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Embedding of complete bipartite graphs into some special classes of chordal graphs

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Abstract

The ability of one architecture to simulate another is the basis for comparing networks. Embedding is used to analyze these simulations. In architectural simulation, graph embedding is considered one of the most powerful applications for executing parallel algorithms and simulating various interconnection networks. In this work, we embed complete bipartite graphs into some special classes of chordal graphs and obtain its wirelength.

Subject Classification (2020): *05C05, 05C60, 05C76, 05C69.*

Keywords: *Embedding, Congestion, Wirelength, Complete bipartite graph, Line graph, Complete m-ary tree, Chordal graph.*

1. Introduction

Recent advancements in integrated circuits have enabled the construction of interconnection networks. Along with these developments, various interconnection topologies have been studied and analyzed [1]. Interconnection networks are frequently represented as graphs, with graph embedding being a crucial factor in evaluating these networks. Graph embedding is especially important for applications related to parallel computing, such as data structures and network design. There are several cost factors that may be used to evaluate embedding, the main one being wirelength. Other important metrics for measuring embedding performance include dilation, edge congestion, load, and expansion [2].

Dilation is the shortest distance in the host graph between any two vertices that correspond to neighboring vertices in the guest graph X. Edge congestion refers to the maximum number of lines from the graph X that are mapped to one line in the host graph. In a number of fields, including data structures, biological models that use parallel architectures, and VLSI design [3], wirelength is crucial [2].

Complete 2-partite graphs are a natural generalization of fully linked networks, which correspond to complete graphs. The vertex set of the graph is divided into two independent sets, with edges connecting all possible pairs of vertices between these two sets. Numerous research works have examined embedding parameters for complete multipartite graphs. K_{n_1,n_2} represents the complete bi-partite graph if the parts have cardinalities of n_1 and n_2 [4].

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A tree is defined as a connected graph that contains no cycles and has one designated root vertex. In this graph, every node apart from the root has exactly one parent and may have zero or more children. An m-ary tree is a type of tree where each point has at most m children. A complete m-ary tree, denoted as CT_l^m, is a tree where all levels are entirely filled except the last, which is filled sequentially from left to right [5].

The line graph of a tree, $L(T)$, is constructed by representing each tree edge as a vertex in $L(T)$. Two nodes in $L(T)$ are connected if and only if the edges they represent share a common node in T. A significant property of the line graph of a tree is that it is always a chordal graph [9]. A chordal graph is a graph in which every cycle of length four or more has a chord. In other words, chordal graphs do not have any induced cycles longer than three vertices. This characteristic of the line graph of a tree arises from the tree's acyclic and hierarchical structure, which naturally prevents the formation of long cycles in the line graph [6].

Studies on graph embedding have been done for complete multipartite graphs into circulant, wheel, path, cycle, hypertree, cylinder, and torus [7]; and certain trees [8] and Cartesian product of paths and cycles [10]; Complete bipartite graph into grid and cylinder [4]. Knödel graphs into a snake, circular necklace, necklace, fan, and windmill [2]; Turan graphs into incomplete hypercubes [11]; Hypercubes into grid [13], and cylinder [16]; Hypercubes into snake, necklace, and windmill [14]; Circulant graph into wheels [15]; and Hypercube into fractal cubic network [17]; Augmented cube into windmill graphs and certain trees [18]; Tur´an graphs into complete binary trees [19].

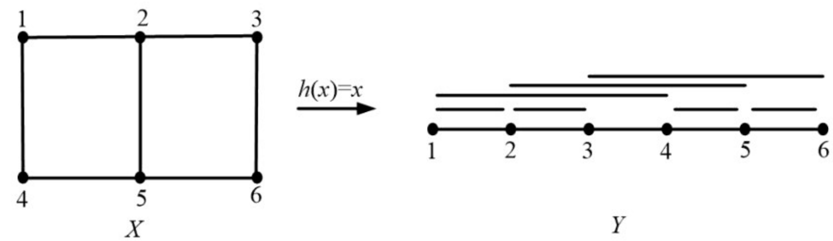

Figure 1
An embedding h from X into Y with $dil_h(X, Y) = 3$, $EC_h(X, Y) = 3$, and $WL_h(X, Y) = 13$

2. Basic concepts

The essential definitions and lemmas are listed below.

Definition 2.1: [2] An embedding of a guest graph X into the host graph Y is the pair (h, P_h) described as:

- h is an injective function from $V(X)$ to $V(Y)$.
- P_h is a injective mapping from $E(X)$ to $P_h(e)$, where $P_h(e)$ represents a path in Y connecting $h(u)$ and $h(v)$ for $e = (uv) \in E(X)$.

For brevity, we denote the pair (h, P_h) as h.

Definition 2.2: [2] Let $h: X \to Y$ be an embedding. If $e = (uv) \in E(X)$, the dilation of e, denoted by $\text{dil}_h(e)$, is the distance of the path $P_h(e)$ in Y. Thus,

$$\text{dil}_h(X, Y) = \max_{e \in E(X)} \text{dil}_h(e),$$
$$\text{dil}(X, Y) = \min_{h: X \to Y} \text{dil}_h(X, Y).$$

Definition 2.3: [2] Let $h: X \to Y$ be an embedding. The edge congestion of a line $e \in E(Y)$, represented as $EC_h(e)$, is defined as the number of lines $(uv) \in E(X)$ for which the path $P_h(uv)$, connecting $h(u)$ and $h(v)$ in Y, passes through the edge e. Formally: $EC_h(e) = |(uv) \in E(X): e \in P_h(uv)|$. Then,

$$EC_h(X, Y) = \max_{e \in E(Y)} EC_h(e),$$
$$EC(X, Y) = \min_{h: X \to Y} EC_h(X, Y).$$

Definition 2.4: [2] Consider an embedding h from X into Y. The wirelength of this embedding is described as

$$WL(X, Y) = \min_{h: X \to Y} \sum_{e \in E(X)} dil_h(e) = \min_{h: X \to Y} \sum_{e \in E(Y)} EC_h(e).$$

If Z is any subset of $E(Y)$, then $EC_h(Z) = \sum_{e \in Z} EC_h(e)$.

Fig. 1 illustrates the concepts of dilation, congestion, and wirelength in an embedding $h: X \to Y$. It is evident that in an embedding of the graph X into Y, the sum of the dilations is equal to the sum of the congestions which is the wirelength. Here, the wirelength is derived by summing the congestion values.

Let $S \subseteq V(X)$, then

- $I_X(S) = \{(uv) \in E | u, v \in S\}, I_X(b) = \max_{S \subseteq V, |S| = b} |I_X(S)|;$
- $\theta_X(S) = \{(uv) \in E | u \in S, v \not\in S\}, \theta_X(b) = \min_{S \subseteq V, |S| = b} |\theta_X(S)|.$

For $b = 1, 2, \dots, n$, the task of identifying a subset $S \subseteq V(X)$ with $|S| = b$ and $|I_X(S)| = I_X(b)$ is referred to as the maximum subgraph problem. Such subsets are called optimal [7]. Similarly, for $b = 1, 2, \dots, n$, the problem of finding a subset $S \subseteq X$ with $|S| = b$ and $|\theta_X(S)| = \theta_X(b)$ is known as the minimum cut problem [2].

In a regular graph, the functions I_X and θ_X are closely related such that solving one provides the solution to the other. Specifically, for any regular graph with regularity r, $2I_X(b) + \theta_X(b) = rb$, $b = 1, 2, \dots, n$. For complete p-partite graphs the maximum subgraph problem was determined in [7] and is stated as follows:

Theorem 2.5: *Let X represent a complete β-partite graph $K_{\alpha,\alpha,\dots,\alpha}$ with $n = \beta\alpha$ vertices, where $\beta \geq 2$ and $\alpha \geq 2$. The maximum number of lines in a subgraph of $X$with $'a'$ nodes is given by:*

$$I_X(a) = \begin{cases} \frac{a(a-1)}{2} & \text{if } a \le \beta - 1; \\ \frac{q^2\beta(\beta-1)}{2} & \text{if } a = q\beta, 1 \le q \le r; \\ \frac{(q-1)^2\beta(\beta-1)}{2} + i(q-1)(\beta-1) + \frac{i(i-1)}{2} & \text{if } a = (q-1)\beta + i, \\ & 1 \le i \le \beta - 1, 2 \le q \le r \end{cases}$$

For our study, we consider $\beta = 2$ in Theorem 2.5.

Definition 2.6: [12] The *line graph* $L(X)$ of a graph X is discribed as follows:

- Each point of $L(X)$ corresponds to an line of X.
- Two points in $L(X)$ are adjacent if and only if the lines they represent in X share a common point.

Definition 2.7: [20] Let X be a graph and $S \subseteq E(X)$ be an edge cut if $X \setminus S$ disconnects the graph X into more than one connected components.

Definition 2.8: [2] Let X be a graph and $S \subseteq E(X)$, $E(X) \setminus S$ disconnect the graph into more than one component, say $C_1, C_2, \ldots, C_l$, where $l \le |V(X)|$. A component C_i, for $1 \le i \le l$, is called a convex component if all shortest paths between every pair of points in C_i must lie within the same component C_i.

Definition 2.9: [5] The complete m-ary tree of level l, denoted by CT_l^m, is a tree in which each internal point has exactly m children, and the tree has a level l. The root is located at level 0, and all leaf nodes are at level l. Thus CT_l^m has exactly $\frac{m^{l+1}-1}{m-1}$ vertices and $\frac{m^{l+1}-m}{m-1}$ edges.

For illustration, refer to the complete 5-ary tree of level 2 shown in Fig. 2.

Definition 2.10: [6] A cycle C of length greater than 3 is said to be chordless cycle if there is no chord in C. A graph X is said to be chordal if there is no chordless cycle of length greater than 3.

Definition 2.11: [20] Let $X = (V, E)$ be a graph, and let $S \subseteq V$. The induced subgraph of X by S, denoted $X[S]$, is the subgraph with node set S and edge set consisting of all lines in X whose endpoints are both in S.

Lemma 2.12: [13] *Let X be an r-regular graph, and let h be an embedding of a guest graph into the host graph. Suppose S is an edge cut in the host graph, such that removing the lines of S separates the host graph into two connected components, Y_1 and Y_2. Define $X_1 = X[h^{-1}(V(Y_1))]$ and $X_2 = X[h^{-1}(V(Y_2))]$. The edge cut S satisfies the following conditions:*

(i) *For every line (ab) that lies entirely within either X_1 or X_2, the path $P_h(ab)$ does not cross any edge in S.*

(ii) *For every line* (ab) *where* $a \in X_1$ *and* $b \in X_2$, *the path* $P_h(ab)$ *includes exactly one line from* S.

(iii) X_1 *is the maximum subgraph with* $k = |V(X_1)|$ *vertices.*

Under these conditions, the congestion on edge cut $EC_h(S)$ is minimized, and it holds that

$$EC_h(S) = r|V(X_1)| - 2|E(X_1)|.$$

Lemma 2.13: [11] *Let* h *be an embedding of* X *into* Y. *Assume that* $\{S_1, S_2, \dots, S_l\}$ *is a partition of* $[2E(Y)]$, *where* $[2E(Y)]$ *is the collection of all edge of* Y *repeated exactly twice and each subset* S_j *represents an edge cut of* Y *and satisfies the Lemma 2.12. Thus*

$$WL_h(X,Y) = \frac{1}{2}\left[\sum_{j=1}^{l} EC_h(S_j)\right].$$

3. Main results

In this section, we embed complete bipartite graphs into certain families of chordal graphs, which is obtained by taking the line graph of the complete m-ary tree of level l. The following result is needed to prove the main result.

Lemma 3.1: *For any* m *and* l, $m \geq 3$, $l \geq 2$ *and* l *is even, the number* $\frac{m^{l+1}-m}{2(m-1)}$ *is a natural number* $\mathbb{N}$.

Proof: For brevity we take $\frac{m^{l+1}-m}{2(m-1)} = p$. Since l is even, $l = 2k$, where k is a non-negative integer. Then $l + 1 = 2k + 1$. Now

$$\begin{aligned} p &= \frac{m^{2k+1}-m}{2(m-1)} = \frac{m(m^k-1)(m^k+1)}{2(m-1)} \\ &= \frac{m(m-1)(m^{k-1}+m^{k-2}+\dots+1)(m^k+1)}{2(m-1)} = \frac{m(m^k+1)(m^{k-1}+m^{k-2}+\dots+1)}{2}. \end{aligned}$$

If m is even, then there is nothing to prove. If m is odd and k is an integer, then m^k is odd, and hence $m^k + 1$ is divisible by 2. This proves our lemma.

Embedding Algorithm

Input: The complete bipartite graph $K_{p,p}$ and a line graph of complete m-ary tree of level l, $L(CT_l^m)$, where $p = \frac{m^{l+1}-m}{2(m-1)} \in \mathbb{N}$, a natural number, $m \geq 3$, $l \geq 2$, and l is even.

Algorithm: Let V_1, V_2 be a partition of the complete bipartite graph and label the points of V_i, $1 \leq i \leq 2$ as $2j + i$, $j = 0,1,\dots,\frac{m^{l+1}-3m+2}{2(m-1)}$. Label the vertices of $L(CT_l^m)$ as follows:

(i) Let $l: E(CT_l^m) \rightarrow \mathbb{N}$, label the edge e between the levels $(i-1, i)$, where $1 \leq i \leq l$, in CT_l^m as $\frac{m^i-m}{m-1}+1, \frac{m^i-m}{m-1}+2, \ldots, \frac{m^{i+1}-m}{m-1}$, from the left-most child to the right-most child, respectively.

(ii) Let $f: V(L(CT_l^m)) \rightarrow \mathbb{N}$, label the vertex $x = L(e) \in V(L(CT_l^m))$ by $f(x = L(e)) = l(e)$.

For illustration, the edge labelling of CT_2^5 and the vertex labelling of $L(CT_2^5)$ are given in Fig. 2 and 3.

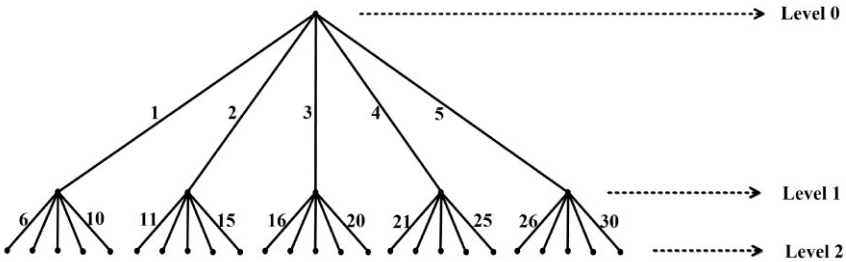

Figure 2
Labelling of 5-ary tree of level 2, CT_2^5

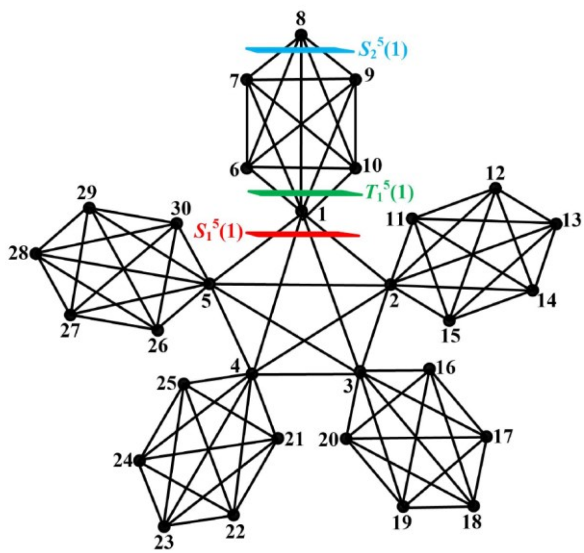

Figure 3
The edge cut of $L(CT_2^5)$

Output: An embedding h of $K_{p,p}$ into $L(CT_l^m)$, defined as $h(x) = x$ with exact wirelength.

Proof of correctness: Let v be a node in level i of CT_l^m, $i = 1,2,\dots,l-1$. Let the subtree rooted at v_j be denoted by $ST_i(v_j)$, $i = 1,2,\dots,l-1$, $j = 1,2,\dots,m^i$. Let $S_i^m(j)$ be the edge cut in $L(CT_l^m)$ whose removal disconnects $L(CT_l^m)$ into two connected components, say $Y_i^m(j)$ and $\overline{Y_i^m(j)}$, where $Y_i^m(j)$ is isomorphic to $L(ST_i(v_j) \cup \{e\})$, e is an edge incident to v_j and $e \not\in E(ST_i(v_j))$. Let $X_i^m(j) = X[h^{-1}(V(Y_i^m(j)))]$ and $\overline{X_i^m(j)} = X[h^{-1}(V(\overline{Y_i^m(j)}))]$. Since $X_i^m(j)$ is an optimal set by the labelling of the complete bipartite graph. Also $S_i^m(j)$ satisfies the Lemma 2.12. Hence, $EC_h(S_i^m(j))$ achieves its minimum value and is given by

$$EC_h(S_i^m(j)) = \frac{m^{l+1}-m}{2(m-1)} \cdot \frac{m^{l-i+1}-1}{m-1} - 2\left(\left\lceil\frac{m^{l-i+1}-1}{2(m-1)}\right\rceil\left\lfloor\frac{m^{l-i+1}-1}{2(m-1)}\right\rfloor\right).$$

Let $T_i^m(j)$, $i = 1,2,\dots,l-1$, $j = 1,2,\dots,m^i$ be the edge cut in $L(CT_l^m)$ whose removal disconnects $L(CT_l^m)$ into two connected components, say $Y_i^m(j)$ and $\overline{Y_i^m(j)}$, where $Y_i^m(j)$ is isomorphic to $L(ST_i(v_j))$. Let $X_i^m(j) = X[h^{-1}(V(Y_i^m(j)))]$ and $\overline{X_i^m(j)} = X[h^{-1}(V(\overline{Y_i^m(j)}))]$. $X_i^m(j)$ is an optimal set by the labelling of the complete bipartite graph. Also $T_i^m(j)$ satisfies the Lemma 2.12. Hence, $EC_h(T_i^m(j))$ achieves its minimum value and is given by

$$EC_h(T_i^m(j)) = \frac{m^{l+1}-m}{2(m-1)} \cdot \frac{m^{l-i+1}-m}{(m-1)} - 2\left(\left\lceil\frac{m^{l-i+1}-m}{2(m-1)}\right\rceil\left\lfloor\frac{m^{l-i+1}-m}{2(m-1)}\right\rfloor\right).$$

Let $S_l^m(j)$, $j = 1,2,\dots,m^l$ be the edge cut in $L(CT_l^m)$ whose removal disconnects $L(CT_l^m)$ into two connected components, say $Y_l^m(j)$ and $\overline{Y_l^m(j)}$, where $Y_l^m(j)$ is isomorphic $L(e_j)$, e_j is the j^{th} pendent edge in CT_l^m. Let $X_l^m(j) = X[h^{-1}(V(Y_l^m(j)))]$ and $\overline{X_l^m(j)} = X[h^{-1}(V(\overline{Y_l^m(j)}))]$. Since $X_l^m(j)$ is an optimal set, each $S_l^m(j)$ satisfies the Lemma 2.12. Hence $EC_h(S_l^m(j))$ achieves its minimum value and is given by

$$EC_h(S_l^m(j)) = \frac{m^{l+1}-m}{2(m-1)}.$$

The edge set $S_i^m(j) \cup T_i^m(j) \cup S_l^m(j)$, $i = 1,2,\dots,l-1$, $j = 1,2,\dots,m^i$ constitutes all the edges of $L(CT_l^m)$ twice and is a partition of $[2E(L(CT_l^m))]$. Then by Lemma 2.13, the wirelength is minimum.

Theorem 3.2: *The minimum wirelength of embedding complete bipartite graph $K_{p,p}$ into the line graph of CT_l^m, $L(CT_l^m)$, where $p = \frac{m^{l+1}-m}{2(m-1)} \in \mathbb{N}$, a natural number, $m \geq 3$, $l \geq 2$, and l even is given by*

$$\begin{aligned} WL(K_{p,p}, L(CT_l^m)) = \frac{1}{2}\Bigg[\sum_{i=1}^{l} \Bigg[&\frac{m^i(m^{l+1}-m)(2\times m^{l-i+1}-m-1)}{2(m-1)^2} \\ &-2\times m^i \times \left(\left\lceil\frac{m^{l-i+1}-1}{2(m-1)}\right\rceil\left\lfloor\frac{m^{l-i+1}-1}{2(m-1)}\right\rfloor\right. \\ &\left.+\left\lceil\frac{m^{l-i+1}-m}{2(m-1)}\right\rceil\left\lfloor\frac{m^{l-i+1}-m}{2(m-1)}\right\rfloor\right)\Bigg]\Bigg]. \end{aligned}$$

Proof: Label the nodes of $K_{p,p}$ and $L(CT_l^m)$ using Embedding Algorithm. Then by Lemma 2.13, the wirelength is given by

$$WL(K_{p,p}, L(CT_l^m))$$

$$= \frac{1}{2}[\sum_{i=1}^{l-1}\sum_{j=1}^{m^i}\left[\frac{m^{l+1}-m}{2(m-1)}\cdot\frac{m^{l-i+1}-1}{m-1}-2\left(\left\lceil\frac{m^{l-i+1}-1}{2(m-1)}\right\rceil\left\lfloor\frac{m^{l-i+1}-1}{2(m-1)}\right\rfloor\right)\right]$$

$$+\sum_{i=1}^{l-1}\sum_{j=1}^{m^i}\left[\frac{m^{l+1}-m}{2(m-1)}\cdot\frac{m^{l-i+1}-m}{m-1}-2\left(\left\lceil\frac{m^{l-i+1}-m}{2(m-1)}\right\rceil\left\lfloor\frac{m^{l-i+1}-m}{2(m-1)}\right\rfloor\right)\right]$$

$$+\sum_{j=1}^{m^l}\frac{m^{l+1}-m}{2(m-1)}]$$

$$= \frac{1}{2}[\sum_{i=1}^{l-1}\left[\frac{m^i(m^{l+1}-m)(m^{l-i+1}-1)}{2(m-1)^2}-2\times m^i\left(\left\lceil\frac{m^{l-i+1}-1}{2(m-1)}\right\rceil\left\lfloor\frac{m^{l-i+1}-1}{2(m-1)}\right\rfloor\right)\right]$$

$$+\sum_{i=1}^{l-1}\left[\frac{m^i(m^{l+1}-m)(m^{l-i+1}-m)}{2(m-1)^2}-2\times m^i\left(\left\lceil\frac{m^{l-i+1}-m}{2(m-1)}\right\rceil\left\lfloor\frac{m^{l-i+1}-m}{2(m-1)}\right\rfloor\right)\right]$$

$$+\frac{m^l(m^{l+1}-m)}{2(m-1)}]$$

$$= \frac{1}{2}\left[\sum_{i=1}^{l}\left[\frac{m^i(m^{l+1}-m)(2\times m^{l-i+1}-m-1)}{2(m-1)^2}\right.\right.$$

$$\left.\left.-2\times m^i\times\left(\left\lceil\frac{m^{l-i+1}-1}{2(m-1)}\right\rceil\left\lfloor\frac{m^{l-i+1}-1}{2(m-1)}\right\rfloor+\left\lceil\frac{m^{l-i+1}-m}{2(m-1)}\right\rceil\left\lfloor\frac{m^{l-i+1}-m}{2(m-1)}\right\rfloor\right)\right]\right].$$

Remark: When m is even and l is odd, we will get the same result as in Theorem 3.2.

4. Implementation of line graphs in real-time applications

The use of line graphs in real-time applications highlights their ability to represent relationships between edges in an original graph as vertices in a derived graph.

In telecommunication networks, the lines of the graph represent physical connections between devices, while the line graph models adjacency or interference relationships between these connections. This method aids in optimizing routing protocols and mitigating channel conflicts by identifying critical connections with significant interactions.

In traffic flow analysis, intersections are represented as nodes and roads as edges, with the line graph providing insights into relationships between roads. It helps identify high-traffic or high-interaction segments, which are crucial for effective traffic management and urban planning.

In power grid systems, substations are modeled as nodes and power lines as edges in the original graph. The line graph captures dependencies between power lines, supporting fault detection and balanced load distribution in real-time operations.

This same concept of line graphs can be extended to the analysis of wirelength in circuit design. In this context, the edges represent the nets connecting components, while the line graph models the relationships between these nets based on shared components (pins). By calculating wirelength using the line graph, it becomes easier to optimize the layout by identifying critical nets with high interaction, thus reducing wirelength, minimizing signal delays, and improving overall circuit performance. Just as in telecommunications, traffic flow, and power grids, leveraging the line graph structure in wirelength analysis aids in streamlining design optimization processes by highlighting essential connections and reducing inefficiencies in the system.

5. Conclusion and open problem

In this paper, we have determined the minimum wirelength for embedding complete bipartite graphs into the line graph of a complete m-ary tree with l levels, for any m and even l; m is even and l is odd. Our results contribute to the broader understanding of graph embedding, offering precise algorithms and insights that enhance the structural analysis of interconnection networks.

The problem of determining the wirelength for embedding complete bipartite graphs into the line graph of CT_l^m remains unresolved when both m and l are odd. Moreover, embedding the architecture discussed in this paper onto complete multipartite graphs presents an intriguing and challenging opportunity for researchers.

Open Problem:

Determining the wirelength of an embedding from a complete multipartite graph into cube-like architectures, interval graphs, chordal graph and the line graph of tree-like architectures.

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