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On graded P_e-J_{gr}-primary submodules

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Abstract

This study investigates the concept of gr-P_e-J_{gr}-PrySups as an extension of gr-J_{gr}-PrySub within the framework of Ω-graded commutative rings. Let Ω be a group with identity e, $W = \oplus_{g\in\Omega} W_g$ a Ω-graded commutative ring, $C = \oplus_{g\in\Omega} C_g$ a graded W-module, and $P = \oplus_{g\in\Omega} P_g$ a graded ideal of W. A proper graded submodule U of C is identified as a graded P_e-J_{gr}-primary submodule if, whenever $r_g c_h \in U \backslash P_e$ U for $r_g \in h(W)$ and $c_h \in h(C)$, then either $c_h \in U + J_{gr}(C)$ or $r_g \in Gr((U + J_{gr}(C) :_W C))$, where $J_{gr}(C)$ denotes the graded Jacobson radical of C. The paper presents some results concerning the properties and behaviors of this class of graded submodules.

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1. Introduction and Preliminaries

The study of graded rings and modules has long been a topic of interest due to their significant applications in mathematics and physics. In differential geometry, graded Lie algebras play a crucial role, notably in the Frolicher-Nijenhuis bracket and the Nijenhuis-Richardson bracket [1]. Similarly, these structures address various physical challenges, such as those involving supermanifolds, supersymmetry, and the quantization of systems exhibiting symmetry [2, 3].

The concept of gr-primary ideals was first introduced in [4], forming the basis for further investigations into graded structures. Atani and Farzalipour [5]

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later extended this idea by defining gr-PrySub, which have since been studied extensively by numerous researchers [6, 7, 8]. Subsequent generalizations have included gr-2-absorbing primary submodules [9], gr-S-2-absorbing primary submodules [10], gr-weakly 2-absorbing primary submodules [11], gr-classical primary submodules [12], gr-weakly classical primary submodules [13], and gr-I_e-primary submodules [14].

More recently, Alnimer and Al-Zoubi [15] introduced the concept of gr-J_{gr}-PrySubs as a new generalization of gr-PrySubs. Building upon this framework, the current paper proposes the notion of gr-$P_e - J_{gr}$-PrySubs, which further extends the theory of gr-J_{gr}-PrySubs. This work explores the fundamental properties of this class of gr-Sub.

A submodule $U \subseteq C$ is called a graded submodule (abbreviated as, gr-Sub) of C if $U = \bigoplus_{g\in\Omega} (U \cap C_g) := \bigoplus_{g\in\Omega} U_g$. For additional details, see [16].

A gr-Sub U of C is called a gr-maximal submodule (abbreviated as, gr-MSub) if $U \neq C$ and if there is a gr-Sub V of C with $U \subseteq V \subseteq C$, then either $U = V$ or $V = C$, see [16]. The gr-Jacobson radical of C, $J_{gr}(C)$, is defined as the intersection of all gr-MSubs of C. If C has no gr-MSubs, then by definition, $J_{gr}(C) = C$ [16]. Additionally, it has been shown [17, Lemma 2.11] that if B is a gr-Sub of W, then $(B :_W C) = \{r \in W : rC \subseteq B\}$ forms a gr-ideal of W. From here on, we adopt the notation established in [14]. For background information and fundamental results related to modules and graded modules, readers may consult [16, 18, 19, 20, 22].

2. Results

The *gr-radical* of a gr-ideal P, $Gr(P)$, is the set of all $p = \sum_{h\in\Omega} p_h \in W$ such that for each $h \in \Omega$ there exists $m_h > 0$ with $p_h^{m_h} \in P$. If $p \in h(W)$, then $p \in Gr(P)$ if and only if $p^m \in P$ for some $m \in \mathbb{N}$, see [4].

Definition 2.1: Let $P = \bigoplus_{g\in\Omega} P_g$ be a gr-Id of W, $U = \bigoplus_{g\in\Omega} U_g$ a gr-Sub of C and $\alpha \in \Omega$.

(i) The submodule U_α is defined to be an α-P_e-J_{gr}-primary submodule (simply, α-P_e-J_{gr}-PrySub) of the W_e-module C_α if $U_\alpha \neq C_\alpha$; and whenever $r_e c_\alpha \in U_\alpha \backslash P_e U_\alpha$ where $r_e \in W_e$ and $c_\alpha \in U_\alpha$, then either $r_e^n \in (U_\alpha + J_{gr}(C_\alpha) :_{W_e} C_\alpha)$ for some $n \in \mathbb{Z}^+$ or $c_\alpha \in U_\alpha + J_{gr}(C_\alpha)$.

(ii) The submodule U is defined to be a graded P_e-J_{gr}-primary submodule (simply, gr-P_e-J_{gr}-PrySub) of C if $U \neq C$; and whenever $r_g c_h \in U \backslash P_e U$ where $r_g \in h(W)$ and $c_h \in h(C)$, then either $r_g^n \in (U + J_{gr}(C) :_W C)$ for some $n \in \mathbb{Z}^+$ or $c_h \in U + J_{gr}(C)$.

A proper gr-Sub U of C is called a gr-J_{gr}-PrySub if $r_g c_h \in U$ where $r_g \in h(W)$ and $c_h \in h(C)$, then $r_g \in Gr((U + J_{gr}(C) :_W C))$ or $c_h \in U + J_{gr}(C)$ [15].

It is evident that every gr-J_{gr}-PrySub is also a gr-P_e-J_{gr}-PrySub. However, the reverse is not always true, as demonstrated by the following example.

Example 2.2; Let $\Omega = \mathbb{Z}_2$ and $W = \mathbb{Z} \times \mathbb{Z}$. Then W is a Ω-graded ring with $W_0 = \mathbb{Z} \times \mathbb{Z}$ and $W_1 = \{(0,0)\}$. Let $C = \mathbb{Z} \times \mathbb{Z}$, then C is a graded W-module with $C_0 = \mathbb{Z} \times \mathbb{Z}$ and $C_1 = \{(\bar{0}, \bar{0})\}$. Consider the gr-Id $P = \{0\} \times \mathbb{Z}$ of W and the gr-Sub $U = \{\bar{0}\} \times 2\mathbb{Z}$ of C. Then, as $U \backslash P_e U = \emptyset$ we get U is a gr-P_e-J_{gr}-PrySub. But U is not a gr-J_{gr}-PrySub of C since $(1,2) \cdot (\bar{0}, \bar{1}) = (\bar{0}, \bar{2}) \in U$ but $(\bar{0}, \bar{1}) \notin U + J_{gr}(C) = \{\bar{0}\} \times 2\mathbb{Z}$ and $(1,2)^n \notin (U + J_{gr}(C):_W C) = \{0\} \times 2\mathbb{Z}$ for all $n \in \mathbb{Z}^+$.

Let $P = \bigoplus_{g\in\Omega} P_g$ be a gr-Id of W. A proper gr-Sub U of C is called a gr-P_e-J_{gr}-PSub if $r_g c_h \in U \backslash P_e U$ where $r_g \in h(W)$ and $c_h \in h(C)$, then $r_g \in (U + J_{gr}(C):_W C)$ or $c_h \in U + J_{gr}(C)$, see [21].

Theorem 2.3: *Let $P = \bigoplus_{g\in\Omega} P_g$ be a gr-Id of W and U a gr-Sub of C such that $(U + J_{gr}(C):_W C)$ is a graded radical ideal. Then U is a gr-P_e-J_{gr}-PrySub of C if and only if U is a gr-P_e-J_{gr}-PSub of C.*

Proof: The proof comes from $(U + J_{gr}(C):_W C) = Gr((U + J_{gr}(C):_W C))$.

Theorem 2.4*: Let $P = \bigoplus_{g\in\Omega} P_g$ be a gr-Id of W and U a gr-P_e-J_{gr}-PrySub of C.*

(i) If L is a gr-Sub of C with $U \varnothing L$ and $J_{gr}(C) \subseteq J_{gr}(L)$, then U is a gr-P_e-J_{gr}-PrySub of L.

(ii) If $I = \bigoplus_{g\in\Omega} I_g$ is a gr-Id of W with $P \subseteq I$, then U is a gr-I_e-J_{gr}-PrySub of C.

Proof: (i) Straightforward. (ii) The proof comes from $U \backslash I_e U \subseteq U \backslash P_e U$.

Theorem 2.5: *Let $P = \bigoplus_{g\in\Omega} P_g$ be a gr-Id of W and U a gr-Sub of C. Then the following statements are equivalent:*

(1) *U is a gr-P_e-J_{gr}-PrySub of C.*

(2) *For any gr-Id I of W and any gr-Sub L of C with $IL \subseteq U \backslash P_e U$, then either $I \subseteq Gr((U + J_{gr}(C):_W C))$ or $L \subseteq U + J_{gr}(C)$.*

Proof: (1) ⇒ (2) Let I be a gr-Id of W and L a gr-Sub of C such that $IL \subseteq U \backslash P_e U$. Let us assume that $I \not\subseteq Gr((U + J_{gr}(C):_W C))$ and $L \not\subseteq U + J_{gr}(C)$. So, there are $i_\alpha \in (I \cap h(W)) \backslash Gr((U + J_{gr}(C):_W C))$ and $l_\lambda \in (L \cap h(C)) \backslash (U + J_{gr}(C))$. But $i_\alpha l_\lambda \in U \backslash P_e U$, this yields that either $i_\alpha \in Gr((U + J_{gr}(C):_W C))$ or $l_\lambda \in U + J_{gr}(C)$ as U is a gr-P_e-J_{gr}-PrySub of C, which is a contradiction. Hence, either $I \subseteq Gr((U + J_{gr}(C):_W C))$ or $L \subseteq U + J_{gr}(C)$.

(2) ⇒ (1) Let $r_g c_h \in U \backslash P_e U$ for some $r_g \in h(W)$ and $c_h \in h(C)$. So $I = W r_g$ is a gr-Id of W generated by r_g and $L = W c_h$ is a gr-Sub of C generated by c_h. So, $IL \subseteq U \backslash P_e U$ which yields that either $I \subseteq Gr((U + J_{gr}(C) :_W C))$ or $L \subseteq U + J_{gr}(C)$. Therefore, either $r_g \in Gr((U + J_{gr}(C) :_W C))$ or $c_h \in U + J_{gr}(C)$.

Theorem 2.6: *Let $P = \bigoplus_{g \in \Omega} P_g$ be a gr-Id of W and U a gr-P_e-J_{gr}-PrySub of C. If $S \subseteq h(W)$ is a multiplicatively closed subset of W with $(U :_W C) \cap S = \emptyset$ and $S^{-1}(P_e U) \subseteq (S^{-1}P)_e (S^{-1}U)$, then $S^{-1}U$ is a gr-$(S^{-1}P)_e$-J_{gr}-PrySub of the $S^{-1}W$-module $S^{-1}C$.*

Proof: It is clear that $S^{-1}U \neq S^{-1}C$ as $(U :_W C) \cap S = \emptyset$. Now, let $\frac{i_\alpha}{z_1}\frac{c_\lambda}{z_2} = \frac{i_\alpha c_\lambda}{z_1 z_2} \in (S^{-1}U) \backslash (S^{-1}P)_e (S^{-1}U) \subseteq S^{-1}(U \backslash P_e U)$ for some $\frac{i_\alpha}{z_1} \in h(S^{-1}W)$ and $\frac{c_\lambda}{z_2} \in h(S^{-1}C)$. Thus, for some $z_3 \in S$ we have $z_3 i_\alpha c_\lambda \in U \backslash P_e U$ which follows that either $i_\alpha \in Gr((U + J_{gr}(C) :_W C))$ or $z_3 c_\lambda \in U + J_{gr}(C)$ as U is a gr-P_e-J_{gr}-PrySub of C. Hence, either $\frac{i_\alpha}{z_1} \in Gr((S^{-1}U + J_{gr}(S^{-1}C) :_{S^{-1}W} S^{-1}C))$ or $\frac{c_\lambda}{z_2} = \frac{z_3 c_\lambda}{z_3 z_2} \in S^{-1}U + J_{gr}(S^{-1}C)$. So, as a result we get $S^{-1}U$ is a gr-$(S^{-1}P)_e$-J_{gr}-PrySub of $S^{-1}C$.

Theorem 2.7: *Let $P = \bigoplus_{g \in \Omega} P_g$ be a gr-Id of W and U a gr-P_e-J_{gr}-PrySub of C. If L is a gr-Sub of C with LÖU and $(U + J_{gr}(C))/L = U/L + J_{gr}(C/L)$, then U/L is a gr-P_e-J_{gr}-PrySub of C/L.*

Proof: Let $r_\alpha \in h(W)$ and $c_\lambda + L \in h(C/L)$ with $r_\alpha(c_\lambda + L) = r_\alpha c_\lambda + L \in (U/L) \backslash P_e(U/L)$. Hence, $r_\alpha c_\lambda \in U \backslash P_e U$ which follows that either $r_\alpha \in Gr((U + J_{gr}(C) :_W C))$ or $c_\lambda \in U + J_{gr}(C)$. So, either $r_\alpha \in Gr((U/L + J_{gr}(C/L) :_W C/L))$ or $c_\lambda + L \in U/L + J_{gr}(C/L)$. Thus, U/L is a gr-P_e-J_{gr}-PrySub of C/L.

Let $U = \bigoplus_{g \in \Omega} U_g$ be a gr-Sub of C and $\alpha \in \Omega$. Then U_α is said to be an α-J_{gr}-PrySub of the W_e-module C_α if $U_\alpha \neq C_\alpha$; and whenever $i_e c_\alpha \in C_\alpha$ where $i_e \in W_e$ and $c_\alpha \in C_\alpha$, then either $c_\alpha \in U_\alpha + J_{gr}(C_\alpha)$ or $i_e^n C_\alpha \subseteq U_\alpha + J_{gr}(C_\alpha)$ for some $n \in \mathbb{Z}^+$, [15].

Theorem 2.8: *Let $P = \bigoplus_{g \in \Omega} P_g$ be a gr-Id of W, $U = \bigoplus_{g \in \Omega} U_g$ a gr-Sub of C and $\alpha \in \Omega$. If U_α is an α-P_e-J_{gr}-PrySub of C_α, then either $(U_\alpha :_{W_e} C_\alpha) U_\alpha \subseteq P_e U_\alpha$ or U_α is an α-J_{gr}-PrySub of C_α.*

Proof: Assume that $(U_\alpha :_{W_e} C_\alpha) U_\alpha \not\subseteq P_e U_\alpha$ and let U_α be an α-P_e-J_{gr}-PrySub of C_α. Now, let $i_e c_\alpha \in U_\alpha$ for some $i_e \in W_e$ and $c_\alpha \in C_\alpha$. If $i_e c_\alpha \notin P_e U_\alpha$, then we get the result as U_α is an α-P_e-J_{gr}-PrySub of C_α. So, assume $i_e c_\alpha \in P_e U_\alpha$. Now, If $i_e U_\alpha \not\subseteq P_e U_\alpha$, then there exists $u_{1_\alpha} \in U_\alpha$ with $i_e u_{1_\alpha} \notin P_e U_\alpha$, so as

$c_\alpha + u_{1_\alpha} \in C_\alpha$ with $i_e(c_\alpha + u_{1_\alpha}) \in U_\alpha \backslash P_e U_\alpha$ we get either $i_e \in Gr((U_\alpha + J_{gr}(C_\alpha) :_{W_e} C_\alpha))$ or $c_\alpha + u_{1_\alpha} \in U_\alpha + J_{gr}(C_\alpha)$. Hence, either $i_e \in Gr((U_\alpha + J_{gr}(C_\alpha) :_{W_e} C_\alpha))$ or $c_\alpha \in U_\alpha + J_{gr}(C_\alpha)$. Also, if $(U_\alpha :_{W_e} C_\alpha) c_\alpha \not\subseteq P_e U_\alpha$, then there exists $r_{1_e} \in (U_\alpha :_{W_e} C_\alpha)$ with $r_{1_e} c_\alpha \notin P_e U_\alpha$ and then as $r_{1_e} + i_e \in W_e$ we get $(r_{1_e} + i_e) c_\alpha \in U_\alpha \backslash P_e U_\alpha$ implies either $r_{1_e} + i_e \in Gr((U_\alpha + J_{gr}(C_\alpha) :_{W_e} C_\alpha))$ or $c_\alpha \in U_\alpha + J_{gr}(C_\alpha)$. Thus, either $i_e \in Gr((U_\alpha + J_{gr}(C_\alpha) :_{W_e} C_\alpha))$ or $c_\alpha \in U_\alpha + J_{gr}(C_\alpha)$. So assume that $i_e U_\alpha \subseteq P_e U_\alpha$ and $(U_\alpha :_{W_e} C_\alpha) c_\alpha \subseteq P_e U_\alpha$. Now, by our assumption there are $r_{2_e} \in (U_\alpha :_{W_e} C_\alpha)$ and $u_{2_\alpha} \in U_\alpha$ with $r_{2_e} u_{2_\alpha} \notin P_e U_\alpha$, Hence, as $i_e + r_{2_e} \in W_e$ and $c_\alpha + u_{2_\alpha} \in C_\alpha$ we get $(i_e + r_{2_e})(c_\alpha + u_{2_\alpha}) \in U_\alpha \backslash P_e U_\alpha$. This yields that either $i_e + r_{2_e} \in Gr((U_\alpha + J_{gr}(C_\alpha) :_{W_e} C_\alpha))$ or $c_\alpha + u_{2_\alpha} \in U_\alpha + J_{gr}(C_\alpha)$ and then either $i_e \in Gr((U_\alpha + J_{gr}(C_\alpha) :_{W_e} C_\alpha))$ or $c_\alpha \in U_\alpha + J_{gr}(C_\alpha)$. Hence, as a result U_α is an α-J_{gr}-PrySub of C_α.

Corollary 2.9: *Let $U = \bigoplus_{g \in \Omega} U_g$ be a gr-Sub of C and $\alpha \in \Omega$. If U_α is an α - 0_e - J_{gr} -PrySub of C_α which is not an α - J_{gr} -PrySub of C_α, then $(U_\alpha :_{W_e} C_\alpha) U_\alpha = \{0\}$.*

Proof: Take $P = \{0\}$ in Theorem 2.8.

A graded W-module C is referred to as a graded local (gr-local) W-module if it contains precisely one gr-MSub (see [16]). A graded W-module C is described as graded semisimple (gr-semisimple) if every graded submodule of C can be expressed as a direct summand of C, [16].

Theorem 2.10: *Let $P = \bigoplus_{g \in \Omega} P_g$ be a gr-Id of W and U a gr-P_e-J_{gr}-PrySub of C.*

(i) *If $J_{gr}(C) \subseteq U$, then U is a gr-P_e-PrySub of C.*
(ii) *If C is a gr-local module and U is a gr-MSub of C, then U is a gr-P_e-PrySub of C.*
(iii) *If C is a gr-semiprime module, then U is a gr-P_e-PrySub of C.*

Proof:

(i) Straightforward.
(ii) The proof comes from $J_{gr}(C) = U$ as C is a gr-local W-module and U is a gr-MSub of C.
(iii) C is a gr-semisimple follows that $J_{gr}(C) = 0$. Hence, we get the result.

Theorem 2.11: *Let W_j be a Ω-graded ring, C_j a graded W_j-module and $P_j = \bigoplus_{g \in \Omega} P_{j_g}$ a gr-Id of W_j, for $j = 1,2$. Let $P = P_1 \times P_2$.*

(i) *If U_1 is a gr-P_{1_e}-J_{gr}-PrySub of C_1 with $P_{1_e} U_1 \times C_2 \subseteq P_e(U_1 \times C_2)$ and $J_{gr}(C_1) \times C_2 \subseteq J_{gr}(C_1 \times C_2)$, then $U_1 \times C_2$ is a gr- P_e - J_{gr} -PrySub of $C_1 \times C_2$.*

(ii) *If U_2 is a gr-P_{2_e}-J_{gr}-PrySub of C_2 with $C_1 \times P_{2_e}U_2 \subseteq P_e(C_1 \times U_2)$ and $C_1 \times J_{gr}(C_2) \subseteq J_{gr}(C_1 \times C_2)$, then $C_1 \times U_2$ is a gr-P_e-J_{gr}-PrySub of $C_1 \times C_2$.*

Proof:

(i) Let $(i_{1_\alpha}, i_{2_\alpha}) \in h(W_1 \times W_2)$ and $(c_{1_\lambda}, c_{2_\lambda}) \in h(C_1 \times C_2)$ such that $(i_{1_\alpha}, i_{2_\alpha})(c_{1_\lambda}, c_{2_\lambda}) = (i_{1_\alpha}c_{1_\lambda}, i_{2_\alpha}c_{2_\lambda}) \in (U_1 \times C_2)\backslash P_e(U_1 \times C_2)$. So, $(i_{1_\alpha}c_{1_\lambda}, i_{2_\alpha}c_{2_\lambda}) \in (U_1 \times C_2)\backslash(P_{1_e}U_1 \times C_2)$ and then $i_{1_\alpha}c_{1_\lambda} \in U_1\backslash P_{1_e}U_1$ which yields that either $c_{1_\lambda} \in U_1 + J_{gr}(C_1)$ or $i_{1_\alpha}^n C_1 \subseteq U_1 + J_{gr}(C_1)$ for some $n \in \mathbb{Z}^+$. Then as $J_{gr}(C_1) \times C_2 \subseteq J_{gr}(C_1 \times C_2)$ we conclude either $(c_{1_\lambda}, c_{2_\lambda}) \in U_1 \times C_2 + J_{gr}(C_1 \times C_2)$ or $(i_{1_\alpha}^n, i_{2_\alpha}^n)(C_1 \times C_2) \subseteq U_1 \times C_2 + J_{gr}(C_1 \times C_2)$ for some $n \in \mathbb{Z}^+$. Therefore, we get the result.

(ii) The proof is similar to that in (i).

Theorem 2.12: *Let W_j be a Ω-graded ring, C_j a graded W_j-module, $P_j = \bigoplus_{g\in\Omega} P_{j_g}$ a gr-Id of W_j and U_j a proper gr-Sub of C_j, for $j = 1,2$. Let $P = P_1 \times P_2$.*

(i) *If $P_{j_e}U_j = U_j$ for $j = 1,2$, then $U_1 \times U_2$ is a gr-P_e-J_{gr}-PrySub of $C_1 \times C_2$.*

(ii) *If U_1 is a gr-PrySub of C_1, then $U_1 \times C_2$ is a gr-P_e-J_{gr}-PrySub of $C_1 \times C_2$.*

(iii) *If U_1 is a gr-P_{1_e}-J_{gr}-PrySub of C_1 with $J_{gr}(C_1) \times C_2 \subseteq J_{gr}(C_1 \times C_2)$ and $P_{2_e}C_2 = C_2$, then $U_1 \times C_2$ is a gr-P_e-J_{gr}-PrySub of $C_1 \times C_2$.*

(iv) *If U_2 is a gr-PrySub of C_2, then $C_1 \times U_2$ is a gr-P_e-J_{gr}-PrySub of $C_1 \times C_2$.*

(v) *If U_2 is a gr-P_{2_e}-J_{gr}-PrySub of C_2 with $C_1 \times J_{gr}(C_2) \subseteq J_{gr}(C_1 \times C_2)$ and $P_{1_e}C_1 = C_1$, then $C_1 \times U_2$ is a gr-P_e-J_{gr}-PrySub of $C_1 \times C_2$.*

Proof:

(i) It is clear that $(U_1 \times U_2)\backslash P_e(U_1 \times U_2) = \emptyset$, so we get the result.

(ii) We have $U_1 \times C_2$ is a gr-PrySub of $C_1 \times C_2$ as U_1 is a gr-PrySub of C_1. So, $U_1 \times C_2$ is a gr-P_e-J_{gr}-PrySub of $C_1 \times C_2$.

(iii) Since $P_{2_e}C_2 = C_2$, $P_{1_e}U_1 \times C_2 \subseteq P_e(U_1 \times C_2)$. The result comes from Theorem 2.11.

(iv) The proof follows a similar approach to the one used in part (ii).

(v) The proof follows a similar approach to the one used in part (iii).

Theorem 2.13: *Let $P = \bigoplus_{g\in\Omega} P_g$ be a gr-Id of W and U a gr-Sub of C. Then U is a gr-P_e-J_{gr}-PrySub of C if and only if U/P_eU is a gr-0_e-J_{gr}-PrySub of C/P_eU.*

Proof: Let $i_\alpha \in h(W)$ and $(c_\lambda + P_eU) \in h(C/P_eU)$ such that $i_\alpha(c_\lambda + P_eU) = i_\alpha c_\lambda + P_eU \in (U/P_eU)\backslash 0_e(U/P_eU)$. Thus, $i_\alpha c_\lambda \in U\backslash P_eU$ which follows that either $c_\lambda \in U + J_{gr}(C)$ or $i_\alpha^n \in (U + J_{gr}(C):_W C)$ for some $n \in \mathbb{Z}^+$. So, either

$c_\lambda + P_eU \in U/P_eU + J_{gr}(C/P_eU)$ or $i_\alpha^n \in (U/P_eU + J_{gr}(C/P_eU) :_W C/P_eU)$ for some $n \in \mathbb{Z}^+$. Hence, we get the result. Conversely, let $i_\alpha \in h(W)$ and $c_\lambda \in h(C)$ with $i_\alpha c_\lambda \in U \backslash P_eU$. So, $i_\alpha c_\lambda + P_eU \in (U/P_eU) \backslash 0_e(U/P_eU)$ which gives either $c_\lambda + P_eU \in U/P_eU + J_{gr}(C/P_eU)$ or $i_\alpha^n \in (U/P_eU + J_{gr}(C/P_eU) :_W C/P_eU)$ for some $n \in \mathbb{Z}^+$. Then we have either $c_\lambda \in U + J_{gr}(C)$ or $i_\alpha^n \in (U + J_{gr}(C) :_W C)$ for some $n \in \mathbb{Z}^+$. Therefore, U is a gr-P_e-J_{gr}-PrySub of C.

References

[1] I. Kolar, P. W. Michor, J. Slovak, "Natural operations in differential geometry," Springer science and business media (2013).

[2] P. Deligne, "Quantum fields and strings: A course for mathematicians," American Mathematical Society (1999).

[3] A. Rogers, "Supermanifolds: Theory and applications," World Scientific (2007).

[4] M. Refai and K. Al-Zoubi, "On graded primary ideals," *Turk. J. Math.*, vol. 28, no. 3, pp. 217-229 (2004).

[5] S. E. Atani and F. Farzalipour, "On graded secondary modules," *Turk. J. Math.*, vol. 31, no. 4, pp. 371-378 (2007).

[6] K. H. Oral, U. Tekir, and A. G. Agargun, "On graded prime and primary submodules," *Turk. J. Math.*, vol. 35, no. 2, pp. 159-167 (2011).

[7] K. Al-Zoubi, "The graded primary radical of a graded submodule," An. Ştiinţ. Univ. Al. I. Cuza Iaşi Mat. (N.S.), vol. 1, no. 2, pp. 395-402 (2016).

[8] S. E. Atani and U. Tekir, "On the graded primary avoidance theorem," *Chiang Mai J. Sci.*, vol. 34, no. 2, pp. 161-164 (2007).

[9] E. Y. Celikel, "On graded 2-absorbing primary submodules," *Int. J. Pure Appl. Math.*, vol. 109, no. 4, pp. 869-879 (2016).

[10] M. Alkhatib and K. Al-Zoubi, "On graded S-2-absorbing primary submodules," *Journal of Discrete Mathematical Sciences and Cryptography*, vol 28, no.3, pp. 641-652 (2025).

[11] K. Al-Zoubi and M. Al-Azaizeh, "On Graded Weakly 2-Absorbing Primary Submodules," *Vietnam J. Math.*, vol. 47, no. 2, pp. 297-307 (2019).

[12] K. Al-Zoubi and M. Al-Dolat, "On graded classical primary submodules," *Adv. Pure Appl. Math.*, vol. 7, no. 2, pp. 93-96 (2016).

[13] K. Al-Zoubi and F. AL-Turman, "On Graded weakly classical primary submodules," *Proc. Jangjeon Math. Soc.,* vol. 21, no. 3, pp. 405-412 (2018).

[14] S. Alghueiri and K. Al-Zoubi, "On graded I_e-primary submodules of graded modules over graded commutative rings," *Sao Paulo J. Math. Sci.*, vol. 16, pp. 862-870 (2022).

[15] M. Alnimer and K. Al-Zoubi, "On graded J_{gr}-primary submodules," submitted.

[16] C. Nastasescu, F. Van Oystaeyen, "Methods of Graded Rings," LNM, 1836, Berlin-Heidelberg: Springer-Verlag (2004).

[17] S.E. Atani, "On graded prime submodules," *Chiang Mai. J. Sci.*, vol. 33, no. 1, pp. 3-7 (2006).

[18] R. Hazrat, "Graded Rings and Graded Grothendieck Groups," Cambridge University Press, Cambridge (2016).

[19] M. Al-Azaizeh and K. Al-Zoubi, "On graded weakly quasi n-absorbing submodule", *Journal of Interdisciplinary Mathematics,* Vol. 28 , no. 7, pp. 2399-2412, (2025).

[20] I. A. Athab and H. Q. Hamdi, "J-primary submodules," *Journal of Discrete Mathematical Sciences and Cryptography*, vol 24, no. 7, pp. 1915-1922 (2021).

[21] S. Alghueiri and K. Al-Zoubi, "Graded L_e-J_{gr}-Prime Submodules in Commutative Graded Rings," submitted.

[22] W.H. Hanoon and A. A. Abboodi, "Large-maximal small submodules", *Journal of Interdisciplinary Mathematics*, Vol. 26, No. 6, pp. 1043-1052 (2023).

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