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On orthogonal inclusion graphs of odd characteristic

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Abstract

An orthogonal inclusion graph is constructed from totally isotropic subspaces of an orthogonal space over a finite field of odd characteristic with a finite dimension. Its vertices correspond to all nontrivial totally isotropic subspaces, with two vertices adjacent whenever one is a proper subset of the other. In this work, we determine the independence number and the domination number of this graph in dimension 4.

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1. Introduction

It is well-known that graph theory, linear algebra and abstract algebra have many connections among themselves. Recent research has focused heavily on studying graphs that emerge from the geometric properties of finite classical groups. Since this paper primarily examines finite orthogonal groups and their

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associated spaces, we begin by reviewing the existing literature on graphs that originate from these mathematical structures.

A paper [8] investigated orthogonal graphs over finite fields of odd characteristic whose vertex set is the set of totally isotropic subspaces of dimension one. In there, the authors computed the graphs' parameters, chromatic numbers and automorphism groups. Some years later, these graphs were extended to the cases of finite Galois rings in [11] and finite commutative rings in [16]. Additionally, there is considerable research on graphs defined by the totally isotropic subspaces of higher dimensions, see [10, 19]. For a similar line of works on symplectic graphs, we refer the reader to [12, 13, 14, 15, 19, 20, 21, 24].

Recently, a new connection between graphs and vector spaces has been established by Das [3, 4]. For a finite-dimensional vector space $\mathbb{V}$ over a finite field, Das introduced the subspace inclusion graph, denoted by $\mathcal{I}n(\mathbb{V})$. Its vertices are the nonzero proper subspaces of $\mathbb{V}$ and two vertices are joining if and only if one is a proper subset of the other. Later, a subgraph of $\mathcal{I}n(\mathbb{V})$ defined from symplectic totally isotropic subspaces over a finite field was introduced and several properties of this subgraph were investigated [9]. A related study on orthogonal spaces over finite fields of odd characteristic was conducted by Fan in her Master's thesis [5]. Beyond finite geometry over fields or rings, inclusion graphs also play important roles in other mathematical structures, such as topological spaces. The reader may consult references [17, 18] for further details.

Motivated by these references, we consider orthogonal inclusion graphs over finite fields of odd characteristic in dimension 4. In particular, the domination number and the independence number of the graphs are calculated. This work is also inspired by [23], in which the exact domination number of $\mathcal{I}n(\mathbb{V})$ was computed when $\mathbb{V}$ is in dimension 3, the smallest dimension in which the problem is nontrivial. However, the problem remains open for all higher dimensions.

Structure of the rest of this paper: Section 2 covers key graph-theoretic terminology and discusses orthogonal spaces over finite fields of odd characteristic. The definition of an orthogonal inclusion graph is presented. Finally, Section 3 concludes by determining the independence and domination numbers of the dimension of the orthogonal space is 4.

2. Graph and orthogonal space

In what follows, we review key definitions and terminology from graph theory. The reader is referred to [1, 2, 7] for more details.

We only consider an undirected simple graph G with the set V of vertices and the set E of edges. Denoted by $x{:}y$, a vertex x is adjacent to a vertex y. The set $N(x) = \{y \in V | x{:}y\}$ is called the *neighbor* of $x \in V$. For $U \subset V$, we define $N(U) = \cup_{x \in U} N(x)$. The *degree* of x is given by $\deg(x) := |N(x)|$.

A path whose terminal vertex coincides with its initial vertex is called a *cycle*, and the length of the shortest cycle is the *girth* of G, denoted by $g(G)$. For x and y in V, we denote their shortest connecting path length as $\mathrm{d}(x, y)$, with $\mathrm{d}iam(G) := \max_{x,y \in V} \mathrm{d}(x, y)$ representing the graph's *diameter*.

An *independent set* in G is a subset of V where no two vertices are joining. The *independence number* $\alpha(G)$ is the maximum possible cardinality of an independent set. A *dominating set* D of G is a subset of vertices such that $V = N(D) \cup D$. The *domination number* $\gamma(G)$ is the minimum cardinality of a vertex set D satisfying the above equation.

In our work here, we consider a graph that arises from an orthogonal space over a finite field of odd characteristic. Thus, we shall first recall some important terminologies about this space. For more details, the reader can consult the reference [22].

Let $n \geq 2$ be an integer and q be an odd prime power. For the finite field $\mathbb{F}_q$, the n-dimensional vector space over this field is given by

$$\mathbb{F}_q^n = \{(a_1, a_2, \dots, a_n) | a_i \in \mathbb{F}_q, i = 1,2, \dots, n\}.$$

Let e_i be an n-dimensional vector whose ith component is 1 and 0 elsewhere for $i = 1,2, \dots, n$. Recall that two matrices A and B are *cogredient* if there exists a nonsingular matrix P satisfying $B = P^t A P$, where P^t denotes the matrix transpose of P.

Consider an invertible symmetric matrix S of size $n \times n$ with entries from the finite field $\mathbb{F}_q$, that is, $S^t = S$ and $\det(S) \neq 0$. By [22], S is cogredient to

$$S_{2\nu+\delta} = \begin{pmatrix} 0 & I_\nu & \\ I_\nu & 0 & \\ & & \Delta \end{pmatrix},$$

where I_ν is the identity matrix of order ν, δ is either 0,1 or 2 and

$$= \begin{pmatrix} \emptyset, & \text{if } \delta = 0, \\ (1)\text{or}(z), & \text{if } \delta = 1, \\ \begin{pmatrix} 1 & \\ & -z \end{pmatrix}, & \text{if } \delta = 2, \end{pmatrix}$$

where $\emptyset$ represents an empty set and z is a fixed nonsquare unit of $\mathbb{F}_q$. To simplify notation, we denote $S_{2\nu+\delta}$ by S. Let $n = 2\nu + \delta$. An $n \times n$ matrix T over $\mathbb{F}_q$ is called *orthogonal* with respect to S if it satisfies the relation

$$TST^T = S.$$

The collection of all such orthogonal matrices forms a group under matrix multiplication, known as the *orthogonal group* of degree n over $\mathbb{F}_q$ relative to S, denoted by $O_n(\mathbb{F}_q, S)$. When S_1 and S_2 are cogredient nonsingular symmetric matrices, their corresponding orthogonal groups $O_n(\mathbb{F}_q, S_1)$ and $O_n(\mathbb{F}_q, S_2)$ are isomorphic. Consequently, we may restrict our attention to a standard representative S and simply write $O_{2\nu+\delta}(\mathbb{F}_q)$ for the orthogonal group.

The group $O_{2\nu+\delta}(\mathbb{F}_q)$ acts naturally on $\mathbb{F}_q^{2\nu+\delta}$ via right multiplication:

$$\begin{aligned} \mathbb{F}_q^{2\nu+\delta} \times O_{2\nu+\delta}(\mathbb{F}_q) &\to \mathbb{F}_q^{2\nu+\delta} \\ ((x_1, \dots, x_{2\nu+\delta}), T) &\mapsto (x_1, \dots, x_{2\nu+\delta})T. \end{aligned}$$

This action gives rise to the *orthogonal space* over $\mathbb{F}_q$ with dimension $(2\nu+\delta)$, which we denote by $\mathbb{O}$.

For any subspace $P \subset \mathbb{O}$ of dimension m, its *dual subspace* $P^{\perp}$ consists of all vectors orthogonal to P with respect to S:

$$P^{\perp} = \{y \in \mathbb{O} | ySx^t = 0 \text{ for all } x \in P\}.$$

The dual subspace $P^{\perp}$ has dimension $2\nu+\delta-m$.

By slight abuse of notation, we also write P for the matrix formed by basis vectors of the subspace P. A subspace P is called *totally isotropic* if it satisfies $PSP^T = 0$, or equivalently, if $P \subset P^{\perp}$. The maximal possible dimension of a totally isotropic subspace is ν, and such maximal subspaces always exist. In fact, every totally isotropic subspace is contained in a maximal one. Concrete examples include $\langle e_1, \dots, e_\nu \rangle$ and $\langle e_{\nu+1}, \dots, e_{2\nu} \rangle$, where the basis vectors satisfy

$$e_i S e_{\nu+j}^T = \delta_{ij} \text{ for } i, j = 1, \dots, \nu,$$

where δ_{ij} is the Kronecker delta.

Let $\mathbb{O}$ be a $(2\nu+\delta)$-dimensional orthogonal space over $\mathbb{F}_q$ (with q odd and $\nu \geq 2$). The *orthogonal inclusion graph* $\mathcal{In}(\mathbb{O})$ has as vertices all nontrivial totally isotropic subspaces, with adjacency given by inclusion:

$$P_1 : P_2 \Leftarrow P_1 \subset P_2 \text{ or } P_2 \subset P_1.$$

This graph embeds naturally as an induced subgraph of the inclusion graph $\mathcal{In}(\mathbb{V})$ for the vector space $\mathbb{V} = \mathbb{F}_q^{2\nu+\delta}$. Note that subspaces of equal dimension are never adjacent in $\mathcal{In}(\mathbb{O})$.

The graph $\mathcal{In}(\mathbb{O})$ was originally studied in [5]. It was shown that $\mathcal{In}(\mathbb{O})$ is a connected graph. Furthermore, several parameters and properties of the graph---including the number of vertices, vertex degree, diameter, girth, clique number, chromatic number, planarity, and Eulerian property---were determined.

As far as we know, several interesting parameters of the graph $\mathcal{In}(\mathbb{O})$ have not yet been investigated. This work focuses on two such parameters: $\alpha(\mathcal{In}(\mathbb{O}))$ and $\gamma(\mathcal{In}(\mathbb{O}))$ of $\mathcal{In}(\mathbb{O})$. While determining these parameters in general dimensions appears to be prohibitively complex, the smallest case of dimension 4 offers a manageable and nontrivial setting in which the results can be obtained.

3. Main results

In this section, we present the smallest case of $\mathcal{In}(\mathbb{O})$, that is, when $n = 2(2) = 4$, over a finite field $\mathbb{F}_q$ of odd characteristic. The independence number $\alpha(\mathcal{In}(\mathbb{O}))$ and the domination number $\gamma(\mathcal{In}(\mathbb{O}))$ for this case are determined. We recall some useful ingredients when $n = 4$ as follows.

Lemma 3.1: [5] *Let* $n = 4$. *Then*

1. $\mathcal{In}(\mathbb{O})$ *is bipartite with partite sets* V_1 *and* V_2, *the set of vertices of dimension* 1 *and* 2, *with* $|V_1| = (q+1)^2$ *and* $|V_2| = 2(q+1)$, *respectively. Moreover, every vertex in* V_1 *has a degree* 2 *and each vertex in* V_2 *has a degree* $q+1$.

2. *$\mathcal{I}n(\mathbb{O})$ is connected with diameter 4.*
3. *The girth of $\mathcal{I}n(\mathbb{O})$ is $g(\mathcal{I}n(\mathbb{O})) = 8$.*

3.1 *Independence number*

To find the independence number of $\mathcal{I}n(\mathbb{O})$, we first recall the following terminologies for a graph G with the vertex set V.

1. The *vertex cover number* $\tau(G)$ represents the minimum number of vertices needed to intersect every edge.
2. The *matching number* $\alpha'(G)$ counts the maximum number of pairwise disjoint edges in G.

From [6], we have the equality:

$$\alpha(G) + \tau(G) = |V|.$$

Moreover, if G is bipartite, then König's Theorem (see [1]) implies $\tau(G) = \alpha'(G)$ and so $\alpha(G) = |V| - \alpha'(G)$. We now recall Hall's theorem as follows.

Lemma 3.2: (P. Hall 1933). *Let G be a bipartite graph with partite sets V_1 and V_2. A matching that saturates every vertex of V_2 exists if and only if, for every subset $S \subseteq V_2$,*

$$|N(S)| \geq |S|.$$

By Lemma 3.1, $\mathcal{I}n(\mathbb{O})$ bipartite without isolated vertices. To obtain the main result on the independence number, it remains to compute $\alpha'(\mathcal{I}n(\mathbb{O}))$.

Theorem 3.3: *If $n = 4$, then $\alpha(\mathcal{I}n(\mathbb{O})) = (q+1)^2$.*

Proof: Note that the matching number $\alpha'(\mathcal{I}n(\mathbb{O}))$ cannot exceed $|V_2| = 2q + 2$ as in Lemma 3.1 (1). Next, we show that for every subset S of V_2,

$$|N(S)| \geq |S|.$$

Let S be a subset of V_2 and $k = |S|$. We see that any two vertices in V_2 share at most one common neighbor in V_1; otherwise, this would create a cycle of length 4, which contradicts $g(\mathcal{I}n(\mathbb{O})) = 8$ stated in Lemma 3.1(3). Then

$$N(S) \geq (q+1)k - \binom{k}{2}.$$

Note that $(q+1)k - k_{2} \geq k$ if and only if $2q + 1 \geq k$. Moreover, if $k = 2q + 2 = |V_2|$, then $N(S) = |V_1| = (q+1)^2 \geq 2q + 2 = |V_2| = k$. So, we have the claim. By Lemma 3.2, $\alpha'(\mathcal{I}n(\mathbb{O})) = |V_2|$. Therefore, $\alpha(\mathcal{I}n(\mathbb{O})) = |V| - \alpha'(\mathcal{I}n(\mathbb{O})) = (q+1)^2$.

3.2 *Domination number*

Finally, we study the domination number of $\mathcal{I}n(\mathbb{O})$. Consider the standard basis $\{e_1, e_2, e_3, e_4\}$ for the orthogonal space $\mathbb{O}$. We describe all vertices of $\mathcal{I}n(\mathbb{O})$ using this basis as follows.

Vertices of dimension 1:

Write a vector $x = c_1e_1 + c_2e_2 + c_3e_3 + c_4e_4$ and suppose that $\langle x \rangle$ is totally isotropic. Then we have $c_1c_3 + c_2c_4 = 0$. If $c_1 \neq 0$, then we can let $c_1 = 1$ and $c_3 = -c_2c_4$. On the other hand, if $c_1 = 0$, then c_2 or $c_4 = 0$. Hence, we have the following distinct vertices.

1. $\langle e_1 + \alpha e_2 - \alpha\beta e_3 + \beta e_4 \rangle$; $\alpha, \beta \in \mathbb{F}_q$.
2. $\langle e_2 + \alpha e_3 \rangle$; $\alpha \in \mathbb{F}_q$.
3. $\langle e_3 + \alpha e_4 \rangle$; $\alpha \in \mathbb{F}_q$.
4. $\langle e_4 \rangle$.

Vertices of dimension 2:

Construct vertices of dimension 2 by choosing two vectors from the above lists and then check the totally isotropic condition. We have all distinct vertices as follows.

1. $\langle e_1 + \alpha e_4, e_2 - \alpha e_3 \rangle$; $\alpha \in \mathbb{F}_q$.
2. $\langle e_1 + \alpha e_2, e_3 - \frac{1}{\alpha} e_4 \rangle$; $\alpha \in \mathbb{F}_q \backslash \{0\}$.
3. $\langle e_1, e_4 \rangle$.
4. $\langle e_2, e_3 \rangle$.
5. $\langle e_3, e_4 \rangle$.

Now, by the adjacency condition, we can find the neighbors of each vertex in dimension 1 as follows. For fixed elements $\alpha, \beta \in \mathbb{F}_q$ where $\alpha \neq 0$, we see that:

1. $N(\langle e_1 + \alpha e_2 - \alpha\beta e_3 + \beta e_4 \rangle) = \{\langle e_1 + \beta e_4, e_2 - \beta e_3 \rangle, \langle e_1 + \alpha e_2, e_3 - \frac{1}{\alpha} e_4 \rangle\}$,
2. $N(\langle e_1 + \beta e_4 \rangle) = \{\langle e_1 + \beta e_4, e_2 - \beta e_3 \rangle, \langle e_1, e_4 \rangle\}$,
3. $N(\langle e_2 + \beta e_3 \rangle) = \{\langle e_1 - \beta e_4, e_2 + \beta e_3 \rangle, \langle e_2, e_3 \rangle\}$,
4. $N(\langle e_3 + \alpha e_4 \rangle) = \{\langle e_1 - \frac{1}{\alpha} e_2, e_3 + \alpha e_4 \rangle, \langle e_3, e_4 \rangle\}$,
5. $N(\langle e_3 \rangle) = \{\langle e_2, e_3 \rangle, \langle e_3, e_4 \rangle\}$,
6. $N(\langle e_4 \rangle) = \{\langle e_1, e_4 \rangle, \langle e_3, e_4 \rangle\}$.

In what follows, we analyze the domination number of $\mathcal{J}n(\mathbb{O})$. Under the above setting, we can find one important dominating set as follows.

Lemma 3.4: *Suppose that*

$$
\begin{aligned}
D_1 &= \{\langle e_1 - e_2 + \beta e_3 + \beta e_4 \rangle : \beta \in \mathbb{F}_q\},\\
D_2 &= \{\langle e_3 + e_4 \rangle\},\\
D_3 &= \{\langle e_1 + \alpha e_2, e_3 - \tfrac{1}{\alpha} e_4 \rangle : \alpha \in \mathbb{F}_q \backslash \{0, -1\}\},\\
D_4 &= \{\langle e_1, e_4 \rangle, \langle e_2, e_3 \rangle\}.
\end{aligned}
$$

Then $D = D_1 \cup D_2 \cup D_3 \cup D_4$ is a minimal dominating set of $\mathcal{J}n(\mathbb{O})$ with cardinality $2q + 1$, that is, any proper subset of D cannot be a dominating set.

Proof: First, we divide the partite set V_1 of vertices of dimension 1 into the following sets:

$V_{11} = \{\langle e_1 + \beta e_4\rangle : \beta \in \mathbb{F}_q\}$,
$V_{12} = \{\langle e_1 - e_2 + \beta e_3 + \beta e_4\rangle : \beta \in \mathbb{F}_q\}$,
$V_{13} = \{\langle e_1 + \alpha e_2 - \alpha\beta e_3 + \beta e_4\rangle : \alpha \in \mathbb{F}_q\backslash\{0, -1\}$ and $\beta \in \mathbb{F}_q\}$,
$V_{14} = \{\langle e_2 + \alpha e_3\rangle : \alpha \in \mathbb{F}_q\}$,
$V_{15} = \{\langle e_3\rangle\}$,
$V_{16} = \{\langle e_3 + e_4\rangle\}$,
$V_{17} = \{\langle e_3 + \alpha e_4\rangle : \alpha \in \mathbb{F}_q\backslash\{0,1\}\}$,
$V_{18} = \{\langle e_4\rangle\}$.

For the partite set V_2 of vertices of dimension 2, we partition it into:

$V_{21} = \left\{\langle e_1 + \alpha e_4, e_2 - \alpha e_3\rangle : \alpha \in \mathbb{F}_q\right\}$,
$V_{22} = \{\langle e_1 - e_2, e_3 + e_4\rangle\}$,
$V_{23} = \{\langle e_1 + \alpha e_2, e_3 - \frac{1}{\alpha} e_4\rangle : \alpha \in \mathbb{F}_q\{0, -1\}\}$,
$V_{24} = \{\langle e_1, e_4\rangle, \langle e_2, e_3\rangle\}$,
$V_{25} = \{\langle e_3, e_4\rangle\}$.

Note that $D_1 = V_{12}$, $D_2 = V_{16}$, $D_3 = V_{23}$ and $D_4 = V_{24}$. We show that any vertex of $\mathcal{J}n(\mathbb{O})$ not in D is adjacent to a vertex in D.

Note that each vertex in $V_1\backslash D$ is adjacent to a vertex in D_3 or D_4. Indeed,

1. any vertex $\langle e_1 + \beta e_4\rangle$ in V_{11}, where $\beta \in \mathbb{F}_q$, is adjacent to $\langle e_1, e_4\rangle$ in D_4;
2. any vertex $\langle e_1 + \alpha e_2 - \alpha\beta e_3 + \beta e_4\rangle$ in V_{13}, where $\alpha \in \mathbb{F}_q\backslash\{0, -1\}$ and $\beta \in \mathbb{F}_q$, is adjacent to $\langle e_1 + \alpha e_2, e_3, -\frac{1}{\alpha} e_4\rangle$ in D_3;
3. any vertex $\langle e_2 + \alpha e_3\rangle$ in V_{14}, where $\alpha \in \mathbb{F}_q$, is adjacent to $\langle e_2, e_3\rangle$ in D_4;
4. the vertex $\langle e_3\rangle$ in V_{15} is adjacent to $\langle e_2, e_3\rangle$ in D_4;
5. any vertex $\langle e_3 + \alpha e_4\rangle$ in V_{17}, where $\alpha \in \mathbb{F}_q\backslash\{0,1\}$ is adjacent to $\langle e_1 - \frac{1}{\alpha} e_2, e_3 + \alpha e_4\rangle$ in D_3;
6. the vertex $\langle e_4\rangle$ in V_{18} is adjacent to $\langle e_1\rangle$ in D_4.

On the other hand, any vertex in $V_2\backslash D$ is adjacent to a vertex in D_1 or D_2. To see this,

1. any vertex $\langle e_1 + \alpha e_1, e_2 - \alpha e_3\rangle$ in V_{21}, where $\alpha \in \mathbb{F}_q$, is adjacent to $\langle e_1 - e_2 + \alpha e_3 + \alpha e_4\rangle$ in D_1;
2. any vertex in V_{22} and V_{25} is adjacent to $\langle e_3 + e_4\rangle$ in D_2.

Therefore, $D = D_1 \cup D_2 \cup D_3 \cup D_4$ is a dominating set of $\mathcal{J}n(\mathbb{O})$ and it is clear that $|D| = 2q + 1$.

To show that D is minimal, we suppose that D' is another dominating set of $\mathcal{J}n(\mathbb{O})$ which is a proper subset of D. Then there exists a vertex P in $D\backslash D'$.

Case 1: $P \in D_1$. Then $P = \langle e_1 - e_2 + \beta e_3 + \beta e_4 \rangle$ for some $\beta \in \mathbb{F}_q$. Note that $N(P) = \{\langle e_1 + \beta e_4, e_2 - \beta e_3 \rangle, \langle e_1 - e_2, e_3 + e_4 \rangle\}$. Then $N(P) \cap D' \subset N(P) \cap D = \emptyset$. Thus, $P \notin D' \cup N(D')$, which contradicts the dominating set D'.

Case 2: $P \in D_2$. Then $P = \langle e_1 + e_4 \rangle$. Since $N(P) = \{\langle e_1 - e_2, e_3 + e_4 \rangle, \langle e_3, e_4 \rangle\}$ and $N(P) \cap D' \subset N(P) \cap D = \emptyset$, it implies that $P \notin D' \cup N(D')$, a contradiction.

Case 3: $P \in D_3$. Then $P = \langle e_1 + \alpha e_2, e_3 - \frac{1}{\alpha} e_4 \rangle$ for some $\alpha \in \mathbb{F}_q \backslash \{0, -1\}$. Consider a vertex $Q = \langle e_3 + \alpha e_4 \rangle$ in V_{17} which is not in D. Note that $N(Q) = \{P, \langle e_3, e_4 \rangle\}$. Hence, $Q \notin D' \cup N(D')$, which is impossible.

Case 4: $P \in D_4$. Then P is either $\langle e_1, e_4 \rangle$ or $\langle e_2, e_3 \rangle$. If $P = \langle e_1, e_4 \rangle$, then the vertex $\langle e_4 \rangle$ in V_{18} doesn't belong $D' \cap N(D')$, a contradiction. Similarly, if $P = \langle e_2, e_3 \rangle$, then the vertex $\langle e_3 \rangle$ in V_{15} doesn't belong $D' \cap N(D')$, a contradiction.

Hence, D is a minimal dominating set of $\mathcal{J}n(\mathbb{O})$.

The following lemma provides some crucial observations.

Lemma 3.5: *Let P be a vertex of dimension 1 in $\mathcal{J}n(\mathbb{O})$ and $N(P) = \{Q_1, Q_2\}$, where Q_1, Q_2 are vertices of dimension 2. Then*

1. $N(Q_1) \cap N(Q_2) = \{P\}$,
2. $|N(N(Q_i) \backslash \{P\})| = q + 1$ for all $i = 1,2$,
3. $N(N(Q_1) \backslash \{P\}) \cap N(N(Q_2) \backslash \{P\}) = \emptyset$, and
4. $V_2 = N(N(Q_1) \backslash \{P\}) \cup N(N(Q_2) \backslash \{P\})$.

Proof: Recall that $\deg(P_1) = q + 1 = \deg(P_2)$. Now, let

$$N(Q_1) = \{P, P_{11}, P_{12}, \dots, P_{1q}\} \text{ and } N(Q_2) = \{P, P_{21}, P_{22}, \dots, P_{2q}\}.$$

If $P_{1i} = P_{2j}$ for some $i, j \in \{1,2,\dots,q\}$, then $P: Q_1: P_{1i} = P_{2j}: Q_2: P$ is a cycle of length 4 which contradicts the girth $g(\mathcal{J}n(\mathbb{O})) = 8$ in Lemma 3.1 (3). Hence, $N(Q_1) \cap N(Q_2) = \{P\}$. This proves (1).

Next, let $i \in \{1,2\}$. Since $\dim(P_{ij}) = 1$, we have $\deg(P_{ij}) = 2$. So we let $N(P_{ij}) = \{Q_i, Q_{ij}\}$, where Q_{ij} is a vertex of dimension 2, for all $j = \{1,2,\dots,q\}$. Hence,

$$N(N(Q_i) \backslash \{P\}) = N(\{P_{i1}, P_{i2}, \dots, P_{iq}\}) = \{Q_i, Q_{i1}, Q_{i2}, \dots, Q_{iq}\}.$$

Suppose that $Q_{ij} = Q_{ij'}$ where $j \neq j'$. Then $Q_i: P_{ij}: Q_{ij} = Q_{ij'}: P_{ij'}: Q_i$ is a cycle of length 4 which contradicts $g(\mathcal{J}n(\mathbb{O})) = 8$. Thus, $|N(N(Q_i) \backslash \{P\})| = q + 1$. This proves (2).

Note that $Q_i \neq Q_{i'j}$ for all $i, i' \in \{1,2\}$ with $i \neq i'$ and $j \in \{1,2,\dots,q\}$; otherwise, there is a cycle of length 4. Suppose that $Q_{1j} = Q_{2j'}$ for some

$j, j' \in \{1,2,\dots,q\}$. Then $P: Q_1: P_{1j}: Q_{1j} = Q_{2j'}: P_{2j'}: Q_2: P$ is a cycle of length 6, which contradicts the girth of $\mathcal{J}n(\mathbb{O})$. Therefore,

$$\{Q_1, Q_{11}, Q_{22}, \dots, Q_{2q}\} \cap \{Q_2, Q_{21}, Q_{22}, \dots, Q_{2q}\} = \emptyset.$$

This completes (3).

Finally, by (2), the set $N(N(Q_i)\backslash\{P\})$ consists of $q+1$ vertices of dimension 2 for each $i = 1,2$. This result together with (3) imply $|N(N(Q_1)\backslash\{P\}) \cup N(N(Q_2)\backslash\{P\})| = 2(q+1)$. Since $N(N(Q_1)\backslash\{P\}) \cup N(N(Q_1)\backslash\{P\}) \subset V_2$ and $|V_2| = 2(q+1)$, we obtain that

$$V_2 = N(N(Q_1)\backslash\{P\}) \cup N(N(Q_2)\backslash\{P\}).$$

Therefore, we have (4).

The following theorem provides the exact value of the domination number of our graph.

Theorem 3.6: *Let* $\mathbb{O}$ *be an orthogonal space over* $\mathbb{F}_q$ *of dimension* 4. *Then*

$$\gamma(\mathcal{J}n(\mathbb{O})) = 2q + 1.$$

Proof: First, the set D in Lemma 3.4 is a dominating set of $\mathcal{J}n(\mathbb{O})$ with $|D| = 2q + 1$. Hence, $\gamma(\mathcal{J}n(\mathbb{O})) \le 2q + 1$.

Next we show that the converse holds. Let D' be a dominating set of $\mathcal{J}n(\mathbb{O})$ with $|D'| = \gamma(\mathcal{J}n(\mathbb{O}))$. Recall that the graph $\mathcal{J}n(\mathbb{O})$ is bipartite with partite sets V_1 of vertices of dimension 1 and V_2 of vertices of dimension 2 in $\mathcal{J}n(\mathbb{O})$. Suppose that $D_1 = D' \cap V_1$, $D_2 = D' \cap V_2$, $|D_1| = d_1$ and $|D_2| = d_2$. Then $d_1 + d_2 = |D'| = \gamma(\mathcal{J}n(\mathbb{O})) \le 2q + 1$. Since D' is a dominating set,

$$V_1 = D_1 \cup N(D_2) \text{ and } V_2 = N(D_1) \cup D_2. \tag{3.1}$$

Note that $|V_2| = 2(q+1)$. Next, we claim that

$$|N(D_1)| \le d_1 + 1 \tag{3.2}$$

If $d_1 = 1$, then it is clear that $|N(D_1)| = 2 = d_1 + 1$. For $d_1 \ge 2$, let $D_1 = \{P^{(1)}, P^{(2)}, \dots, P^{(d_1)}\}$. Suppose the contrary that $|N(D_1)| > d_1 + 1$. Then there exists $P^{(i_0)} \in D_1$ such that

$$N(P^{(i_0)}) \cap (\cup_{i \ne i_0} N(P^{(i)})) = \emptyset. \tag{3.3}$$

WLOG, we assume that $i_0 = 1$ and $N(P^{(1)}) \cap N(P^{(2)}) = \emptyset$. Let $N(P^{(1)}) = \{Q_1^{(1)}, Q_2^{(1)}\}$ and $N(P^{(2)}) = \{Q_1^{(2)}, Q_2^{(2)}\}$ where $Q_1^{(1)}$, $Q_2^{(1)}$, $Q_1^{(2)}$ and $Q_2^{(2)}$ are four distinct vertices of dimension 2. By Lemma 3.5, we have

$$V_2 = N(N(Q_1^{(1)})\backslash\{P^{(1)}\}) \cup N(N(Q_2^{(1)})\backslash\{P^{(1)}\}).$$

Since $Q_1^{(2)} \in V_2$, WLOG, we suppose that $Q_1^{(2)} \in N(N(Q_1^{(1)})\backslash\{P^{(1)}\})$. Then $Q_1^{(2)}: P_1$ for some vertex $P_1 \in N(Q_1^{(1)})$ and $P_1 \ne P^{(1)}$. Hence, $Q_1^{(2)}: P_1: Q_1^{(1)}$. It can be shown by Lemma 3.1 (2) that there exists a vertex $P_2 \in N(Q_2^{(1)}) \cap N(Q_2^{(2)})$ and $P_1 \ne P_2$. Note that (3.3) implies that $P_1, P_2 \notin D_1$.

Thus, at least two vertices from $\{Q_1^{(1)},\ Q_2^{(1)},\ Q_1^{(2)}, Q_2^{(2)}\}$ must be in D_2. If $N(D_1) = d_1 + 2$, then we would have

$$2q + 1 \geq d_2 + d_1 \geq (|V_2| - N(D_1) + 2) + d_1 = 2q + 2,$$

a contradiction. Assume that $N(D_1) = d_1 + 2 + i$, where $i \geq 1$. Consider the set of $d_1 - 2$ vertices $D_{1\prime} = \{P^{(3)}, P^{(4)}, \cdots, P^{(d_1)}\}$ and the set of $d_1 - 2 + i$ vertices $N' = N(D_1)\backslash\{Q_1^{(1)},\ Q_2^{(1)},\ Q_1^{(2)}, Q_2^{(2)}\}$. Since $\deg(P^{(i)}) = 2$ for each i, there are at least i vertices $P^{(j_1)}, P^{(j_2)}, \dots, P^{(j_i)}$ such that each one is adjacent to two vertices in N' with no common neighbor for each pair, i.e., $N(P^{(j_s)}) \cap N(P^{(j_t)}) = \emptyset$ for all $s \neq t$. If i is even, then, similarly to the previous arguments for $P^{(1)}$ and $P^{(2)}$, at least i vertices from N' must be in D_2. Thus,

$$2q + 1 \geq d_2 + d_1 \geq (|V_2| - N(D_1) + 2 + i) + d_1 = 2q + 2,$$

a contradiction. Assume that i is odd and suppose that $N(P^{j_1}) = \{Q_1^{(j_1)}, Q_2^{(j_1)}\}$. Again we can argue that there exist three distinct vertices $P_{1\prime}, P_{2\prime}, P_{3\prime} \in V_1\backslash D_1$ such that $Q_2^{(1)}: P_{1\prime}: Q_1^{(j_1)}, Q_1^{(2)}: P_{2\prime}: Q_1^{(j_1)}$ and $Q_2^{(2)}: P_{3\prime}: Q_2^{(j_1)}$. Then at least three vertices from $\{Q_1^{(1)},\ Q_2^{(1)},\ Q_1^{(2)}, Q_2^{(2)}, Q_1^{(j_1)}, Q_2^{(j_1)}\}$ must be in D_2. Consider the set $\{P^{(j_2)}, P^{(j_3)}, \dots, P^{(j_i)}\}$. As the previous case, at least $i - 1$ vertices from this set must be in D_2. Hence,

$$2q + 1 \geq d_2 + d_1 \geq (|V_2| - N(D_1) + 3 + (i - 1)) + d_1 = 2q + 2,$$

which is a contradiction. Thus, we have the claim $|N(D_1))| \leq d_1 + 1$. It implies that $|V_2| = |N(D_1) \cup D_2| \leq |N(D_1)| + d_2 \leq (d_1 + 1) + d_2$. Therefore,

$$2q + 1 = |V_2| - 1 \leq d_1 + d_2 = \gamma(\mathcal{J}n(\mathbb{O})).$$

This completes the proof.

Note that if $q = 3$, then the above theorems imply $\alpha(\mathcal{J}n(\mathbb{O})) = 16$ and $\gamma(\mathcal{J}n(\mathbb{O})) = 7$.

Example 1: Let $\mathbb{O}$ be an orthogonal space over $\mathbb{F}_3$ of dimension 4. The following list is for 1-dimensional totally isotropic subspaces of $\mathbb{O}$:

$P_1 = \langle e_1\rangle, P_2 = \langle e_4\rangle, P_3 = \langle e_2\rangle, P_4 = \langle e_3\rangle,$
$P_5 = \langle e_1 + e_4\rangle, P_6 = \langle e_1 + 2e_4\rangle, P_7 = \langle e_2 + 2e_3\rangle, P_8 = \langle e_2 + e_3\rangle,$
$P_9 = \langle e_1 + 2e_2\rangle, P_{10} = \langle e_1 + e_2\rangle, P_{11} = \langle e_3 + e_4\rangle, P_{12} = \langle e_3 + 2e_4\rangle,$
$P_{13} = \langle e_1 + 2e_2 + e_3 + e_4\rangle, P_{14} = \langle e_1 + e_2 + 2e_3 + e_4\rangle,$
$P_{15} = \langle e_1 + 2e_2 + 2e_3 + 2e_4\rangle, P_{16} = \langle e_1 + e_2 + e_3 + 2e_4\rangle,$

where denotes the vector with a 1 in the ith coordinate and 0's elsewhere. Next, we describe all 2-dimensional totally isotropic subspaces of $\mathbb{O}$ as follows:

$Q_1 = \langle e_1, e_4\rangle, Q_2 = \langle e_2, e_3\rangle, Q_3 = \langle e_1, e_2\rangle, Q_4 = \langle e_3, e_4\rangle,$
$Q_5 = \langle e_1 + e_4, e_2 + 2e_3\rangle, Q_6 = \langle e_1 + 2e_4, e_2 + e_3\rangle,$
$Q_7 = \langle e_1 + 2e_2, e_3 + e_4\rangle, Q_8 = \langle e_1 + e_2, e_3 + 2e_4\rangle.$

The graph of $\mathcal{I}n(\mathbb{O})$ is illustrated in Figure 1. The independence number is

$$\alpha(\mathcal{I}n(\mathbb{O})) = 16,$$

by Theorem 3.3, with an independent set $\{P_1, P_2, \ldots, P_{16}\}$. From Theorem 3.6, we have

$$\gamma(\mathcal{I}n(\mathbb{O})) = 7$$

with a dominating set $\{P_9, P_{11}, P_{13}, P_{15}, Q_1, Q_2, Q_8\}$.

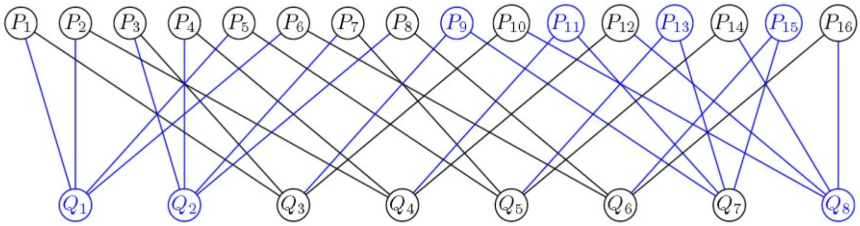

Figure 1
$\mathcal{I}n(\mathbb{O})$ in dimension 4 over $\mathbb{F}_3$

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