

Codes from k-resolving sets for some products

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Abstract

In this paper, codes for Cartesian product of graphs such as $K_t \square P_s$ and strong product of graphs such as $K_t \boxtimes P_s$ using k-resolving sets are determined. We also obtain infinite families of $K_t \square P_s$ and $K_t \boxtimes P_s$ of k-dimension. Also, the k-metric dimension of $K_t \square P_s$ and $K_t \boxtimes P_s$ is studied, an exact formula for their dimensions is determined and the codes are obtained.

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1. Introduction

Coding Theory begun from the year 1948 through the publication of Claude Shannon [1]. This subject is focused on the development of error-correcting codes. These codes are used in detecting and correcting transmission errors caused by some noisy channel.

For the past half century, coding theory has grown into an immense discipline. It intersects with mathematics and engineering. It has applications in communication systems such as satellite, cellular telephone transmissions and in storing data. Good codes have practical applications. As a result, researchers have

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shifted beyond block codes to other types of codes such as convolutional codes, space-time codes, quantum codes and so on [2].

Many mathematical results have been used in Coding Theory. The important role played by algebra, combinatorics and geometry in the development of Coding Theory is an acknowledged fact [2]. For more information refer [2, 3, 4].

The concept of k -metric dimension was introduced by Estrada-Moreno, Rodri'guez- Vela'zquez, and Yero [5]. Despite being a new topic, significant research has taken place. The k -metric dimension of trees [5], cycles [5], paths [5], unicyclic graphs [6], splice and link, hierarchical products [7], corona products [8], lexicographic products [9], NP-completeness [10], some bounds for k [11], grid graphs [12], stacked prisms [13] and Rook's graphs [14] have been studied. For basic definitions and results on k -metric dimension see [5, 12, 13, 14, 15, 16].

Product of graphs being a fundamental concept, forms an important part of graph theory. They have applications in chemical graph theory [17, 18], domination [19], coding theory [12, 13, 14] etc. For basic definitions on products see [20].

Bailey and Yero [12], began a new trend that utilises graph parameters in developing codes. In this paper, the same concept is extended to Cartesian products and strong products involving paths and complete graphs.

2. Main Results

Here we obtain our main results.

Let H denote the graph $K_t \square P_s$ and $\{a_1, a_2, \dots, a_t\} = A$ and $\{b_1, b_2, \dots, b_s\} = B$ be the vertex sets of K_t and P_s , respectively. To obtain the k -metric dimension of $K_t \square P_s$, we first determine for which k values, a k -metric basis exists. The next result answers this.

Theorem 2.1: *Graph $H = K_t \square P_s$ is $2s$ -metric dimensional for any integers $s \geq 2, t \geq 3$.*

Proof: First we show $k \leq 2s$. For this, let us consider (a_1, b_1) and (a_2, b_1) . Then, $\mathcal{D}_H((a_1, b_1), (a_2, b_1)) = (\{a_1, a_2\} \times B)$

$$(\because \mathcal{D}(a, b) = \{z \in V(H) : d_H(a, z) \neq d_H(b, z)\}).$$

Therefore, $|\mathcal{D}_H((a_1, b_1), (a_2, b_1))| = 2s$. Thus H is k -metric dimensional where $k \leq 2s$. Next we show $k \geq 2s$. Consider two distinct vertices of H namely, (a_i, b_j) and (a_p, b_q) .

Case I: $i = p$.

Thus $j \neq q$ and $\{a_i\} \times \{b_j, b_q\} \subseteq \mathcal{D}_H((a_i, b_j), (a_p, b_q))$ where $i = 1, 2, \dots, t$.

Also $|(\{a_i\} \times B) \cap \mathcal{D}_H((a_i, b_j), (a_p, b_q))| \geq s - 1$ when $s \geq 3$ and $|(\{a_i\} \times B) \cap \mathcal{D}_H((a_i, b_j), (a_p, b_q))| \geq s$ when $s = 2$ and $t = 3$, for $i = 1, 2, \dots, t$.

Thus, we have

$$\begin{aligned}
 |\mathcal{D}_H((a_i, b_j), (a_p, b_q))| &\geq |\{a_i\} \times \{b_j, b_q\}| + |(\{a_i\} \times B) \cap \mathcal{D}_H((a_i, b_j), (a_p, b_q))| \\
 &\quad - 2, \text{ for } i = 1, 2, \dots, t \\
 &\geq s - 1, \text{ for } i = 1, 2, \dots, t.
 \end{aligned}$$

Hence in general,

$$\begin{aligned}
 |\mathcal{D}_H((a_i, b_j), (a_p, b_q))| &\geq st - t. \\
 &\geq ts - s \quad \text{if } s \geq t. \\
 &\geq 2s, \quad \text{since } t \geq 3.
 \end{aligned}$$

Hence $|\mathcal{D}_H((a_i, b_j), (a_p, b_q))| \geq 2s$, if $s \geq t$.

If $s \leq t$ then $st - t \geq s^2 - s = s(s - 1)$.

This implies $st - t \geq s$ and $s - 1$.

Consider $st - t \geq s$. This implies that $st \geq s + t$. Hence $st \geq 2s$ (because $s \leq t$). Hence, $|\mathcal{D}_H((a_i, b_j), (a_p, b_q))| \geq st - t \geq 2s$, if $s \leq t$.

When $s = 2$, $t = 3$, $|\mathcal{D}_H((a_i, b_j), (a_p, b_q))| \geq s$, for $i = 1, 2, \dots, t$

Thus,

$$\begin{aligned}
 |\mathcal{D}_H((a_i, b_j), (a_p, b_q))| &\geq s + s + s \quad (t = 3 \text{ times}). \\
 &\geq 3s \\
 &\geq 2s
 \end{aligned}$$

Case II: $j = q$.

Can be proved in similar manner as Case I.

Case III: $i \neq p$, $j \neq q$.

Let $i < p$. Depending on whether j is strictly greater or lesser than q , consider the following cases:

Case III (A): $j < q$.

We note that at the most two vertices of $(\{a_1, \dots, a_p\} \times \{b_j\}) \cup (\{a_i, \dots, a_t\} \times \{b_q\}) \cup (\{a_i\} \times \{b_1, \dots, b_q\}) \cup (\{a_p\} \times \{b_j, \dots, b_s\})$ do not belong to the set $\mathcal{D}_H((a_i, b_j), (a_p, b_q))$. Consequently

$$\begin{aligned}
 |\mathcal{D}_H((a_i, b_j), (a_p, b_q))| &\geq |\{a_1, \dots, a_p\} \times \{b_j\}| + |\{a_i, \dots, a_t\} \times \{b_q\}| \\
 &\quad + |\{a_i\} \times \{b_1, \dots, b_q\}| + |\{a_p\} \times \{b_j, \dots, b_s\}| - 6 \\
 &\geq p + (t - i + 1) + q + (s - j + 1) - 6 \\
 &= p + t + s + q - i - j - 4 \\
 &\geq s + t - 4 \quad (\because i < p, j < q).
 \end{aligned}$$

We observe that $s + t - 4 \geq s$ if $t \geq 4$ and $s + t - 4 \geq s - 1$ when $t = 3$. When $t \geq 4$, we have $s + t \geq s + 4$. This implies, $2(s + t) \geq 2s + 8 \geq 2s$. When $t = 3$, we have $s + t \geq s + 3$. This means, $2(s + t) \geq 2s + 6 \geq 2s$. Thus, $|\mathcal{D}_H((a_i, b_j), (a_p, b_q))| \geq 2s$.

Case III (B): $j > q$.

Similar to previous case.

From all the cases, we conclude $k \geq 2s$.

Thus $k = 2s$. Therefore $K_t \square P_s$ is $2s$ -metric dimensional.

The result given below gives an interesting formula for k -metric dimension of $K_t \square P_s$.

Theorem 2.2:

$$\dim_k(H) = \dim_k(K_t \square P_s) = \begin{cases} \left(\frac{k+1}{2}\right)t - 1, & k \text{ is odd,} \\ \frac{kt}{2} & k \text{ is even} \end{cases}$$

where $t \geq 3$ and $s \geq 2$ are any integers and $k \in \{1, 2, \dots, 2s\}$.

Proof: When $k = 1$, from [21], $\beta(K_t \square P_s) = t - 1$. Thus $\dim_1(H) = \dim(H) = t - 1$, so we consider the proof only for $k \geq 2$. Let (a_i, b_j) and (a_p, b_q) be distinct and arbitrary vertices of H .

Case I: $t \leq s$.

Depending on k being odd or even, consider the following cases.

Case IA: k is odd.

Here the metric basis is

$$S = (\{a_1, \dots, a_{t-1}\} \times \{b_1, \dots, b_s\}) \cup (\{a_t\} \times \{b_1, \dots, b_{s-1}\}).$$

(This is obtained using the k -metric dimension algorithm given in section 3). For every k , the metric representations is distinct, for every vertex and every such distinct pairs differs in minimum k positions. Thus S becomes a k -resolving set. Hence $\dim_k(H) \leq \left(\frac{k+1}{2}\right)t - 1$.

Now we prove that $\dim_k(H) \geq \left(\frac{k+1}{2}\right)t - 1$.

Case IA.I: Let t be even. When $k = 2s - 1$, $\dim_{2s-1}(H) \leq st - 1$. Let S' be a k -metric basis of $H = K_t \square P_s$ with cardinality $< st - 1$. That is $S' = \{(a_1, b_1), \dots, (a_1, b_s), (a_2, b_1), \dots, (a_2, b_s), \dots, (a_t, b_1), \dots, (a_t, b_{s-c})\}$ where $c \geq 3$ is a constant. Consider the following vertices $(a_{2e-1}, b_{2e-1}), (a_{2e}, b_{2e-1}), (a_{t-1}, b_s)$ and (a_t, b_s) . Let $L = \mathcal{D}_H((a_{2e-1}, b_{2e-1}), (a_{2e}, b_{2e-1})) \cap S'$ and

$M = \mathcal{D}_H((a_{t-1}, b_s), (a_t, b_s)) \cap S'$. Now

$$|L| = |\mathcal{D}_H((a_{2e-1}, b_{2e-1}), (a_{2e}, b_{2e-1})) \cap S'| = |(\{a_{2e-1}, a_{2e}\} \times B) \cap S'|$$

$$\geq \frac{kt}{2u+2} + 1.$$

This repeats e times where $1 \leq e \leq t$ and $1 \leq u \leq t$ and

$$\begin{aligned} |M| &= |\mathcal{D}_H((a_{t-1}, b_s), (a_t, b_s)) \cap S'| = |(\{a_{t-1}, a_t\} \times B) \cap S'| \\ &\geq \frac{kt}{2u+2} - (e + 1). \end{aligned}$$

We have

$$\begin{aligned} |S'| &\geq |L| + |M| \\ &\geq \left(\frac{kt}{2u+2} + 1\right)t + \frac{kt}{2u+2} - (e + 1) \\ &= \frac{kt}{2u+2}(e + 1) - 1 \\ &\geq \frac{kt}{2} - 1. \quad (\because e \leq t, u \leq t) \end{aligned}$$

Suppose $|S'| = \frac{kt}{2} - 1$. Then $|L| = \left(\frac{kt}{2u+2} + 1\right)e$ and $|M| = \frac{kt}{2u+2} - (e + 1)$. Consider $(a_{2e-1}, b_s), (a_t, b_{2e-1}) \in S$ and $S = L \cup M$. Let P_1 denote all the vertices of H except (a_{2e-1}, b_s) and P_2 denote all the vertices of H except (a_t, b_{2e-1}) . That is, $P_1 = (\{a_{2e-1}\} \times \{b_1, \dots, b_{s-1}\} \cup \{a_{2e}, \dots, a_{t-1}\} \times B \cup (\{a_t\} \times \{b_1, \dots, b_{s-3}\}))$ and $P_2 = (\{a_1, \dots, a_{t-1}\} \times B) \cup (\{a_t\} \times \{b_{2e}, \dots, b_{s-1}\})$. So

$$|P_1 \cap S'| = \left(\frac{kt}{2u+2} + 1\right)e - 1$$

and

$$|P_2 \cap S'| = \frac{kt}{2u+2} - e - 2.$$

Consider the vertices $(a_1, b_1), (a_2, b_2), \dots, (a_{t-1}, b_{s-1}), (a_t, b_s)$.

Let $Z = \mathcal{D}_H((a_1, b_1), (a_2, b_2))$. Then,

$|Z \cap S'| \geq \left(\frac{kt}{2u+2} + 1\right)e$ and $|Z \cap S'| \geq \frac{kt}{2u+2} - (e + 1)$ for all the pairs of vertices along the diagonal.

Now,

$$|Z \cap S'| \leq |P_1 \cap S'| = \left(\frac{kt}{2u+2} + 1\right)e - 1,$$

and

$$|Z \cap S'| \leq |P_2 \cap S'| = \frac{kt}{2u+2} - e - 2.$$

This is a contradiction.

Thus, $|S'| \neq st - 3$ and hence $|S'| \geq st - 1$.

Applying the same process when $k = 2s - 3, 2s - 5, \dots, 3$, we obtain similar contradictions. Therefore $|S'| \geq \left(\frac{k+1}{2}\right)t - 1$.

Case IAI.II: Let t be odd. When $k = 2s - 1$, $\dim_{2s-1}(H) \leq st - 1$.

Depending on $st - 1$ being odd or even, we consider the following subcases.

Subcase IAI.II.I: Let $st - 1$ be even. We shall prove that no smaller k -resolving set exists. Let S' be a k -metric basis for $H = K_t \square P_s$ of cardinality less than $st - 1$. We consider the vertices $(a_{2e-1}, b_{2e-1}), (a_{2e}, b_{2e-1}), (a_{t-2}, b_s)$ and (a_{t-1}, b_s) . Let $L = \mathcal{D}_H((a_{2e-1}, b_{2e-1}), (a_{2e}, b_{2e-1})) \cap S'$ and

$$M = \mathcal{D}_H((a_{t-2}, b_s), (a_{t-1}, b_s)) \cap S'.$$

Now

$$\begin{aligned} |L| &= |\mathcal{D}_H((a_{2e-1}, b_{2e-1}), (a_{2e}, b_{2e-1})) \cap S'| \\ &= |(\{a_{2e-1}, a_{2e}\} \times B) \cap S'| \geq \left(\frac{k-(4u-3)}{4}\right)t - w \end{aligned}$$

which repeats e times where $1 \leq e \leq t$ and w is an arbitrary constant and

$$\begin{aligned} |M| &= |\mathcal{D}_H((a_{t-2}, b_s), (a_{t-1}, b_s)) \cap S'| \\ &= |(\{a_{t-2}, a_{t-1}\} \times B) \cap S'| \\ &\geq \left(\frac{k-(4u-3)}{4}\right)t \pm w. \end{aligned}$$

Now we have,

$$\begin{aligned} |S'| &\geq |L| + |M| \\ &\geq \left[\left(\frac{k-(4u-3)}{4}\right)t - w\right]e + \left(\frac{k-(4u-3)}{4}\right)t \pm w. \end{aligned}$$

$$\text{Hence } |S'| = \left(\frac{k-(4u-3)}{4}\right)t(e+1) - we \pm w.$$

Observe that $|S'| = \left(\frac{k-1}{2}\right)t - 1$. If $|S'| = \left(\frac{k-1}{2}\right)t - 1$. Then $|L| = \left(\frac{k-(4u-3)}{4}\right)te - we$, $|M| = \left(\frac{k-(4u-3)}{4}\right)t \pm w$.

Consider $(a_{2e-1}, b_s), (a_{t-1}, b_1) \in S$ and $S = L \cup M$.

Let P_1 represent all the vertices of H except (a_{2e-1}, b_s) and P_2 represent all the vertices of H except (a_{t-1}, b_1) . That is, $P_1 = (\{a_{2e-1}, a_{t-1}\} \times \{b_1, \dots, b_{s-1}\}) \cup (\{a_{2e}, \dots, a_{t-2}\} \times B)$ and $P_2 = (\{a_{2e-1}, \dots, a_{t-2}\} \times B) \cup (\{a_{t-1}\} \times \{b_2, \dots, b_{s-1}\})$. Then

$$|P_1 \cap S'| = \left(\frac{k-(4u-3)}{4}\right)te - we - 1$$

and

$$|P_2 \cap S'| = \left(\frac{k-(4u-3)}{4}\right)t \pm w - 1.$$

Consider the vertices $(a_1, b_1), (a_2, b_2), \dots, (a_{t-1}, b_{s-1}), (a_t, b_s)$.

Let $Z = \mathcal{D}_H((a_1, b_1), (a_2, b_2))$.

$|Z \cap S'| \geq \left(\frac{k-(4u-3)}{4}\right)te - we$ and $|Z \cap S'| \geq \left(\frac{k-(4u-3)}{4}\right)t \pm w$ for all the pairs of vertices along the diagonal.

Thus,

$$|Z \cap S'| \leq |P_1 \cap S'| = \left(\frac{k-(4u-3)}{4}\right)te - we - 1,$$

and

$$|Z \cap S'| \leq |P_2 \cap S'| = \left(\frac{k-(4u-3)}{4}\right)t \pm w - 1.$$

This is a contradiction.

Applying the same process for $k = 2s - 3, 2s - 5, \dots, 3$, we obtain similar contradictions. Hence $|S'| \geq \left(\frac{k+1}{2}\right)t - 1$.

Subcase IAI.II.II: Let $st - 1$ be odd. Proof is similar to Subcase IAI.II.I.

From all the cases, we conclude, $\dim_k(H) \geq \left(\frac{k+1}{2}\right)t - 1$.

Thus $\dim_k(H) = \left(\frac{k+1}{2}\right)t - 1$.

Case IAI. k is even.

First it will be shown that $\dim_k(H) \leq \frac{kt}{2}$. Here the metric basis is

$S = (\{a_1, \dots, a_t\} \times \{b_1, \dots, b_s\})$ (which is obtained from the k -metric dimension algorithm). Clearly all vertices have unique representation and each pair of these representations differs in atleast k positions. Thus S becomes a k -resolving set. Hence $\dim_k(H) \leq \frac{kt}{2}$.

Next it will be shown that $\dim_k(H) \geq \frac{kt}{2}$. When $k = 2s$, $\dim_{2s}(H) \leq st$. We have $\mathcal{D}_{2s}(H) = V(K_t \square P_s)$. We know that $K_t \square P_s$ is $2s$ -metric dimensional. Thus, it follows that, $\dim_{2s}(H) \geq |\mathcal{D}_{2s}(H)| = st$. Hence $\dim_{2s}(H) = st$. When $k < 2s$, that is, $k = 2s - 2, 2s - 4, \dots, 2$ we follow the same procedure as when k is odd and arrive at similar contradictions. Thus we get $\dim_k(H) \geq \frac{kt}{2}$. Hence $\dim_k(H) = \frac{kt}{2}$.

Case II: $t > s$. Similar arguments as in Case I hold.

Hence the proof.

The next theorem gives the maximum value of k for which $K_t \boxtimes P_s$ becomes k -metric dimensional.

Theorem 2.3: Graph $H = K_t \boxtimes P_s$ is 2-metric dimensional for any integers $t \geq 2$ and $s \geq 3$.

Proof: First, consider the vertices (a_1, b_1) and (a_2, b_1) . Note,

$$\mathcal{D}_H((a_1, b_1), (a_2, b_1)) = \{(a_1, b_1), (a_2, b_1)\}.$$

The set of non-trivial distinctive vertices is

$$\mathcal{D}_H^*((a_1, b_1), (a_2, b_1)) = \phi.$$

Thus, (a_1, b_1) and (a_2, b_1) are twins. We note that every vertex pair of the complete graph K_t gives twins. Thus the total number of such vertex pairs of K_t is $\frac{t(t-1)}{2}$. Since we have s copies of K_t , it follows that there are $\frac{st(t-1)}{2}$ pairs of twins. Hence $K_t \boxtimes P_s$ is 2-metric dimensional.

The next result gives the general formula for $\dim_k(K_t \boxtimes P_s)$ for any $k \in \{1, 2\}$.

Theorem 2.4:

$$\dim_k(H) = \dim_k(K_t \boxtimes P_s) = (k + t - 2)s$$

where $s \geq 3, t \geq 2$ are any integers and $k \in \{1, 2\}$.

Proof: Case 1. $k = 1$. We will first show that $\dim_1(H) \leq (t - 1)s$. The metric basis is $S_1 = (\{a_1, a_2, \dots, a_{t-1}\} \times B)$ (this is obtained using k -metric dimension algorithm in section 3). Clearly each vertex has a unique representation, that is, no two vertices have the same representation and each pair of metric representations differs in minimum 1 position. Thus S_1 is a 1-resolving set. Hence $\dim_1(H) \leq (1 + t - 2)s$.

Now, it will be shown that $\dim_1(H) \geq (t - 1)s$, using contradiction. Suppose $S' = (\{a_1, \dots, a_{t-2}\} \times B)$ is a 1-metric basis of cardinality $(t - 2)s$. We note that $\mathcal{D}_H((a_{t-1}, B), (a_t, B)) = \emptyset$. This means that (a_{t-1}, B) and (a_t, B) have the same metric representations. Thus S' cannot be a 1-resolving set, which contradicts the assumption. Hence our assumption that $|S'| = (t - 2)s$ is wrong. Therefore, $\dim_1(K_t \boxtimes P_s) \geq (t - 2 + 1)s = (t - 1)s$. Hence $\dim_1(K_t \boxtimes P_s) = (t - 1)s$.

Case 2: $k = 2$. We will first show that $\dim_2(K_t \boxtimes P_s) \leq ts$. In this case, the metric basis is $S_2 = (\{a_1, a_2, \dots, a_t\} \times B)$. Clearly every vertex has distinct representation and every pair differs in minimum 2 positions. Thus $\dim_2(K_t \boxtimes P_s) \leq ts$.

Next we will show that $\dim_2(K_t \boxtimes P_s) \geq ts$. We see that $\mathcal{D}_2(H) = V(K_t \boxtimes P_s)$. From Theorem 2.3, we know that $K_t \boxtimes P_s$ is 2-metric dimensional. Thus, it follows that $\dim_2(K_t \boxtimes P_s) \geq |\mathcal{D}_2(H)| = ts$. Hence $\dim_2(K_t \boxtimes P_s) = ts$.

The family of $K_t \square P_s$ and $K_t \boxtimes P_s$ is utilised, to obtain $(K_t \square P_s, k)$ -codes and $(K_t \boxtimes P_s, k)$ -codes.

The following result gives the codewords obtained from k -resolving sets of $K_t \square P_s$.

Theorem 2.5: A $(K_t \square P_s, k)$ -code contains nm codewords having length $= \dim_k(K_t \square P_s)$ over an alphabet of size $\text{diam}(K_t \square P_s) = s$ having minimum distance k , capable of correcting $\lfloor (k - 1)/2 \rfloor$ errors.

Proof: Follows from previous theorems of $K_t \square P_s$.

This result is illustrated with an example given below.

Example 2.6: Consider $K_3 \square P_3$ as in Fig. 1. By Theorem 2.1, we know that $K_3 \square P_3$ is 6-metric dimensional. By Theorem 2.2, $\dim_6(K_3 \square P_3) = \frac{6t}{2} = 9$. Thus $S = V(K_3 \square P_3)$ is a 6-resolving set. The distance matrix or metric representations with respect to basis S is

$$\begin{bmatrix} 0 & 1 & 2 & 1 & 2 & 3 & 1 & 2 & 3 \\ 1 & 0 & 1 & 2 & 1 & 2 & 2 & 1 & 2 \\ 2 & 1 & 0 & 3 & 2 & 1 & 3 & 2 & 1 \\ 1 & 2 & 3 & 0 & 1 & 2 & 1 & 2 & 3 \\ 2 & 1 & 2 & 1 & 0 & 1 & 2 & 1 & 2 \\ 3 & 2 & 1 & 2 & 1 & 0 & 3 & 2 & 1 \\ 1 & 2 & 3 & 1 & 2 & 3 & 0 & 1 & 2 \\ 2 & 1 & 2 & 2 & 1 & 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 3 & 2 & 1 & 2 & 1 & 0 \end{bmatrix}.$$

Next determine the minimum Hamming distance(HD) for these.

$$\begin{bmatrix} 9 & 6 & 6 & 6 & 8 & 6 & 6 & 8 \\ 9 & 9 & 6 & 6 & 6 & 6 & 6 & 6 \\ 6 & 9 & 8 & 6 & 6 & 8 & 6 & 6 \\ 6 & 6 & 8 & 9 & 6 & 6 & 6 & 8 \\ 6 & 6 & 6 & 9 & 9 & 6 & 6 & 6 \\ 8 & 6 & 6 & 6 & 9 & 8 & 6 & 6 \\ 6 & 6 & 8 & 6 & 6 & 8 & 9 & 6 \\ 6 & 6 & 6 & 6 & 6 & 6 & 9 & 9 \\ 8 & 6 & 6 & 8 & 6 & 6 & 6 & 9 \end{bmatrix}$$

Clearly the minimum HD is 6. Thus by Theorem 2.5, there exists a $(K_3 \square P_3, 6)$ -code having 9 codewords namely 012123123, 101212212, 210321321, 123012123, 212101212, 321210321, 123123012, 212212101, 321321210.

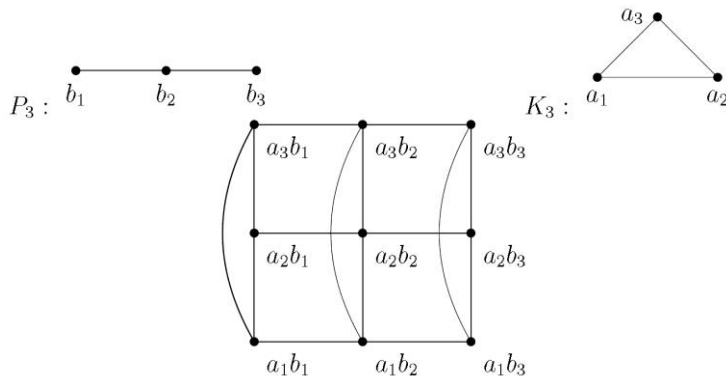


Figure 1
 $K_3 \square P_3$

The following result gives the codewords obtained from k -resolving sets of $K_t \square P_s$.

Theorem 2.7: A $(K_t \boxtimes P_s, k)$ -code contains ts codewords having length $= \dim_k(K_t \boxtimes P_s)$ over an alphabet of size $\text{diam}(K_t \boxtimes P_s) = s - 1$ and with minimum distance k .

Proof: Follows from the previous theorems of $K_t \boxtimes P_s$.

This result is illustrated using the example given below.

Example 2.8: Consider $K_3 \boxtimes P_3$ as in Fig. 2. By Theorem 2.3, we know that $K_3 \boxtimes P_3$ is 2-metric dimensional. By Theorem 2.4, $\dim_2(K_3 \boxtimes P_3) = (2 + 3 - 2)3 = 9$. Thus $S = (\{a_1, a_2, a_3\} \times B)$ is a 2-resolving set. The distance matrix of $K_3 \boxtimes P_3$.

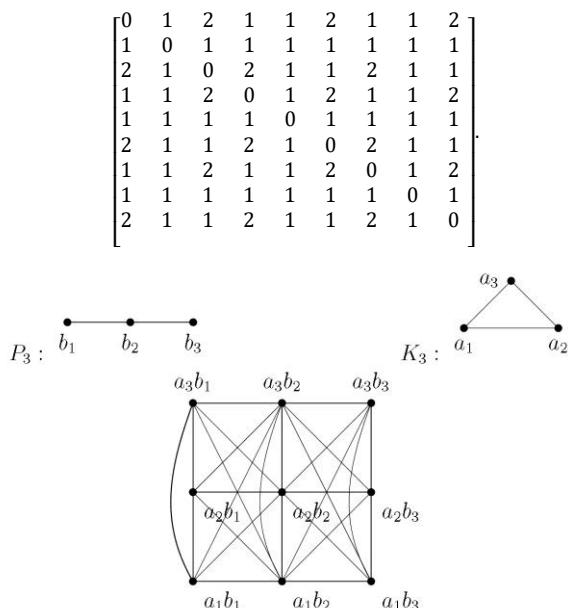


Figure 2
 $K_3 \boxtimes P_3$

Now let us determine the minimum Hamming distance for these.

$$\begin{bmatrix} 5 & 6 & 2 & 5 & 6 & 2 & 5 & 6 \\ 5 & 5 & 5 & 2 & 5 & 5 & 2 & 5 \\ 6 & 5 & 6 & 5 & 2 & 6 & 5 & 2 \\ 2 & 5 & 6 & 5 & 6 & 2 & 5 & 6 \\ 5 & 2 & 5 & 5 & 5 & 5 & 2 & 5 \\ 6 & 5 & 2 & 6 & 5 & 6 & 6 & 2 \\ 2 & 5 & 6 & 2 & 5 & 6 & 5 & 6 \\ 5 & 2 & 5 & 5 & 2 & 6 & 5 & 5 \\ 6 & 5 & 2 & 6 & 5 & 2 & 6 & 5 \end{bmatrix}$$

It is seen that the minimum HD is 2. Thus by Theorem 2.7, there exists a $(K_3 \boxtimes P_3, 2)$ -code having 9 codewords namely 012112112, 101111111, 210211211, 112012112, 111101111, 211210211, 112112012, 111111101, 211211210.

Remark 2.9: In our paper [13, 14], a new decoding algorithm was developed for stacked prisms and Rook's graph and its complexity was analyzed. The same algorithm works for $(K_t \square P_s, k)$ -codes.

3. Algorithm

Here an algorithm for obtaining the k -metric basis of products is provided. The algorithm involves the following steps:

Step 1: Denote the distance matrix of $K_t \square P_s / K_t \boxtimes P_s$ by d .

Step 2: A function $rSubset(arr, r)$ is defined. It lists all combinations of the array depending on r where r denotes the r -element sets of vertex sets of $K_t \square P_s / K_t \boxtimes P_s$.

Step 3: Denote the dimensions of the distance matrix d by t and the total combinations possible for any t and r by s , that is, $s = tCr$.

Step 4: Consider an empty matrix A . For each i belonging to t , determine all the $rSubset(d.row(i), r)$ and add them to A .

Step 5: For i in $range(s)$:

For k in $range(\text{length of } A)$:

Add $(A[k][i])$ to an empty matrix L .

Step 6: For each row x and row y in matrix L , we compute its minimum hamming distance (which is nothing but the k -metric dimension). Define four empty matrices namely H_1, H_2, E and F . If row x is same as row y then add these rows to H_2 otherwise add them to H_1 . If length of H_2 equals t it implies that minimum hamming distance of self loops is considered. To avoid this, we always consider length of $H_2 > t$. When length of H_2 is greater than t , it implies that there are rows in L whose minimum hamming distance is zero. In such a case, we add all these rows to E otherwise add minimum value of H_1 to F .

Step 7: print $(E+F)$.

This gives the metric dimensions of the r -set (including 0).

Step 8: The k -metric basis is obtained by printing the matrix L .

4. Conclusion

In this paper, the open cases for k -metric dimension of $K_t \square P_s$ and $K_t \boxtimes P_s$ are studied. An exact formula for $\dim_k(K_t \square P_s)$ and $\dim_k(K_t \boxtimes P_s)$ is obtained and the codes that arise from their resolving sets are developed. Further, an algorithm developed to compute the k -metric dimension of products is given.

References

- [1] C. Shannon, "A mathematical theory of communication", *Bell System Tech. J.*, vol. 27, no. 3, pp. 379–423, Jul. (1948).
- [2] P. J. Cameron and J. H. van Lint, *Designs, Graphs, Codes and their Links*, Cambridge University Press (1991).
- [3] C. Ling and C. Xing, *Coding Theory, A First Course*, Cambridge University Press, (2004).
- [4] W. C. Huffman. and V. Pless, *Fundamentals of Error Correcting Codes*, Cambridge University Press (2003).
- [5] A. Estrada-Moreno, J. A. Rodri'guez- Vela'zquez and I. G. Yero, "The k -metric dimension of a graph", *Applied Mathematics and Information Sciences*, vol. 9, no. 6, pp. 2829-2840 (2015).
- [6] A. Estrada-Moreno, I. G. Yero and J. A. Rodri'guez-Vela'zquez, "The k -metric dimension of corona product graphs", *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 39, no. 1, pp. 135-156 (2016).
- [7] S. Klavzar, F. Rahbarnia and M. Tavakoli, " Some binary products and integer linear programming for computing k -metric dimension of graphs", *Applied Mathematics and Computation*, vol. 409, pp. 126420 (2021).
- [8] A. Estrada-Moreno, I. G. Yero and J. A. Rodri'guez-Vela'zquez, "The k -metric dimension of the lexicographic product of graphs", *Discrete Mathematics*, vol. 339, no. 7, pp. 1924-1934 (2016).
- [9] A. Estrada-Moreno, "The k -metric dimension of a unicyclic graph", *Mathematics*, vol. 9, no. 21 (2021).
- [10] I. G. Yero, A. Estrada- Moreno and J. A. Rodri'guez-Vela'zquez, "Computing the k -metric dimension of graphs", *Applied Mathematics and Computation*, vol. 300, no. 1, pp. 60-69 (2017).

- [11] S. G. Corregidor and A. Marti'nez-Pe'rez, "A note on k-metric dimensional graphs", *Discrete Applied Mathematics*, vol. 289, no. 31, pp. 523-533 (2021).
- [12] R. F. Bailey and I.G. Yero, "Error-correcting codes from k-resolving sets", *Discussiones Mathematicae Graph Theory*, vol. 309, pp. 4253-4265 (2009).
- [13] M. I. Huilgol and G. D. D'Souza, "Codes from k-resolving sets for stacked prism graphs", *Journal of Mathematical and Computational Science*, vol. 14 (2024).
- [14] M. I. Huilgol and G. D. D'Souza, "Codes from k-resolving sets for some Rook's graphs", *Journal of Systems Engineering and Electronics*, vol. 34, no. 4, pp. 123-143 (2024).
- [15] P. J. Slater, "Leaves of trees", *Congr. Numer.*, vol. 14, pp. 549-559 (1975).
- [16] F. Harary and R. A. Melter, "On the metric dimension of a graph", *Ars Combin*, vol. 2, pp. 191-195 (1976).
- [17] M. I. Huilgol, G. D. D'Souza and I. N. Cangul, "Omega indices of strong and lexicographic products", *Current Organic Synthesis*, vol. 22, no. 5, pp. 143-158 (2025).
- [18] V. Loksha, K. Z. Yasmeen and T. Deepika, "Edge version of SDD and ISI index for rooted product graphs", *Journal of Discrete and Mathematical Sciences and Cryptography*, vol. 22, no. 6, pp. 1077-1090 (2019).
- [19] A. D. Akwu, O. Oyewumi and D.O.A. Ajayi, "The outer-connected vertex edge domination number in cartesian product graphs", *Journal of Discrete Mathematical Sciences and Cryptography*, vol. 25, no. 5, pp. 1275-1287 (May 2020).
- [20] D. B. West, *Introduction to Graph Theory, Second Edition*, Pearson Education, Inc. (2001).
- [21] J. Ca'ceres, C. Hernando, M. Mora, I. M. Pelayo, M. L. Puertas, C. Seara and D.R. Wood, "On the metric dimension of Cartesian product of graphs", *SIAM Journal on Discrete Mathematics*, vol. 21, no. 2 (2007).

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