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Core of almost distributive lattice

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Abstract

The purpose of this article is to establish a sub algebra of an almost distributive lattice $\mathfrak{R}$ by defining the Core of $\mathfrak{R}$ notated by $\mathfrak{I}_\pi$ for any $\pi \in \mathfrak{R}$. The features of $\mathfrak{I}_\pi$ are investigated. Some properties of R is characterised according to the structure of J_π. Moreover, the sets $\overline{\mathfrak{I}}_\mathfrak{R}$, $\underline{\mathfrak{I}}_\mathfrak{R}$ are introduced and $\underline{\mathfrak{I}}_\mathfrak{R}$ is proved to be a distributive lattice with 0. The relation between the set $\overline{\mathfrak{I}}_\mathfrak{R}$ and Birkhoff centre of $\mathfrak{R}$ is demonstrated.

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1. Introduction

In 1980, the launch of the almost distributive lattice was made by G. C. Rao and U. M. Swamy to generalise the notion of lattices and to imitate sorts of rings such as regular, biregular, associate rings and triple system. The proposal of G. C. Rao and U. M. Swamy contained the definition of almost distributive lattices abbreviated as ADL. The concept was attached with many properties that extend the characteristics of lattices. Further, several aspects of almost distributive were studied by many authors. See [6] and [8], [9].

A lattice is defined to be a nonempty set gathered with two the binary operations join and meet which are associative and commutative and satisfy the absorption identities. The genesis of lattice was established by chales S. Pierice and Ernst Schroder. In the mid 1930s, Birkhoff's developed the general lattice

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theory [5] . Nowadays, lattice theory extended into fuzzy, algebra and logic. See [1], [3] and [4].

In this proof, we investigate the utility of constructing a lattice from ADL by introducing elements with the property of commutativity. For an ADL $\mathfrak{R}$ and an element $\pi \in \mathfrak{R}$, we outline the Core of $\mathfrak{R}$ notated by $\mathfrak{I}_\pi$. Several properties of $\mathfrak{I}_\pi$ are illustrated with many examples. Also, the distributivity of $\mathfrak{R}$ is tested by the concept of $\mathfrak{I}_\pi$.

Furthermore, the sets $\overline{\mathfrak{I}}_{\mathfrak{R}}, \underline{\mathfrak{I}}_{\mathfrak{R}}$ are defined to be the union and the intersection of $\mathfrak{I}_\pi$ for any $\pi \in \mathfrak{R}$. The scoop of this article is to study the lattice parts of $\mathfrak{R}$ which turns out to be $\underline{\mathfrak{I}}_{\mathfrak{R}}$ and those $\pi \in \mathfrak{R}$ satisfying $\mathfrak{I}_{\mathfrak{I}_\pi} = \mathfrak{I}_\pi$. Moreover, we illustrate the relation between the set $\overline{\mathfrak{I}}_{\mathfrak{R}}$ and Birkhoff centre of $\mathfrak{R}$ which is a kind of generation.

2. Preliminaries

For the sake of consistency, we quote previous definitions and results which illustrate many concepts in the paper. A Lattice is an algebra $(\mathfrak{R};^\wedge,^\vee)$ satisfying that the idempotent, commutativity, associativity and the absorption laws for both operation. A lattice $\mathfrak{R}$ is said to be distributive if

$$\mathfrak{s} \vee (\mathfrak{p} \wedge \mathfrak{t}) = (\mathfrak{s} \vee \mathfrak{p}) \wedge (\mathfrak{s} \vee \mathfrak{t}),$$
$$\mathfrak{s} \wedge (\mathfrak{p} \vee \mathfrak{t}) = (\mathfrak{s} \wedge \mathfrak{p}) \vee (\mathfrak{s} \wedge \mathfrak{t}),$$

for $\mathfrak{s}, \mathfrak{p}, \mathfrak{t} \in \mathfrak{R}$.

In [11], the authors characterised the algebra $(\mathcal{R},^\wedge,^\vee, 0)$ of type $(2,2,0)$ as an almost distributive lattice if the operations satisfy the given axioms for any $\mathfrak{p}, \mathfrak{t}, \mathfrak{s}\ in\ \mathcal{R}$

$\mathfrak{s} \vee 0 = \mathfrak{s}$, $0 \wedge \mathfrak{s} = 0$, $(\mathfrak{s} \wedge \mathfrak{t}) \vee (\mathfrak{p} \wedge \mathfrak{t}) = (\mathfrak{s} \vee \mathfrak{p}) \wedge \mathfrak{t}$, $(\mathfrak{t} \wedge \mathfrak{s}) \vee (\mathfrak{t} \wedge \mathfrak{p}) = \mathfrak{t} \wedge (\mathfrak{s} \vee \mathfrak{p})$, $(\mathfrak{t} \vee \mathfrak{s}) \wedge (\mathfrak{t} \vee \mathfrak{p}) = \mathfrak{t} \vee (\mathfrak{s} \wedge \mathfrak{p})$, $(\mathfrak{t} \vee \mathfrak{s}) \wedge \mathfrak{s} = \mathfrak{s}$.

We use the abbreviation ADL for almost distributive lattice. In the following context we refer to $\mathfrak{R}$ to be almost distributive lattice unless otherwise stated. In the following lemma, we mention the absorption laws which are valid in ADLs.

Lemma 1 [11] *Assume that* $\mathfrak{t}, \mathfrak{s}, \mathfrak{p} \in \mathcal{R}$ *the following equalities are satisfied*

(i) $\mathfrak{s} \wedge \mathfrak{s} = \mathfrak{s} = \mathfrak{s} \vee \mathfrak{s}$,

(ii) $\mathfrak{s} \wedge 0 = 0$,

(iii) $0 \vee \mathfrak{s} = \mathfrak{s} = (\mathfrak{t} \wedge \mathfrak{s}) \vee \mathfrak{s}$,

(iv) $(\mathfrak{t} \vee \mathfrak{s}) \wedge \mathfrak{p} = (\mathfrak{s} \vee \mathfrak{t}) \wedge \mathfrak{p}$,

(v) $(\mathfrak{t} \wedge \mathfrak{s}) \wedge \mathfrak{p} = (\mathfrak{s} \wedge \mathfrak{t}) \wedge \mathfrak{p}$,

(vi) $\mathfrak{t} \wedge (\mathfrak{t} \vee \mathfrak{s}) = \mathfrak{t} = \mathfrak{t} \vee (\mathfrak{t} \wedge \mathfrak{s})$,

(vii) $(\mathfrak{t} \vee \mathfrak{s}) \wedge \mathfrak{t} = \mathfrak{t} = \mathfrak{t} \vee (\mathfrak{s} \wedge \mathfrak{t})$,

(vii) $(\mathfrak{s} \wedge \mathfrak{t}) \wedge \mathfrak{p} = \mathfrak{s} \wedge (\mathfrak{t} \wedge \mathfrak{p})$

(ix) $\mathfrak{s} \wedge \mathfrak{t} = \mathfrak{t}$ if and only if $\mathfrak{s} \vee \mathfrak{t} = \mathfrak{s}$,

(x) $\mathfrak{t} \wedge \mathfrak{s} = \mathfrak{t}$ if and only if $\mathfrak{t} \vee \mathfrak{s} = \mathfrak{s}$.

Definition 2.2: [11] Let $\mathfrak{s}, \mathfrak{t} \in \mathcal{R}$, $\mathfrak{s}$ is said to be less than or equal to $\mathfrak{t}$ in symbols we write $\mathfrak{s} \leq \mathfrak{t}$ provided that $\mathfrak{s} \wedge \mathfrak{t} = \mathfrak{s}$ or $\mathfrak{s} \vee \mathfrak{t} = \mathfrak{t}$.

From the previous definition we note that the relation $\leq$ constructs a partial order on $\mathcal{R}$.

Proposition 2.3: Let $\mathfrak{s}, \# \in \mathfrak{R}$, we have the following equalities;

(1) $\mathfrak{s} \wedge \# = (\mathfrak{s} \wedge \#) \vee (\# \wedge \mathfrak{s})$,

(2) $\mathfrak{s} \vee \# = (\# \vee \mathfrak{s}) \wedge (\mathfrak{s} \vee \#)$.

For more information see [2] and [7].

3. Core of Almost Distributive Lattice

For a fixed element $\pi \in \mathfrak{R}$, we define

$$\mathfrak{I}_\pi = \{\mathfrak{s} \in \mathfrak{R};\ \mathfrak{s} \vee \pi = \pi \vee \mathfrak{s}\}.$$

Obviously, $\mathfrak{I}_\pi$ is a non empty set since $\mathfrak{I}_\pi$ at least contains $0, \pi$.

Example 3.1:

(1) Consider a lattice $\mathcal{L}$, we have that $\mathfrak{I}_\pi = \mathcal{L}$ for any $\pi \in \mathcal{L}$.

(2) Set $\mathcal{R}$ to be an ADL, hence $\mathfrak{I}_0 = \mathcal{R}$.

(3) Consider $\mathcal{R} = \{0, \mathfrak{t}, \mathfrak{s}, \mathfrak{p}\}$ endowed with the following operations

$\vee$	0	$\mathfrak{t}$	$\mathfrak{s}$	$\mathfrak{p}$
0	0	$\mathfrak{t}$	$\mathfrak{s}$	$\mathfrak{p}$
$\mathfrak{t}$	$\mathfrak{t}$	$\mathfrak{t}$	$\mathfrak{t}$	$\mathfrak{t}$
$\mathfrak{s}$	$\mathfrak{s}$	$\mathfrak{s}$	$\mathfrak{s}$	$\mathfrak{s}$
$\mathfrak{p}$	$\mathfrak{p}$	$\mathfrak{t}$	$\mathfrak{s}$	$\mathfrak{p}$

$\wedge$	0	$\mathfrak{t}$	$\mathfrak{s}$	$\mathfrak{p}$
0	0	0	0	0
$\mathfrak{t}$	0	$\mathfrak{t}$	$\mathfrak{s}$	$\mathfrak{p}$
$\mathfrak{s}$	0	$\mathfrak{t}$	$\mathfrak{s}$	$\mathfrak{p}$
$\mathfrak{p}$	0	$\mathfrak{p}$	$\mathfrak{p}$	$\mathfrak{p}$

We see that $(\mathcal{R}; {}^\wedge, {}^\vee, 0)$ is an ADL. By simple calculation we get that $\mathfrak{I}_\mathfrak{p} = \mathcal{R}$ and $\mathfrak{I}_\mathfrak{t} = \{0, \mathfrak{t}, \mathfrak{p}\}$.

Lemma 3.2: *For $\pi, \mathfrak{t} \in \mathfrak{R}$ and $\mathfrak{s} \in \mathfrak{I}_\pi$, the following are satisfied*

(1) $\pi \vee \mathfrak{t},\ \mathfrak{t} \wedge \pi \in \mathfrak{I}_\pi$,

(2) $\mathfrak{s} \vee (\pi \vee \mathfrak{s}) = \pi \vee \mathfrak{s} = (\mathfrak{s} \vee \pi) \vee \mathfrak{s}$,

(3) $\mathfrak{s} \wedge (\pi \vee \mathfrak{s}) = \mathfrak{s} = (\mathfrak{s} \vee \pi) \wedge \mathfrak{s}$,

(4) $(\pi \wedge \mathfrak{s}) \vee \pi = \pi$,

(5) $(\pi \wedge \mathfrak{s}) \wedge \pi = \pi \wedge \mathfrak{s}$,

(6) $(\pi \wedge \mathfrak{s}) \vee \mathfrak{s} = \mathfrak{s} = (\mathfrak{s} \wedge \pi) \vee \mathfrak{s}$.

Theorem 3.3: *Let $\mathcal{R}$ be an ADL, then*

(i) $\mathfrak{s} \in \mathfrak{I}_\pi$ *if and only if* $\pi \in \mathfrak{I}_\mathfrak{s}$,

(ii) *If $\mathfrak{s}$ and π are comparable elements, then $\mathfrak{s} \in \mathfrak{I}_\pi$ and $\pi \in \mathfrak{I}_\mathfrak{s}$.*

(iii) *If $\pi_2 \leq \pi_1$, then $\mathfrak{I}_{\pi_1} \subseteq \mathfrak{I}_{\pi_2}$,*

(iv) *If $\mathfrak{s} \in \mathfrak{I}_\pi$, then there exist $\partial, þ \in \mathfrak{I}_\pi$ such that $\partial \leq \mathfrak{s} \leq þ$,*

(v) $\mathfrak{I}_{\pi_1} = \mathfrak{I}_{\pi_2}$ *if and only if* $\pi_1 \in \mathfrak{I}_{\mathfrak{I}_{\pi_2}}$ *and* $\pi_2 \in \mathfrak{I}_{\mathfrak{I}_{\pi_1}}$.

Proof:

(ii) Assume that $\mathfrak{s} \leq \pi$, we get $\mathfrak{s} \wedge \pi = \mathfrak{s}$. $\pi \wedge \mathfrak{s} = \pi \wedge (\mathfrak{s} \wedge \pi) = \mathfrak{s} \wedge \pi$. Hence we get the required.

(iii) Since $\pi_2 \leq \pi_1$, then $\pi_2 \wedge \pi_1 = \pi_1 \wedge \pi_2 = \pi_2$. Let $\mathfrak{s} \in \mathfrak{J}_{\pi_1}$, then

$$\begin{aligned}
\mathfrak{s} \wedge \pi_2 &= \mathfrak{s} \wedge ((\pi_2 \wedge \pi_1) \vee \pi_2) = (\mathfrak{s} \wedge (\pi_2 \wedge \pi_1)) \vee (\mathfrak{s} \wedge \pi_2) \\
&= (\pi_2 \wedge \mathfrak{s} \wedge \pi_1) \vee (\mathfrak{s} \wedge \pi_2) = (\pi_2 \wedge \pi_1 \wedge \mathfrak{s}) \vee (\mathfrak{s} \wedge \pi_2) \\
&= (\pi_2 \wedge \mathfrak{s}) \vee (\mathfrak{s} \wedge \pi_2) = \pi_2 \wedge \mathfrak{s}.
\end{aligned}$$

(iv) Assume that $\partial = \mathfrak{s} \wedge \pi$. We see $\partial \wedge \mathfrak{s} = (\partial \wedge \mathfrak{s}) \wedge \mathfrak{s} = \pi \wedge \mathfrak{s} = \partial$. Thus $\partial \leq \mathfrak{s}$. Similarly, by letting $\eth = \mathfrak{s} \vee \pi$.

(iv) If $\pi_1 \in \mathfrak{J}_{\mathfrak{J}_{\pi_2}}$, then $\pi_1 \wedge \mathfrak{s} = \mathfrak{s} \wedge \pi_1$ for every $\mathfrak{s} \in \mathfrak{J}_{\pi_2}$. Therefore $\mathfrak{J}_{\pi_2} \subseteq \mathfrak{J}_{\pi_1}$. By same manner, we get that $\mathfrak{J}_{\pi_1} \subseteq \mathfrak{J}_{\pi_2}$. Conversely, assume that $\mathfrak{J}_{\pi_1} = \mathfrak{J}_{\pi_2}$. Then every element commutes with π_1 commutes also with π_2 and vice versa. Hence we get the required.

Corollary 3.4: *For $\pi \in \mathfrak{R}$, $\mathfrak{J}_\pi$ is a lattice with 0 if one the following conditions is satisfied*

(i) *each pair of elements in $\mathfrak{J}_\pi$ are comparable elements.*

(ii) *for elements $\mathfrak{s}, \mathrm{t} \in \mathfrak{J}_\pi$ implies that $\mathfrak{s} \in \mathfrak{J}_{\mathrm{t}}$ or $\mathrm{t} \in \mathfrak{J}_{\mathfrak{s}}$.*

Proposition 3.5: For $\mathfrak{s}, \mathrm{t} \in \mathfrak{R}$. If $\mathfrak{s} \wedge \mathrm{t} \in \mathfrak{J}_{\mathrm{t} \wedge \mathfrak{s}}$ or $\mathfrak{s} \vee \mathrm{t} \in \mathfrak{J}_{\mathrm{t} \vee \mathfrak{s}}$ then $\mathfrak{R}$ is a distributive lattice with 0.

Proof: Since $\mathfrak{s} \wedge \mathrm{t} \in \mathfrak{J}_{\mathrm{t} \wedge \mathfrak{s}}$, then $(\mathfrak{s} \wedge \mathrm{t}) \vee (\mathrm{t} \wedge \mathfrak{s}) = (\mathrm{t} \wedge \mathfrak{s}) \vee (\mathfrak{s} \wedge \mathrm{t})$. From Proposition 2.3, we get that $\mathfrak{s} \wedge \mathrm{t} = \mathrm{t} \wedge \mathfrak{s}$. Hence $\mathfrak{R}$ is a distributive lattice with 0.

Theorem 3.6: *If $\pi \in \mathfrak{R}$, then $\mathfrak{s} \in \mathfrak{J}_\pi$ if and only if $\mathfrak{s} \wedge \pi = \pi \wedge \mathfrak{s}$.*

Proof: Let $\mathfrak{s} \in \mathfrak{J}_\pi$, then $\mathfrak{s} \vee \pi = \pi \vee \mathfrak{s}$. Hence $\pi \wedge \mathfrak{s} = (\pi \wedge \mathfrak{s}) \wedge \pi = \pi \wedge (\mathfrak{s} \wedge \pi) = \mathfrak{s} \wedge \pi$. On the other hand, let $\mathfrak{s} \wedge \pi = \pi \wedge \mathfrak{s}$. We have that $\pi \wedge (\mathfrak{s} \vee \pi) = (\pi \wedge \mathfrak{s}) \vee (\pi \wedge \pi) = (\mathfrak{s} \wedge \pi) \vee \pi = \pi$. Hence $\pi \vee \mathfrak{s} = \pi \vee (\mathfrak{s} \wedge (\mathfrak{s} \vee \pi)) = (\pi \vee \mathfrak{s}) \wedge (\pi \vee (\mathfrak{s} \vee \pi)) = (\pi \vee \mathfrak{s}) \wedge (\mathfrak{s} \vee \pi) = \mathfrak{s} \vee \pi$.

In the upcoming, we prove that $\mathfrak{J}_\pi$ is a sub ADL.

Corollary 3.7: *The set $\mathfrak{J}_\pi$ is closed under meet and joint.*

Proof: Suppose that $\mathfrak{s}, \mathrm{t} \in \mathfrak{J}_\pi$, we deduce the following

$$\begin{aligned}
(\mathfrak{s} \wedge \mathrm{t}) \wedge \pi &= \mathfrak{s} \wedge (\mathrm{t} \wedge \pi) = \mathfrak{s} \wedge (\pi \wedge \mathrm{t}) \\
&= (\mathfrak{s} \wedge \pi) \wedge \mathrm{t} = \pi \wedge (\mathfrak{s} \wedge \mathrm{t}). \\
(\mathfrak{s} \vee \mathrm{t}) \wedge \pi &= (\mathfrak{s} \wedge \pi) \vee (\mathrm{t} \wedge \pi) = (\pi \wedge \mathfrak{s}) \vee (\pi \wedge \mathrm{t}) \\
&= \pi \wedge (\mathfrak{s} \vee \mathrm{t}).
\end{aligned}$$

Hence $\mathfrak{s} \wedge \mathrm{t} \in \mathfrak{J}_\pi$ and $\mathfrak{s} \vee \mathrm{t} \in \mathfrak{J}_\pi$.

Corollary 3.8: *For* $\pi \in \mathfrak{R}$, *let* $\mathfrak{s}, \mathrm{t} \in \mathfrak{I}_{\pi}$. *Then* $(\mathfrak{s} \wedge \mathrm{t}) \vee \pi = (\mathfrak{s} \vee \pi) \wedge (\mathrm{t} \vee \pi)$.

Corollary 3.9: *If* $\mathfrak{I}_{\pi} = \mathfrak{R}$ *for every* $\pi \in \mathfrak{R}$, *then* $\mathfrak{R}$ *is a distributive lattice with* 0.

Theorem 3.10: *For* $\pi \in \mathfrak{R}$, $\mathfrak{I}_{\pi} = \{\mathfrak{s} \in \mathfrak{R};$ *there exist* ∂ *such that* $\partial \wedge \pi = \pi \wedge \partial$ *satisfying* $\mathfrak{s} \leq \partial\}$.

Proof: Let $\mathfrak{s} \in \mathfrak{I}_{\pi}$, then there exists $\partial = \mathfrak{s} \vee \pi = \pi \vee \mathfrak{s}$ such that $\mathfrak{s} \leq \partial$ and $\partial \in \mathfrak{I}_{\pi}$. Hence $\mathfrak{s} \in \{\mathfrak{s} \in \mathfrak{R};$ there exist ∂ such that $\partial \wedge \pi = \pi \wedge \partial$ satisfying $\mathfrak{s} \leq \partial\}$. Let $\mathfrak{s} \in \mathfrak{R}$ such that $\mathfrak{s} \leq \partial$ with $\partial \wedge \pi = \pi \wedge \partial$, then $\mathfrak{s} \wedge \partial = \mathfrak{s}$. Hence $(\mathfrak{s} \wedge \partial) \wedge \pi = (\mathfrak{s} \wedge \pi) \wedge \partial = (\pi \wedge \mathfrak{s}) \wedge \partial = \pi \wedge (\mathfrak{s} \wedge \partial) = \pi \wedge \mathfrak{s}$.

Proposition 3.11: Let $\mathrm{t} \in \mathfrak{R}$ and $\mathfrak{s} \in \mathfrak{I}_{\pi}$, then there exists $\Lambda \in \mathfrak{R}$ such that $\mathfrak{s} \wedge \mathrm{t} \in \mathfrak{I}_{\Lambda}$.

Proof: We intent to prove that $\mathfrak{s} \wedge \mathrm{t} \in \mathfrak{I}_{\pi \wedge \mathrm{t}}$.

$$\begin{aligned}(\mathfrak{s} \wedge \mathrm{t}) \vee (\pi \wedge \mathrm{t}) &= (\mathfrak{s} \vee \pi) \wedge \mathrm{t} = (\pi \vee \mathfrak{s}) \wedge \mathrm{t} \\ &= (\pi \wedge \mathrm{t}) \vee (\mathfrak{s} \wedge \mathrm{t}).\end{aligned}$$

We define the following notion

$$\mathfrak{I}_{\pi_1,\pi_2} = \{\mathfrak{s} \in \mathfrak{R};\ \mathfrak{s} \vee \pi_1 = \pi_1 \vee \mathfrak{s}, \mathfrak{s} \vee \pi_2 = \pi_2 \vee \mathfrak{s}\}$$

Lemma 3.12: *Let* $\pi_1, \pi_2 \in \mathfrak{R}$, *then*

(1) $\mathfrak{I}_{\pi_1,\pi_2} = \mathfrak{I}_{\pi_1} \cap \mathfrak{I}_{\pi_2}$,

(2) $\mathfrak{I}_{\pi_1,\pi_2} = \mathfrak{I}_{\pi_2,\pi_1}$,

(3) $\mathfrak{I}_{\pi_1,\pi_2} \subseteq \mathfrak{I}_{\pi_1 \wedge \pi_2}$.

Theorem 3.13: *For every* $\mathfrak{s}, \mathrm{t} \in \mathfrak{R}$. *If* $\mathrm{t} \wedge \mathfrak{s} \in \mathfrak{I}_{\mathrm{t}}$, *then* $\wedge$ *is commutative over* $\mathfrak{R}$.

Proof: Suppose that $\mathfrak{s}, \mathrm{t} \in \mathfrak{R}$, we have that $\mathrm{t} \wedge \mathfrak{s} \in \mathfrak{I}_{\mathfrak{s}} \cap \mathfrak{I}_{\mathrm{t}}$. Therefore $\mathrm{t} \wedge \mathfrak{s} \in \mathfrak{I}_{\mathfrak{s},\mathrm{t}} \subseteq \mathfrak{I}_{\mathfrak{s} \wedge \mathrm{t}}$. Then $(\mathrm{t} \wedge \mathfrak{s}) \wedge (\mathfrak{s} \wedge \mathrm{t}) = (\mathfrak{s} \wedge \mathrm{t}) \wedge (\mathrm{t} \wedge \mathfrak{s})$. We see that $(\mathfrak{s} \wedge \mathrm{t}) \wedge (\mathrm{t} \wedge \mathfrak{s}) = \mathfrak{s} \wedge (\mathrm{t} \wedge \mathrm{t}) \wedge \mathfrak{s} = \mathfrak{s} \wedge (\mathrm{t} \wedge \mathfrak{s}) = \mathrm{t} \wedge \mathfrak{s}$. Also, we have that $(\mathrm{t} \wedge \mathfrak{s}) \wedge (\mathfrak{s} \wedge \mathrm{t}) = \mathrm{t} \wedge (\mathfrak{s} \wedge \mathfrak{s}) \wedge \mathrm{t} = \mathrm{t} \wedge (\mathfrak{s} \wedge \mathrm{t}) = \mathfrak{s} \wedge \mathrm{t}$. Hence $\mathfrak{s} \wedge \mathrm{t} = \mathrm{t} \wedge \mathfrak{s}$.

By a similar manner, we deduce the upcoming theorem;

Theorem 3.14: *Consider* $\pi, \mathfrak{s} \in \mathfrak{R}$. *Suppose that* $\mathfrak{s} \vee \pi \in \mathfrak{I}_{\pi}$, *then* $\vee$ *is commutative over* $\mathfrak{R}$.

Theorem 3.15: *For* $\mathfrak{s}, \mathrm{t} \in \mathfrak{R}$, *we have the following;*

(1) $\mathfrak{s} \wedge \mathrm{t} \in \mathfrak{I}_{\mathfrak{s} \vee \mathrm{t}}$ *if and only if* $\mathfrak{s} \vee \mathrm{t} \in \mathfrak{I}_{\mathfrak{s} \wedge \mathrm{t}}$,

(2) $\mathfrak{s} \wedge \mathrm{t} = \mathrm{t} \wedge \mathfrak{s}$ *if and only if* $\mathfrak{s}, \mathrm{t} \in \mathfrak{I}_{\mathfrak{s} \vee \mathrm{t}}$.

Proof:

(1) We note the following

$$\mathfrak{s} \wedge \mathfrak{t} \in \mathfrak{I}_{\mathfrak{s}\vee\mathfrak{t}} \Leftarrow (\mathfrak{s} \wedge \mathfrak{t}) \vee (\mathfrak{s} \vee \mathfrak{t}) = (\mathfrak{s} \vee \mathfrak{t}) \vee (\mathfrak{s} \wedge \mathfrak{t})$$
$$\Leftarrow \mathfrak{s} \vee \mathfrak{t} \in \mathfrak{I}_{\mathfrak{s}\wedge\mathfrak{t}}.$$

(2) We have that

$$\mathfrak{s} \wedge \mathfrak{t} = \mathfrak{t} \wedge \mathfrak{s} \Leftarrow \mathfrak{s} \in \mathfrak{I}_{\mathfrak{t}}, \mathfrak{t} \in \mathfrak{I}_{\mathfrak{s}}$$
$$\Leftarrow \mathfrak{s}, \mathfrak{t} \in \mathfrak{I}_{\mathfrak{s}} \cap \mathfrak{I}_{\mathfrak{t}} \subseteq \mathfrak{I}_{\mathfrak{s}\vee\mathfrak{t}}.$$

On the other hand, let $\mathfrak{s}, \mathfrak{t} \in \mathfrak{I}_{\mathfrak{s}\vee\mathfrak{t}}$. Then $(\mathfrak{s} \wedge \mathfrak{t}) \wedge (\mathfrak{s} \vee \mathfrak{t}) = (\mathfrak{s} \vee \mathfrak{t}) \wedge (\mathfrak{s} \wedge \mathfrak{t})$. Now, we have that $(\mathfrak{s} \wedge \mathfrak{t}) \wedge (\mathfrak{s} \vee \mathfrak{t}) = ((\mathfrak{s} \wedge \mathfrak{t}) \wedge \mathfrak{s}) \vee ((\mathfrak{s} \wedge \mathfrak{t}) \wedge \mathfrak{t}) = (\mathfrak{s} \wedge (\mathfrak{t} \wedge \mathfrak{s})) \vee (\mathfrak{s} \wedge \mathfrak{t}) = (\mathfrak{t} \wedge \mathfrak{s}) \vee (\mathfrak{s} \wedge \mathfrak{t}) = \mathfrak{t} \wedge \mathfrak{s}$. Also, we observe that $(\mathfrak{s} \vee \mathfrak{t}) \wedge (\mathfrak{s} \wedge \mathfrak{t}) = ((\mathfrak{s} \vee \mathfrak{t}) \wedge \mathfrak{s}) \wedge \mathfrak{t} = \mathfrak{s} \wedge \mathfrak{t}$.

4. Related sets of $\mathfrak{I}_\pi$

We set the notations $\overline{\mathfrak{I}}_{\mathfrak{R}}$ and $\underline{\mathfrak{I}}_{\mathfrak{R}}$ for the union and the intersection of all $\mathfrak{I}_\pi$. We observe the following;

Lemma 4.1: *The following are valid*

(1) $\overline{\mathfrak{I}}_{\mathfrak{R}} = \{\mathfrak{t} \in \mathfrak{R};$ *there exists* $\mathfrak{s} \in \mathfrak{R}$ *such that* $\mathfrak{t} \wedge \mathfrak{s} = \mathfrak{s} \wedge \mathfrak{t}\}$,

(2) $\underline{\mathfrak{I}}_{\mathfrak{R}} = \{\mathfrak{t} \in \mathfrak{R};$ *satisfying that* $\mathfrak{t} \wedge \mathfrak{s} = \mathfrak{s} \wedge \mathfrak{t}$ *for every* $\mathfrak{s} \in \mathfrak{R}\}$.

Proof:

(1) Set $\mathfrak{t} \in \overline{\mathfrak{I}}_{\mathfrak{R}}$, then $\mathfrak{t} \in \mathfrak{I}_\pi$ for some $\pi \in \mathfrak{R}$. Therefore $\mathfrak{t} \wedge \pi = \pi \wedge \mathfrak{t}$ for some $\pi \in \mathfrak{R}$, then $\mathfrak{t} \in \{\mathfrak{t} \in \mathfrak{R};$ there exists $\mathfrak{s} \in \mathfrak{R}$ such that $\mathfrak{t} \wedge \mathfrak{s} = \mathfrak{s} \wedge \mathfrak{t}\}$. On the other hand, let $\mathfrak{t} \in \mathfrak{R}$ satisfying that there exists $\mathfrak{s} \in \mathfrak{R}$ satisfying that $\mathfrak{t} \wedge \mathfrak{s} = \mathfrak{s} \wedge \mathfrak{t}$. By the annotation of $\overline{\mathfrak{I}}_{\mathfrak{R}}$, we get that $\mathfrak{t} \in \overline{\mathfrak{I}}_{\mathfrak{R}}$.

(2) Consider $\mathfrak{t} \in \underline{\mathfrak{I}}_{\mathfrak{R}}$, then $\mathfrak{t} \in \mathfrak{I}_\pi$ for every $\pi \in \mathfrak{I}_\pi$. We conclude that $\mathfrak{t} \wedge \pi = \pi \wedge \mathfrak{t}$ for every $\pi \in \mathfrak{R}$, hence $\mathfrak{t}$ within the intersection of all $\mathfrak{I}_\pi$. Conversely, assume that $\mathfrak{t} \in \{\mathfrak{t} \in \mathfrak{R};$ for every $\mathfrak{s} \in \mathfrak{R}$ satisfying that $\mathfrak{t} \wedge \mathfrak{s} = \mathfrak{s} \wedge \mathfrak{t}\}$. Directly from the definition of $\underline{\mathfrak{I}}_{\mathfrak{R}}$, we get the required.

In the upcoming lemma we study essential properties of $\overline{\mathfrak{I}}_{\mathfrak{R}}$ and $\underline{\mathfrak{I}}_{\mathfrak{R}}$.

Lemma 4.2: *Let* $\mathfrak{s} \in \underline{\mathfrak{I}}_{\mathfrak{R}}$ *and* $\pi \in \mathfrak{R}$, *then*

(1) $\mathfrak{s} \wedge \pi$, $\pi \vee \mathfrak{s} \in \underline{\mathfrak{I}}_{\mathfrak{R}}$,

(2) $^\wedge$, $^\vee$ *are closed over* $\underline{\mathfrak{I}}_{\mathfrak{R}}$,

(3) $^\vee$ *is right distributive over* $^\wedge$ in $\underline{\mathfrak{I}}_{\mathfrak{R}}$.

Proposition 4.3: For any $\pi \in \mathfrak{R}$, $\mathfrak{s} \in \underline{\mathfrak{I}}_{\mathfrak{R}}$ if and only if $\mathrm{b} \vee \pi = \pi \vee \mathrm{b}$.

In the following, we introduce the elements $\pi \in \mathfrak{R}$ which construct lattice by using the notion of $\mathfrak{I}_\pi$.

Theorem 4.4: *For* $\pi \in \mathfrak{R}$, $\mathfrak{I}_\pi$ *is distributive lattice if and only if* $\mathfrak{I}_{\mathfrak{I}_\pi} = \mathfrak{I}_\pi$.

Proof: Consider $\mathfrak{J}_{\mathfrak{J}_\pi} = \mathfrak{J}_\pi$, assume that $\mathfrak{s}, \mathfrak{t} \in \mathfrak{J}_\pi$. Therefore $s \wedge \mathfrak{p} = \mathfrak{p} \wedge \mathfrak{s}$ for every $\mathfrak{p} \in \mathfrak{J}_\pi$. Then $s \wedge \mathfrak{t} = \mathfrak{t} \wedge \mathfrak{s}$. Now, assume that $\mathfrak{s}, \mathfrak{t}, \mathfrak{p} \in \mathfrak{J}_{\mathfrak{J}_\pi}$, then

$$\begin{aligned} \mathfrak{t} \vee (\mathfrak{s} \vee \mathfrak{p}) &= [\mathfrak{t} \wedge ((\mathfrak{t} \vee \mathfrak{s}) \vee \mathfrak{p})] \vee (\mathfrak{s} \vee \mathfrak{p}) \\ &= [\mathfrak{t} \vee (\mathfrak{s} \vee \mathfrak{p})] \wedge [((\mathfrak{t} \vee \mathfrak{s}) \vee \mathfrak{p}) \vee (\mathfrak{s} \vee \mathfrak{p})] \\ &= [\mathfrak{t} \vee (\mathfrak{s} \vee \mathfrak{p})] \wedge [(\mathfrak{t} \vee \mathfrak{s}) \vee \mathfrak{p}] \\ &= [(\mathfrak{t} \vee (\mathfrak{s} \vee \mathfrak{p})) \wedge (\mathfrak{t} \vee \mathfrak{s})] \vee [(\mathfrak{t} \vee (\mathfrak{s} \vee \mathfrak{p})) \wedge \mathfrak{p}] \\ &= (\mathfrak{t} \vee \mathfrak{s}) \vee \mathfrak{p}. \end{aligned}$$

Conversely, let $\mathfrak{J}_\pi$ be a distributive lattice. It is obvious to see that $\mathfrak{J}_{\mathfrak{J}_\pi} \subseteq \mathfrak{J}_\pi$ and assume that $\mathfrak{s} \in \mathfrak{J}_\pi$. Then $\mathfrak{s} \wedge \mathfrak{t} = \mathfrak{t} \wedge \mathfrak{s}$ for every $\mathfrak{t} \in \mathfrak{J}_\pi$ since $\mathfrak{J}_\pi$ is a lattice.

Similarly, we can prove that $\underline{\mathfrak{J}}_{\mathfrak{R}}$ is a distributive lattice with 0. $\underline{\mathfrak{J}}_{\mathfrak{R}}$ is a distributive lattice with 0.

Corollary 4.5: $\mathfrak{R}$ is a distributive lattice if and only if $\mathfrak{R} = \underline{\mathfrak{J}}_{\mathfrak{R}}$.

In [10], Swamy et al. introduced Birkhoff centre of ADL which notated by $B(\mathfrak{R})$ which stands for the following definition;

$$\begin{aligned} B(\mathfrak{R}) = \{\rho \in \mathfrak{R} \mid & \text{ there exists } \pi \in \mathfrak{R} \text{ satisfying} \\ & \text{that } \rho \wedge \pi = 0 \text{ and } \rho \vee \pi \text{ is maximal } \} \end{aligned}$$

It is obvious that if $\rho \in B(\mathfrak{R})$, then $\rho \wedge \sigma = 0 = \sigma \wedge \rho$ for some $\sigma \in \mathfrak{R}$. Hence $\rho \in \overline{\mathfrak{J}}_{\mathfrak{R}}$, which is considered to be a generalisation of $B(\mathfrak{R})$. However, It is remarkable that $\overline{\mathfrak{J}}_{\mathfrak{R}} = \mathfrak{R}$ since any element $\rho \in \mathfrak{R}$ commute with itself, therefore $\rho \in \mathfrak{J}_\rho \subseteq \overline{\mathfrak{J}}_{\mathfrak{R}}$. As a future work instead of dealing with $\overline{\mathfrak{J}}_{\mathfrak{R}}$, we investigate the following set

$$\begin{aligned} \overline{\mathfrak{J}}_{\mathfrak{R}}^{\,*} = \{\mathrm{b} \in \mathfrak{R} \mid & \text{ there exists an non zero } \pi \in \mathfrak{R} \\ & \text{satisfying that } \pi \neq \mathrm{b} \text{ and } \pi \vee \mathrm{b} = \mathrm{b} \vee \pi\} \end{aligned}$$

We see that $\overline{\mathfrak{J}}_{\mathfrak{R}}^{\,*}$ not necessarily to be $\mathfrak{R}$, consider the following example

Example 4.6: Let Γ be a non empty set, set $\partial \in \Gamma$. Define for any $\sigma, \delta \in \Gamma$ the following operation

$$\mathrm{b} \vee \delta = \begin{cases} \delta, & \text{if } \mathrm{b} = \partial \\ \mathrm{b}, & \text{if } \mathrm{b} \neq \partial \end{cases} \text{ and } \mathrm{b} \wedge \delta = \begin{cases} \partial, & \text{if } \mathrm{b} = \partial \\ \delta, & \text{if } \mathrm{b} \neq \partial \end{cases}$$

We see that $(\mathfrak{R}; ^{\vee}, ^{\wedge}, \partial)$ is an almost distributive lattice and ∂ is the zero element. In this example, $\mathfrak{R}$ is said to be the discrete almost distributive lattice. Now, It is easy to prove that $\overline{\mathfrak{J}}_{\mathfrak{R}}^{\,*} = \{\partial\}$.

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