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Burning number of triangular-like architectures

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Abstract

The study of graph burning is a crucial area in graph theory, providing insight into the spread of influence, information, or fire across networks. This paper investigates the burning number of triangular grid graphs and Sierpinski gasket graphs, which represent a class of lattice-based and fractal architectures. For triangular grids, we derive a lower bound for the burning number and provide a mathematical framework supporting its validity. Similarly, for Sierpinski gasket graphs, we establish new lower bounds by analyzing their recursive structure and boundary-vertex distribution. Our results emphasize the relationship between the graph's structural properties and the derived lower bounds on their burning numbers. These findings improve the theoretical understanding of burning dynamics in triangular-like architectures, with potential applications in network design and resource allocation.

Subject Classification: *05C30, 05C35, 05C90.*

Keywords: *Burning number, Triangular grids, Sierpinski gasket graph.*

1. Introduction

The concept of the *burning number* was introduced as a graph-theoretic parameter to represent the diffusion of information, influence across a network. It was first proposed by Bonato, Janssen, and Roshanbin [1, 14], where the burning number $b(G)$ of a graph G denotes the smallest number of discrete time steps required to ensure that every vertex becomes "burned." This process imitates the dynamic spread of influence in social or communication networks. In a subsequent study in 2015, Bonato, Janssen, and Roshanbin [2] investigated its mathematical properties in depth, obtained bounds for $b(G)$, and determined exact values for several well-known graph classes, including networks generated by the Iterated Local Transitivity (ILT) model. Their analysis led to the formulation of the celebrated *burning number conjecture*, asserting that for any connected graph G with n vertices, the inequality $b(G) \leq \lceil \sqrt{n} \rceil$ holds.

Following this foundational work, a variety of extensions have been explored in different contexts. In [3], the burning process was studied for product graphs, yielding new upper bounds and insights into fire propagation under graph product operations. Another notable improvement was presented in [4], where an algorithmic framework established a refined upper bound $b(G) \leq \sqrt{\frac{4n}{3}} + O(1)$, which further advanced progress toward resolving the conjecture.

The computational complexity of finding the burning number has also been examined extensively. It was shown in [5] that computing $b(G)$ is NP-complete even for restricted families such as connected cubic and proper interval graphs. In addition, enhanced bounds for P_k-free graphs and extensions like edge-burning and total-burning variants were proposed. Interestingly, determining $b(T)$ for trees T with maximum degree three also remains NP-complete [6], and the problem persists for spider trees containing a central vertex of degree at least three [6].

Further studies have focused on grid-based and specialized graph structures. For instance, upper and lower bounds were obtained for the burning number of Cartesian grids of size $m \times n$ [7], and a 2-approximation algorithm for burning $n \times n$ square grids with $n \geq 403$ was developed [7]. In the case of generalized Petersen graphs, tight bounds on $b(G)$ have been derived [8]. Moreover, for specific graph families such as split graphs and co-graphs, the burning number can be computed efficiently in linear time [9].

For interval graphs with diameter d, it has been shown that $\lceil\sqrt{d+1}\rceil \leq b(G) \leq \lceil\sqrt{d+1}\rceil + 1$ [9]. Another key research area involves message dissemination in high-dimensional cube networks [15]. This paper studies domination parameters of graphs and characterizes graphs based on the forcing connected domination number [15]. The authors analyze structural properties such as bipartiteness, Hamiltonicity, and chromatic number of strong double graphs [16]. The study in [10] investigates the efficiency of message transmission from an external sender to all vertices in an n-dimensional cube. It establishes the minimum number of rounds required for complete dissemination under given transmission constraints. A significant result demonstrates that for any integer n and a sequence $(a_1, a_2, \ldots, a_k)$ of $k = \left\lceil\frac{n}{2}\right\rceil$ binary vectors of length n, there exists a binary vector z whose Hamming distance from each a_i is strictly greater than $k - i$ for all $1 \leq i \leq k$. This finding provides insights into efficient message transmission strategies in large-scale network structures. This survey [17, 18] discusses several applications of graph theoretic concepts in cryptography and networks.

2. Basic Concepts

This section presents fundamental graph theory concepts necessary for understanding this paper. The *closed neighborhood* of y within distance ℓ is $N_\ell[y] = \{x \in V(G): d(x,y) \leq \ell\}$; for $\ell = 1$, it is denoted by $N[y]$. An *isometric cycle* is a cycle where the distance between any two of its vertices equals their distance in G. The *perimeter* of G is the length of its longest isometric cycle. For paths $P_1: u_1u_2 \cdots u_n$ and $P_2: v_1v_2 \cdots v_m$, their *concatenation* is $P_1 \circ P_2: u_1u_2 \cdots u_n(= v_1)v_2 \cdots v_m$.

Definition 2.1: [2] Let $G = (V, E)$ be a graph. The *burning process* on G is a discrete-time spreading procedure defined as follows:

1. **Initial Step:** Initially, every vertex of G is unburned. At time step $t = 1$, one vertex is chosen and set on fire, marking it as burned.
2. **Recursive Step:** For each subsequent time step $t = 2,3, \ldots$, the following operations occur:
   (a) a new unburned vertex is selected and ignited, and
   (b) each vertex that was burned in any previous step propagates the fire to all its neighboring vertices, which then become burned.

A vertex, once burned, remains in that state for the rest of the process. The procedure terminates when every vertex in the graph has been burned.

The *burning number* of G, denoted by $b(G)$, is the smallest number of time steps required for the entire graph to become burned.

According to Bonato, Janssen, and Roshanbin [2], a sequence $(y_1, y_2, \dots, y_r)$ represents a valid sequence of burning sources in a graph G if and only if

$$N_{r-1}[y_1] \cup N_{r-2}[y_2] \cup \cdots \cup N_0[y_r] = V(G). \tag{2.1}$$

Definition 2.2: [11] A *triangular grid* refers to a regular tessellation of the Euclidean plane using identical equilateral triangles. The associated *triangular grid graph*, denoted by T_ℓ for $\ell \geq 1$, is derived from a triangular lattice of order $(\ell + 1)$. In this graph, vertices represent the intersection points of the lattice lines, and edges connect pairs of points that are joined by the lattice segments.

Formally, the vertex set of T_ℓ consists of all ordered triples (i, j, k), where i, j, and k are nonnegative integers satisfying $i + j + k = \ell$. Two vertices of T_ℓ are adjacent if and only if the sum of the absolute differences of their respective coordinates equals 2, see Fig.1

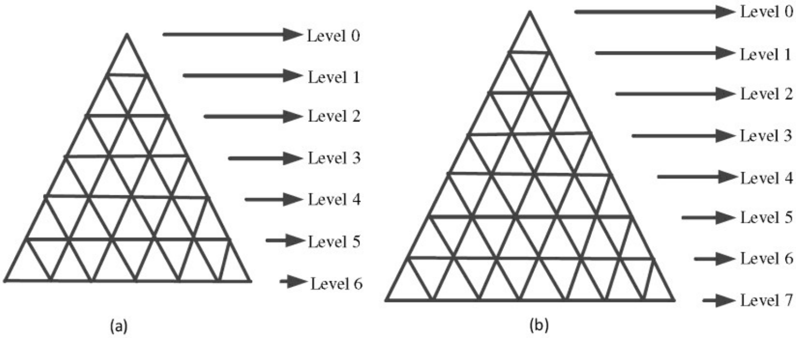

Figure 1
Triangular grids: (a) T_6 (b) T_7

Definition 1 [11] In the *Sierpiński gasket graph* S_ℓ, the *anchor vertices* are the three outermost vertices that define the triangular boundary of the graph. These vertices remain unchanged at each level of recursion and serve as connection points between different smaller copies of the graph.

Role in Construction:

1. **Base Case ($\ell = 1$):** The three vertices of the triangle are the anchor vertices.
2. **Recursive Case ($\ell \geq 2$):** The graph S_ℓ consists of three smaller copies of $S_{\ell-1}$.
   - Each copy inherits the anchor vertices of $S_{\ell-1}$.
   - The three anchor vertices of S_ℓ remain the same as those in S_1 and do not change across recursive levels. For example the Sierpinski gasket graph is shown in Fig. 2.

These anchor vertices help in maintaining the *self-similar structure* of the graph as it expands to higher levels.

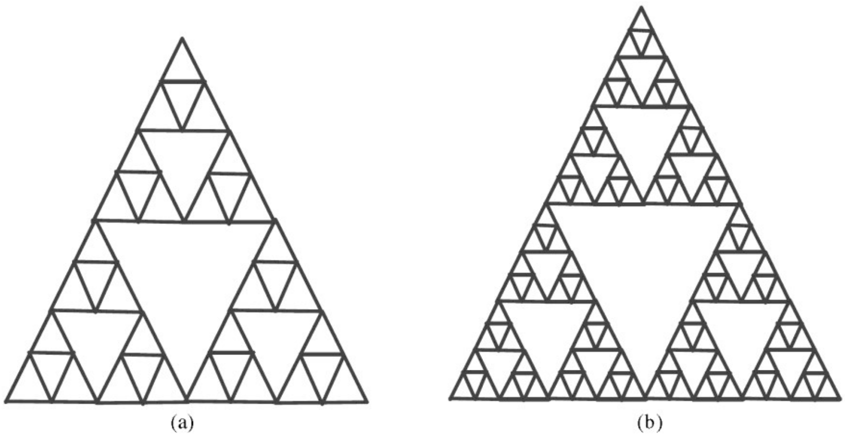

Figure 2
Sierpinski Gasket Graph: (a) S_4 (b) S_5

Remark 2.4: The following are some of the properties of the Sierpinski gasket graph: The number of vertices, $|V(S_\ell)| = \frac{3}{2}\left(3^{\ell-1}+1\right)$, number of edges, $|E(S_\ell)| = 3^\ell$, with diameter $diam(S_\ell) = 2^{\ell-1}$.

The Sierpinski gasket graph exhibits a self-similar fractal structure, with each level ℓ containing three smaller subgraphs $S_{\ell-1}$ connected in a triangular arrangement.

Definition 2.5: Let G be a graph with n vertices, and let $A \subseteq V(G)$. The quantity $b_G(A)$ denotes the size of the smallest burning sequence in G that successfully burns every vertex in the subset A.

Definition 2.6: Let w_1, w_2, w_3 be the vertices of degree 2 in $T_\ell(\text{orin}S_\ell)$. Let P_1, P_2 and P_3 be the shortest paths between w_1 and w_2; w_2 and w_3; w_3 and w_1 respectively. Then the concatenation $P_1 \circ P_2 \circ P_3$ of paths is defined as the boundary of $T_\ell(\text{orin}S_\ell)$.

3. Main Results

We present our findings, in this section on the lower bounds for the burning number of triangular grids and Sierpinski gasket graphs. These results provide a theoretical framework for understanding the burning process in structured graph architectures. By analyzing the unique properties of these graphs, we establish optimal bounds that serve as benchmarks for further studies in graph-burning problems.

Lemma 3.1: *Let G be a graph with n vertices and let $A \subseteq V(G)$. Then the inequality $b(G) \geq b_G(A)$ holds.*

Proof: Since A is a subset of $V(G)$, any burning sequence that completely burns G will necessarily burn all vertices of A as well. Therefore, the burning number of G cannot be smaller than $b_G(A)$. This establishes the result.

Using Lemma 3.1, we have established the lower bound for the burning number of triangular grids in the following theorem.

Theorem 3.2: *Let T_ℓ denote the triangular grid. Then the burning number of T_ℓ satisfies*

$$b(G) \geq \lceil -1 + \sqrt{3\ell + 2} \rceil, \ell \geq 1.$$

Proof: Let P_1, P_2, and P_3 be paths between the vertices (0,0) to $(\ell, 0)$, $(\ell, 0)$ to (ℓ, ℓ), and (ℓ, ℓ) to (0,0) respectively in T_ℓ. Let B be set of all vertices in $P_1 \circ P_2 \circ P_3$. By Lemma 3.1, $b(G) \geq b_G(B)$. In order to find the lower bound for $b(G)$, we find the lower bound for $b_G(B)$.

Consider an optimal burning sequence say $(x_1, x_2, \dots, x_r)$, where r represents the least number of time steps needed to burn B. As per the definition of burning process, x_1 burns at most $3r$ vertices in B by time steps r, x_k burns at most $2[r - (k-1)]$ vertices for $2 \leq k \leq r - 1$ in B by time step r and x_r burns at most 1 vertex in B by time step r. Hence in total $(x_1, x_2, \dots, x_r)$ burns at most $r^2 + 2r - 1$ vertices in B by time step r. To attain the lower bound for $b_G(B)$, $r^2 + 2r - 1 \geq |B| = 3\ell$, which implies $r \geq \lceil -1 + \sqrt{3\ell + 2} \rceil$. Hence $b_G(B) \geq \lceil -1 + \sqrt{3\ell + 2} \rceil$. Thus $b(G) \geq b_G(B) \geq \lceil -1 + \sqrt{3\ell + 2} \rceil$.

Remark 3.3: Let us consider a triangular grid T_9, for which $b(T_9) \geq 5$ by the above Theorem 3.2. We claim that $b(T_9) = 5$. Let us consider the optimal burning sequence $(x_1, x_2, x_3, x_4, x_5)$ for T_9. Let us choose x_1 as the vertex (3,3,3), x_2 as the vertex (0,9,0), x_3 as the vertex (0,0,9), x_4 as the vertex (9,0,0) and due to the non availability of the vertices in time step 5, we can not choose x_5 see Fig. 3. From the Fig. 3, it is clear that $N[x_i]$ are pairwise disjoint. Now, we have $|N_4[x_1]| = 43$ (vertices shown in red), $|N_3[x_2]| = 6$ (vertices shown in blue), $|N_2[x_3]| = 3$ (vertices shown in yellow), $|N_1[x_4]| = 3$ (vertices shown in green), and $|N_0[x_5]| = 0$, as illustrated in Fig. 3. Therefore,

$$\sum_{i=1}^{5} |N_{5-i}[x_i]| = 55 = |V(T_9)|.$$

Consequently, the burning number of T_9 is $b(T_9) = 5$. This observation leads us to propose the following conjecture.

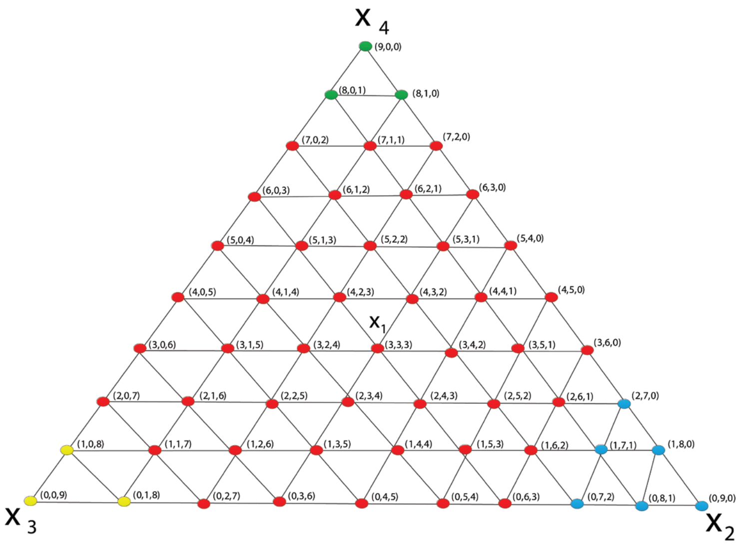

Figure 3

Triangular Grid T_9:Vertices burned by x_1 are colored in red, x_2 by blue color, x_3 by yellow color and x_4 by green color.

Conjecture 3.4: Let G be be a triangular grid T_ℓ, then $b(G) = \lceil -1 + \sqrt{3\ell + 2} \rceil$, $\ell \geq 1$.

By applying Lemma 3.1, we derive a lower bound for the burning number of the Sierpiński gasket graph, which is stated in the following theorem.

Theorem 3.5: *Let G denote the Sierpiński gasket graph S_ℓ. Then the burning number of G satisfies*

$$b(G) \geq \left\lceil 3 + \sqrt{9 + 15 \times 2^{\ell-4}} \right\rceil, \ell \geq 7.$$

Proof: Let the vertices with degree 2 be represented by v_1, v_2 and v_3 respectively in G. Also let P_1, P_2 and P_3 be the shortest paths between v_1 and v_2; v_2 and v_3; v_3 and v_1 respectively. Let B be set of all vertices in $P_1 \circ P_2 \circ P_3$. By Lemma 3.1 $b_G(B) \leq b(G)$. In order to find the lower bound for $b(G)$, we find the lower bound for $b_G(B)$. Consider an optimal burning sequence say $(x_1, x_2, \dots, x_r)$, where r represents the least number of time steps needed to burn B. Next we see that in time step r, x_j burns at most $3 \times 2^{\ell-4} + 1 - 4(j-1)$ vertices in B for $1 \leq j \leq 3$; x_k burns at most $2k+1$ vertices in B for $4 \leq k \leq r$. Hence in total $(x_1, x_2, \dots, x_r)$ burns at most $r^2 - 6r + 9 \times 2^{\ell-4}$ vertices in B. To attain the lower bound for $b_G(B)$, we have $r^2 - 6r + 9 \times 2^{\ell-4} \geq |B| = 3 \times 2^{\ell-1}$, which implies $r \geq \lceil 3 + \sqrt{9 + 15 \times 2^{\ell-4}} \rceil$. Hence $b_G(B) \geq \lceil 3 + \sqrt{9 + 15 \times 2^{\ell-4}} \rceil$. Thus $b(G) \geq b_G(B) \geq \lceil 3 + \sqrt{9 + 15 \times 2^{\ell-4}} \rceil$.

In fact finding the burning number for the Sierpinski gasket graph is really challenging.

4. Applications of Burning Number in Triangular-like Architectures

4.1 *Triangular Grid Graph*

In a forest environment, suppose wireless sensors are deployed in a triangular grid pattern T_ℓ to monitor parameters such as temperature, humidity, and fire outbreaks [12]. Each sensor represents a vertex, and communication between sensors is represented by edges. The burning number in this context can be applied to optimize the spread of alerts across the network. If a fire is detected by one sensor, the alert needs to propagate to all other sensors in the grid as quickly as possible. The burning number denotes the fewest steps necessary to guarantee that all sensors receive the notification, assuming optimal placement of initial alerts. By understanding the burning number of triangular grids, the network design can be improved to minimize latency in alert propagation. In some cases, certain sensors may have limited battery life or transmission range. Determining the burning sequence helps identify the optimal starting points for data relays to minimize energy consumption while ensuring complete coverage. For example, sensors placed at critical boundary vertices (where connectivity is lower) can be prioritized for boosting signal strength. Wireless sensor networks deployed for wildfire monitoring in regions like California or Australia often use triangular grid layouts for efficiency. Using insights from the burning number, emergency response teams can estimate how quickly alerts can spread across the network in case of a fire and critical sensors in boundary areas can be identified and reinforced to ensure full coverage in minimal time.

4.2 *Sierpinski Gasket Graph*

Sierpinski gasket graphs closely resemble hierarchical and fractal population structures. In many real-world scenarios, populations or contact networks are not uniform but exhibit fractal-like patterns, such as towns connected in hierarchical arrangements or social structures with varying levels of interaction (e.g., families, communities, and regions). Imagine a scenario where an infectious disease starts spreading in a population that follows a hierarchical pattern similar to the Sierpinski gasket. The vertices in the graph represent individuals, and edges represent potential contact between them. The boundary vertices represent individuals who are more likely to interact with the outside the group (e.g., travelers or market workers). The burning number in this context helps answer the question: How quickly can the disease reach every individual in this population, assuming optimal efforts are made to start controlling the spread at strategic points? Each burning step simulates how the disease spreads from an infected person to their direct contacts [13]. Analyzing the burning number of a graph provides an estimate of the minimum rate at which a disease can spread within a population, independent of any control measures [13]. By identifying the burning sequence, health officials can determine which key individuals or groups to monitor or vaccinate first to slow down or stop the spread. For example, in a

Sierpinski gasket graph of order S_ℓ, boundary vertices often represent the most connected individuals or groups. Protecting these vertices could delay or prevent the spread to the rest of the population. During the COVID-19 pandemic, public health authorities faced challenges in curbing disease spread within and between hierarchical social groups. Using insights from the burning number of graphs like the Sierpinski gasket, health authorities could prioritize vaccinations or restrictions in regions represented by boundary vertices and also the time required to control the spread across all regions could be predicted more effectively, allowing resource optimization.

5. Conclusion and Outlook

This study examined the burning process in specific classes of graph structures, with particular emphasis on triangular grids and Sierpiński gasket graphs. By establishing rigorous lower bounds for their burning numbers, we have provided fundamental insights into the spread dynamics within these networks. For the triangular grid, we not only derived a lower bound for its burning number but also conjectured that this bound precisely characterizes its burning number. This conjecture, if proven, would establish the exact burning number for triangular grids, offering a deeper understanding of fire propagation in such lattice structures. For the Sierpinski gasket graph, we successfully determined a lower bound for its burning number, laying the groundwork for future studies to explore tighter bounds or exact values. Given the fractal nature of the Sierpinski gasket, our findings contribute to understanding the burning process in self-similar and recursive graph structures.

Moving forward, several open problems remain that can deepen our understanding of graph burning in structured networks:

- Determining the exact burning number for Sierpinski-type graphs, extending our current lower bound results.
- Establishing the precise burning number for cube-like architectures, which play a crucial role in network structures and parallel computing.

Solving these problems would further deepen our understanding of graph burning dynamics in self-similar and high-dimensional structures, contributing to both theoretical and applied aspects of network spread models.

Conflict of Interest

There is no conflict of interest for the authors.

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