

A graph theoretic investigation of edge odd gracefulfulness in the family of Petersen graphs

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Abstract

Solairaju and Chithra [2] presented the concept of edge odd graceful labeling (EOGL) for finite simple graphs. A graph is said to be EOGL if such a labeling exists a bijective mapping a function f from edge set to the set of odd integers from 1 to $(2k - 1)$ so that every vertex is allocated, the total of the labeled edges incident to it mod $2k$, are distinct. Here k is the maximum value among p vertices and q edges. The existence of EOGL for Family of Petersen graphs is shown in this work.

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1. Introduction

The assignment of integers to a graph's vertices or edges under specified circumstances is known as graph labeling. In this work, we focus only on simple, undirected and finite graphs. here p and q are denoted as the cardinalities of vertex set $V(G)$ and edges set $E(G)$ respectively.

Lo [7] introduced the idea of edge labeling, in which each edge of a graph is given a unique number between 1 to q . All such induced vertex labels must be distinct. The corresponding vertex labels are derived by adding all the incident edges and reducing the result modulo p . The necessary and sufficient requirements for some families of pathways and cycles to admit odd edge labeling were then derived by Daoud [8].

Rosa suggested an alternative labeling method called β -valuation [1]. Solomon W. Golomb later dubbed this labeling "graceful labeling." Within this framework, an injective function $f: V(G) \rightarrow \{1, 3, 5, \dots, q\}$ is defined so that no two edges have the same label and the absolute difference between the end vertices are labeled to the edges.[5] Marimuthu, Kavitha and Balakrishnan are motivated to use the generalized Petersen graph, they discussed about the super edge magic graceful labeling of generalized Petersen graph.[3] Solairaju and Senthil Kumar are explain the EOGL of path and cycle related multi graphs. Further other notation of diagrams terminology we refer to Chartrand and Lesniak [6].

2. Preliminary

Definition 2.1: [4] A Family of Petersen graph $P(a, b)$, $a \geq 3, 1 \leq b < \frac{a}{2}$ is a $3 -$ regular graph with $2a$ vertices $\{c_1, c_2, \dots, c_a, d_1, d_2, \dots, d_a\}$ and edges $\{c_j, d_j\}, \{c_j, c_{j+1}\}, \{d_j, d_{j+b}\}$, for all $j \in \{1, 2, \dots, a\}$ where the index vaules are considered modulo a .

Proposition 2.2: The Graph G with Parameters p and q is EOGL.

Proof: Assume that the $G(p, q)$ any graph admits an EOGL f .

The sum of all the edge weight is $q^2 (p - 2)(q - 2) \leq$ The sum of all the induced vertex obtained from the edge is $\leq 2pq$

Observation 2.3: The Family of Petersen graph admits an EOGL then we have

- (i) All the vertices and edges are odd numbers less than $2k$.
- (ii) Sum of all the vertices equal to pq
- (iii) Sum of all the edges equal to q^2

3. Main Result:

In this section, We have proved that the Family of Petersen graphs $P(a, b)$ are EOGL. Here we will discuss about all the positive possible integers for a and b in the following theorem.

Theorem 3.1: For every integer $a \geq 5$, the Family of Petersen graphs $P(a, b)$ admits an EOGL.

Proof: Let $P(a, b)$ be the Family of Petersen graph with $2a$ vertices $\{c_1, c_2, \dots, c_a\}$, $\{d_1, d_2, \dots, d_a\}$ and edges $\{c_j, d_j\}$, $\{c_j, c_{j+1}\}$, $\{d_j, d_{j+b}\}$ for all $j \in \{1, 2, \dots, a\}$ where the index values are considered modulo a . Then $p = 2a$, $q = 3a$ and $k = 3a$:

Case (i): $P(a, 1)$, a is any positive integer.

Construct a Edge set function $f: E(G) \rightarrow \{1, 3, 5, \dots, (2k - 1)\}$.

$$f(c_i c_{i+1}) = 2i - 1; \text{ for } i = 1, 2, 3, \dots, a.$$

$$f(d_i d_{i+1}) = 4a + (2i - 1); \text{ for } i = 1, 2, 3, \dots, a.$$

$$f(d_{i+1} c_{i+1}) = 4a - (2i - 1); \text{ for } i = 1, 2, 3, \dots, a.$$

Hence, the induced labeling on the vertices is given below:

$f^*: V(G) \rightarrow \{1, 3, 5, \dots, (2k - 1)\}$ in the various steps as follows:

$$f^*(d_i) = 2i - 1; \text{ for } i = 1, 2, 3, \dots, a.$$

$$f^*(c_i) = 4a + (2i - 1); \text{ for } i = 1, 2, 3, \dots, a.$$

Case (ii): $P(a, b)$, a and b are any positive integer but it must satisfies the condition $1 \leq b \leq \frac{a}{2}$

Construct a Edge set function $f: E(G) \rightarrow \{1, 3, 5, \dots, (2k - 1)\}$.

$$f(c_i c_{i+b}) = (2i - 1); \text{ for } i = 1, 2, 3, \dots, (a - 1) \text{ where } i \neq (a - b) + 1$$

$$f(c_a c_b) = 2(a - b) + 1.$$

$$f(c_{(a-b)+1} c_1) = (2a - 1).$$

$$f(d_i d_{i+1}) = 2a + (2i - 1); \text{ for } i = 1, 2, 3, \dots, a.$$

$$f(d_{i+1} d_{i+1}) = 6a - (2i - 1); \text{ for } i = 1, 2, 3, \dots, a.$$

Therefore the corresponding labels of vertices will be :

$f^*: V(G) \rightarrow \{1, 2, 3, \dots, (2k - 1)\}$ in the various steps as follows:

$$f^*(d_i) = 4a + (2i - 1); \text{ for } i = 1, 2, 3, \dots, a.$$

$$f^*(c_1) = 1$$

$$f^*(c_i) = 2(a - b) + 2 + (2i - 1); \text{ for } i = 2, 3, \dots, b - 1$$

$$f^*(c_b) = 2(a - b) + 3$$

$$f^*(c_{b+i}) = 2i + 1; \text{ for } i = 1, 2, 3, \dots, (a - 1) \text{ where } i \neq (a - b) + 1.$$

$$f^*(c_{(a-b)+1}) = 2(a - b) + 1.$$

$f^*(c_a) = 5$ whenever the Family of Petersen graph satisfies this condition $\left(a, \frac{a-1}{2}\right)$

= 7 whenever the Family of Petersen graph satisfies this condition $\left(a, \frac{a-2}{2}\right)$

= 9 whenever the Family of Petersen graph satisfies this condition $\left(a, \frac{a-3}{2}\right)$

⋮
⋮
⋮

$= 2x + 3$ whenever the Family of Petersen graph satisfies this condition $\left(a, \frac{a-x}{2}\right)$ the value is fixed.

Hence the collections of edge labels and vertex labels are mutually distinct, and every edge label is an odd integer. Therefore the Family Petersen graph $P(a, b)$ is an EOGL of graphs.

Example: Figure 1 Shows that the Petersen graph $P(7, 3)$ is an Edge Odd Graceful graph.

Here $p = 14, q = 21$ with Sum of all weighted vertices are 294 and Sum of all weighted Edges 441.

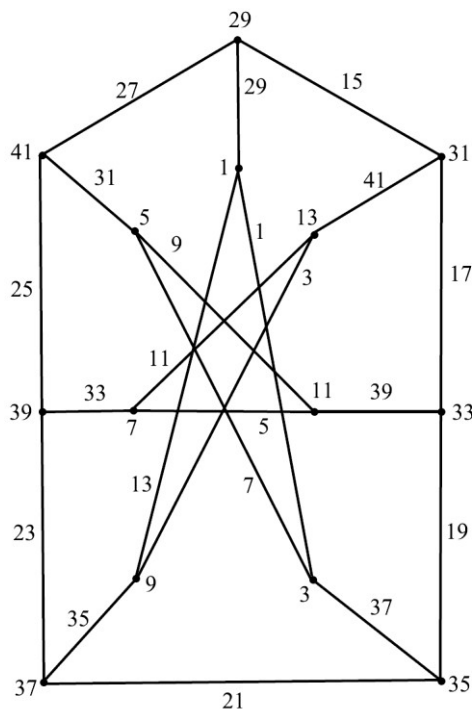


Figure 1
The Petersen graph $P(7, 3)$ is an Edge Odd Graceful graph.

Conclusion

This paper presents an EOGL of the Family of Petersen graph, while the corresponding problem for combinations of different graphs remains unresolved.

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