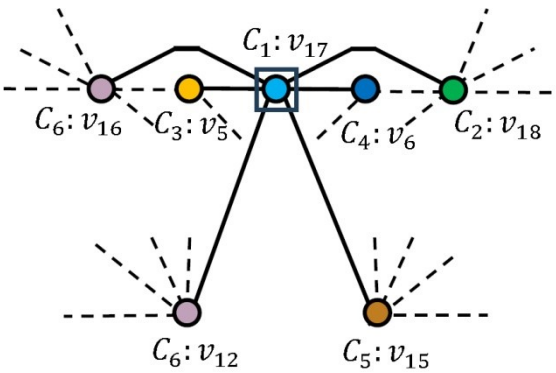

Figure 15
v_{17} is the *b*-vertex of the color class C_1

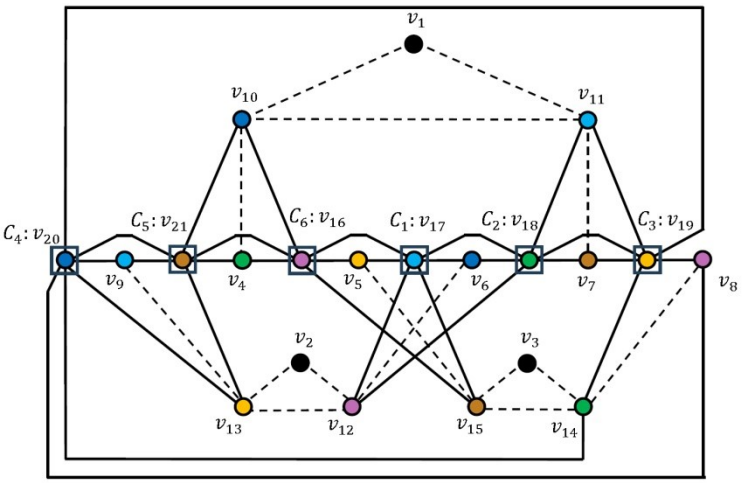

Figure 16
The b-vertex of each color class from C_1 to C_6

Similarly, b-coloring is done for each of the remaining five vertices v_{18}, v_{19}, v_{20}, v_{21}, and v_{16} together with its neighbours using these six colors. These six vertices form the b-vertex set of the b-coloring of the middle graph $\mathrm{M}(G)$.

The b-coloring of these six vertices is shown in Figure 16.

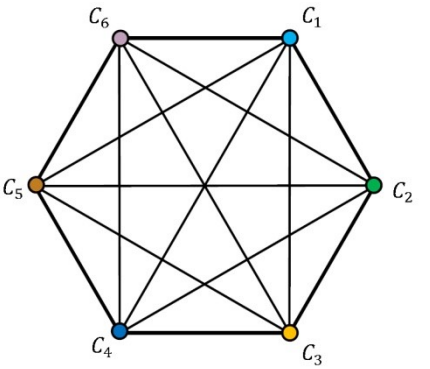

Figure 17
The Color Class Complete Graph K_6 proves that $\varphi(M(G)) = 6$, when $a = 3$

Consider the color classes C_i, $i = 1$ to 6.

1. $C_1 = \{v_9, v_{11}, v_{17}\}$; The b - vertex is v_{17}
2. $C_2 = \{v_1, v_4, v_{14}, v_{18}\}$; The b - vertex is v_{18}
3. $C_3 = \{v_3, v_5, v_{13}, v_{19}\}$; The b - vertex is v_{19}

4. $C_4 = \{v_2, v_6, v_{10}, v_{20}\}$; The b - vertex is v_{20}
5. $C_5 = \{v_7, v_{15}, v_{21}\}$; The b - vertex is v_{21}
6. $C_6 = \{v_8, v_{12}, v_{16}\}$; The b - vertex is v_{16}

Hence the b-chromatic number of the middle graph of the 3-petal graph G with number of petals $a = 3$ is six. Refer Figure 17.

Case 2: $a > 3$

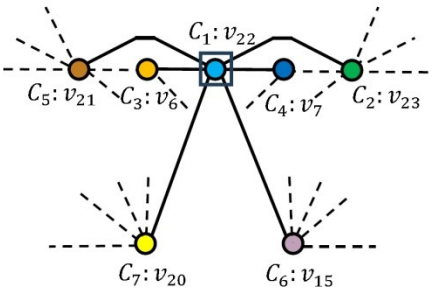

Figure 18
v_{22} is the b-vertex of the color class C_1

The middle graph has minimum 16 vertices on the core of the graph. This is the minimum requirement for the formation of the color class complete graph K_7. We include these vertices in the color classes A_i. These sets of vertices will have the b-vertices and their neighbours only. The graph also has more vertices which are not elements of A_i. We include these vertices in separate color classes denoted B_i. The union C_i of the color classes A_i and B_i will have all the vertices of the graph and form the color class complete graph.

The sets A_i will be the same for graphs with number of petals $a > 4$. The sets B_i will be distinct for graphs with respect to the number of petals.

Case 2a: $a = 4$

Let us consider the vertices v_{22}, v_{23}, v_{24}, v_{25}, v_{26}, v_{27} and v_{28} of degree six. We assign a proper coloring with seven colors to the vertex v_{22} and its neighbours. Refer Figure 18.

The same is done for all the remaining vertices mentioned above using the same seven colors.

This proper coloring creates the set of color classes A_i as follows.

1. $A_1 = \{v_{11}, v_{13}, v_{17}, v_{22}\}$; The b - vertex is v_{22}
2. $A_2 = \{v_{12}, v_{16}, v_{23}\}$; The b - vertex is v_{23}
3. $A_3 = \{v_5, v_6, v_{19}, v_{24}\}$; The b - vertex is v_{24}

4. $A_4 = \{v_7, v_{18}, v_{25}\}$; The b - vertex is v_{25}
5. $A_5 = \{v_8, v_{21}, v_{26}\}$; The b - vertex is v_{26}
6. $A_6 = \{v_9, v_{15}, v_{27}\}$; The b - vertex is v_{27}
7. $A_7 = \{v_{10}, v_{14}, v_{20}, v_{28}\}$; The b - vertex is v_{28}

A proper coloring is done for the remaining vertices using the same colors and the color classes B_i are constructed as follows:

1. $B_1 = \{v_2, v_4\}$
2. $B_2 = \{v_1, v_3\}$
3. $B_3 = \{\ \}$
4. $B_4 = \{\ \}$
5. $B_5 = \{\ \}$
6. $B_6 = \{\ \}$
7. $B_7 = \{\ \}$

The color classes $C_i = A_i \cup B_i$, $i = 1$ to 7 form the color class complete graph K_7.

The color classes A_i, when the number of petals $a \geq 5$ will be the same. The color classes B_i will be different for graphs with number of petals $a = 5$, $a = 6$ and $a \geq 7$. This is because, the number of vertices will be increasing when the number of petals is inceasing.

Case 2b: $a \geq 5$

Let us consider the vertices v_{5a+2}, v_{5a+3}, v_{5a+4}, v_{5a+5}, v_{5a+6}, v_{5a+7} and v_{5a+8} of degree six. We assign a proper coloring with seven colors to the vertex v_{5a+2} and its neighbours. The same is done for all the remaining vertices mentioned above using the same seven colors. Refer Figure 19.

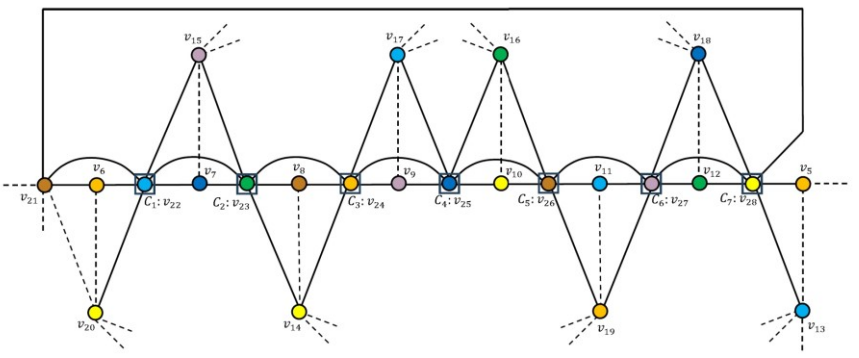

Figure 19
The b-vertex of each color class from C_1 to C_7

This proper coloring creates the set of color classes A_i as follows.

1. $A_1 = \{v_{a+7}, v_{3a+5}, v_{3a+9}, v_{5a+2}\}$; The b - vertex is v_{5a+2}
2. $A_2 = \{v_{a+8}, v_{3a+4}, v_{5a+3}\}$; The b - vertex is v_{5a+3}
3. $A_3 = \{v_{a+2}, v_{a+9}, v_{3a+7}, v_{5a+4}\}$; The b - vertex is v_{5a+4}
4. $A_4 = \{v_{a+3}, v_{3a+6}, v_{5a+5}\}$; The b - vertex is v_{5a+5}
5. $A_5 = \{v_{a+4}, v_{5a+1}, v_{5a+6}, v_{5a+9}\}$; The b - vertex is v_{5a+6}
6. $A_6 = \{v_{a+5}, v_{3a+3}, v_{5a+7}\}$; The b - vertex is v_{5a+7}
7. $A_7 = \{v_{a+6}, v_{3a+2}, v_{5a}, v_{5a+8}\}$; The b - vertex is v_{5a+8}

We construct the color classes B_i as follows:

When $a = 5$

1. $B_1 = \{v_1, v_2, v_4, v_6, v_{15}\}$
2. $B_2 = \{v_3, v_5, v_{35}\}$
3. $B_3 = \{v_{16}\}$
4. $B_4 = \{v_{23}\}$
5. $B_5 = \{\ \}$
6. $B_6 = \{\ \}$
7. $B_7 = \{\ \}$

When $a = 6$

1. $B_1 = \{v_1, v_2, v_4, v_6, v_7, v_{16}, v_{17}, v_{18}\}$
2. $B_2 = \{v_3, v_5, v_{40}, v_{42}\}$
3. $B_3 = \{v_{19}, v_{41}\}$
4. $B_4 = \{v_{26}, v_{28}, v_{29}\}$
5. $B_5 = \{\ \}$
6. $B_6 = \{\ \}$
7. $B_7 = \{\ \}$

When $a \geq 7$

1. $B_1 = \{v_1, v_2, v_4, v_6, v_7, v_8, \dots, v_a, v_{a+1}, v_{a+10}, v_{a+11}, v_{a+12}, \dots, v_{3a-1}, v_{3a}\}$
2. $B_2 = \{v_3, v_5, v_{5a+10}, v_{5a+12}, v_{5a+14}, \dots, v_{7a-2}, v_{7a}\}$
3. $B_3 = \{v_{3a+1}, v_{5a+11}, v_{5a+13}, \dots, v_{7a-3}, v_{7a-1}\}$
4. $B_4 = \{v_{3a+8}, v_{3a+10}, v_{3a+11}, v_{3a+13}, \dots, v_{5a-3}, v_{5a-1}\}$
5. $B_5 = \{v_{3a+12}, v_{3a+14}, \dots, v_{5a-4}, v_{5a-2}\}$
6. $B_6 = \{\ \}$
7. $B_7 = \{\ \}$

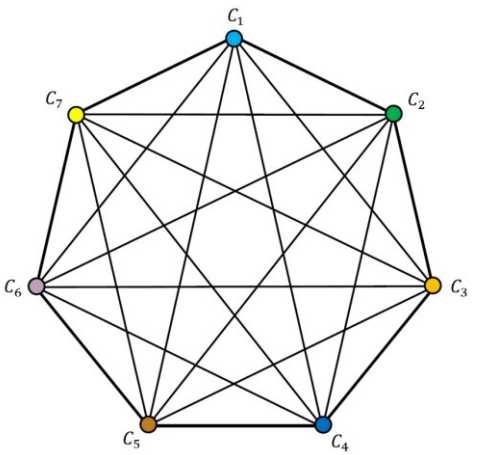

Figure 20
The Color Class Complete Graph K_7 proves that $\varphi(M(G)) = 7$, when $a \geq 4$

Thus, when $a \geq 4$, the b-chromatic number of the middle graph $M(G)$ of the 3-petal graph G is seven. Refer Figure 20.

Theorem 2.4: *The b-chromatic number of the total graph of the* 3*-petal graph* $G = P_{n,3}$ *is given by* $\varphi(T(G)) = 7$

Proof: Consider G with a petals. Then $T(G)$ has $4a$ vertices v_{a+1}, v_{a+2}, v_{a+3}, …, v_{3a-1}, v_{3a}, v_{5a+1}, v_{5a+2}, v_{5a+3}, …, v_{7a-1} and v_{7a} of degree six. Hence, the maximum possible complete graph is K_7.

Let us consider the vertices v_{5a+2}, v_{a+3}, v_{5a+3}, v_{a+1}, v_{a+2}, v_{5a+1} and v_{a+4} of degree six. We assign a proper coloring to the vertex v_{5a+2} as shown in Figure 21.

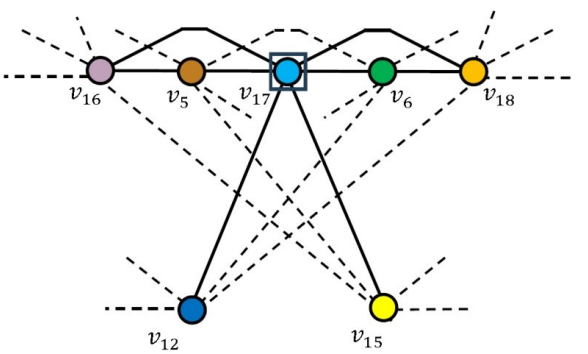

Figure 21
v_{17} is the b-vertex of the color class C_1

Similarly, b-coloring is done for each of the remaining six vertices v_{a+3}, v_{5a+3}, v_{a+1}, v_{a+2}, v_{5a+1} and v_{a+4} together with its neighbours using these seven colors. These seven vertices form the b-vertex set of the b-coloring of the total graph $T(G)$.

The b-coloring of these seven vertices is shown in Figure 22.

Consider the color classes $C_i = A_i \cup B_i$, $i = 1$ to 7.

We construct the color classes A_i as follows:

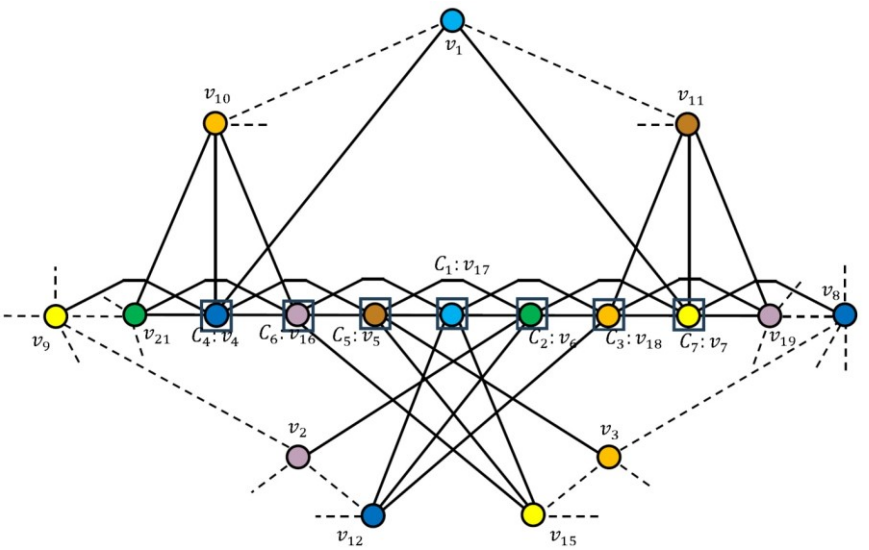

Figure 22
The b-vertex of each color class from C_1 to C_7

1. $A_1 = \{v_1, v_{5a+2}\}$; The b - vertex is v_{5a+2}
2. $A_2 = \{v_{a+3}, v_{7a}\}$; The b - vertex is v_{a+3}
3. $A_3 = \{v_a, v_{3a+1}, v_{5a+3}\}$; The b - vertex is v_{5a+3}
4. $A_4 = \{v_{a+1}, v_{a+5}, v_{3a+3}\}$; The b - vertex is v_{a+1}
5. $A_5 = \{v_{a+2}, v_{3a+2}\}$; The b - vertex is v_{a+2}
6. $A_6 = \{v_2, v_{5a+1}, v_{5a+4}\}$; The b - vertex is v_{5a+1}
7. $A_7 = \{v_{a+4}, v_{3a}, v_{5a}\}$; The b - vertex is v_{a+4}

We construct the color classes B_i as follows:

1. $B_1 = \{v_{a+7}, v_{a+9}, \dots, v_{3a-3}, v_{3a-1}, v_{3a+4}, v_{3a+6}, \dots, v_{5a-4}, v_{5a-2}\}$
2. $B_2 = \{v_{a+6}, v_{a+8}, \dots, v_{3a-2}, v_{3a+5}, v_{3a+7}, \dots, v_{5a-3}, v_{5a-1}\}$
3. $B_3 = \{v_3, v_4, v_5, \dots, v_{a-1}, v_{5a+5}, v_{5a+7}, \dots, v_{7a-1}\}$
4. $B_4 = \{v_{5a+6}, v_{5a+8}, \dots, v_{7a-2}\}$

5. $B_5 = \{\ \}$
6. $B_6 = \{\ \}$
7. $B_7 = \{\ \}$

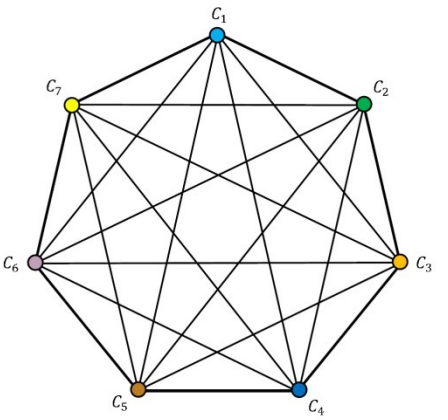

Figure 23
The Color Class Complete Graph K_7 proves that $\varphi(T(G)) = 7$

Thus, $\varphi(T(G)) = 7$. Refer Figure 23.

3. Conclusions and Recommendations

When $p > 3$ the p-petal graph is not planar. Its also interesting because of its feasibility in getting the results.

We hope that the following recommendations give important results:

A. We can extend our work to the complementary graph, split graph and the star graph of the 3-petal graph. We can find the bounds for b-chromatic number of G, when $p > 3$.

B. Ghanbari and AlikhanI [9] have studied about the TDC number and given some important results with reference to the neighbourhood of two graphs. They have investigated about the TDC number of r-gluing of two graphs. It will be quite interesting to investigate the TDC number of families of petal graphs.

C. The Geo chromatic number $\chi_{gc}(G)$ introduced by Samli and Chelladurai [10] is the minimum cardinality among all geo chromatic sets of Paul and Mary [11] discussed about the geo chromatic number of various graphs such as sunlet graph, prism graph and book graph. We can extend the work on the families of petal graphs.

References

[1] Robert W. Irving and David F. Manlove, "The b-chromatic number of a graph", *Discrete Applied Mathematics*, vol. 91, pp. 127 – 141 (1999).

[2] R. S. Nadeem Ansari, Chandel and Rizwana Jamal, "On b-chromatic number of Prism Graph families", *Applications and Applied Mahematics,* vol. 13, no. 1, pp. 286 – 295 (2018).

[3] V. Kolappan and R. Selva Kumar, "Petal Graphs", *International Journal of Pure and Applied Mathematics,* vol. 75, no. 3, pp. 269 – 278 (2012).

[4] Hassler Whitney, "Congruent graphs and the connectivity of graphs", *American Journal of Mathematics,* vol. 54, no. 1, pp. 150 – 168 (1932).

[5] T. Hamada and I. Yoshimura, "Traversability and Connectivity of the middle graph of a graph", *Discrete Mathematics*, vol. 14, pp. 247 – 255 (1976).

[6] M. Behzad and G. Chartrand, "Total graphs and traversability" *Proceedings of the Edinburgh Mathematical Society,* pp. 117 – 120 (1966).

[7] Mekkia Kouider and Maryvonne Maheo, "Some bounds for the b-chromatic number of a graph", *Discrete Mathematics*, vol. 256, pp. 267 – 277 (2002).

[8] D. Vijayalakshmi, K. Thilagavathy and N. Roopesh, "b-chromatic number of $M[C_n]$, $M[P_n]$, $M[F_{1,n}]$ and $M[W_n]$," *Open Journal of Discrete Mathematics*, vol. 1, pp. 85 – 88 (2011).

[9] N. Ghanbari and S. Alikhani, "More on the total dominator chromatic number of a graph," *Journal of Information and Optimization Sciences*, vol. 40, no. 1, pp. 157 – 169 (2018).

[10] S. Beulah Samli and S. Robinson Chelladurai, "Geo Chromatic Number of a Graph," *International Journal of Scientific Research in Mathematical and Statistical Sciences*, vol. 5, no. 6, pp. 259 – 264 (2018).

[11] R. Joseph Paul and U. Mary, "Geo chromatic number of certain graphs," *Journal of Interdisciplinary Mathematics*, vol. 26, no. 1, pp. 11 – 16 (2023).

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