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Some new classes of graceful coronas and its 2nd kind

Sushree Shradhanjali Dash [†]
Amiya Kumar Behera *
Debi Prasad Bhatta [§]
Chapala Bohidar [‡]
Department of Mathematics
Siksha 'O' Anusandhan (Deemed to be University)
Bhubaneswar 751030
Odisha
India

Mohammed Siddique [@]
Department of Mathematics
Centurion University of Technology and Management
Bhubaneswar 752050
Odisha
India

Abstract

A graph labeling assigns integers to the components of a graph (vertices, edges, or both), adhering to specific conditions. Let G be a graph with q edges. Then the graceful labeling is defined as one-one mapping from vertex set of G to $\{0, 1, ..., q\}$ in this way each edge label arises due to absolute difference of vertex labels of its corresponding vertices, making all edge labels dissimilar. This paper shows that a class of coronas is graceful and of its 2nd kind under certain conditions. The corona graph considered here is a cycle graph with each or alternate vertex attached with two pendant edges.

Subject Classification: *05C78, 05C10, 05C38.*

Keywords: *Graceful labeling, 2nd kind graceful labeling, Cycle, Corona, Pendant edge.*

[†] *E-mail:* **shradhanjalidash2000@gmail.com**
* *E-mail:* **amiyabehera@soa.ac.in (Corresponding Author)**
[§] *E-mail:* **debiprasadbhatta@soa.ac.in**
[‡] *E-mail:* **chapalabohidar@soa.ac.in**
[@] *E-mail:* **siddique1807@gmail.com**

1. Introduction

Out of all the graph labeling methods, graceful labeling is the most popular, introduced by Rosa [14]. A graceful graph is allocation of integers from $\{0, 1, ..., q\}$ to its vertices such that each edge labeling, which is produced by taking the absolute difference of its corresponding vertex labeling, is a distinct integer forming the set $\{1, 2, ..., q\}$, where q is the cardinality of the edge set of the graph. Gallian [7] proposed a conjecture related to graceful labeling, which states that all trees exhibit graceful labeling. The survey highlights two well-known methods: graceful and harmonious labelings. Lau et al. [13] studied the existence of this labeling for graphs with the highest possible maximum vertex degrees. Kumar, Anbarasan and Pushparaj [11] investigated whether the Cartesian product of two paths can be edge-odd graceful. Basher [3] showed how certain types of derived graphs (splitting graphs) from common graphs like paths, cycles, and stars can be labeled in a specific way using odd integers for vertices, which in turn leads to even integers for edges when calculating their differences. This property, called odd-even graceful, is a specific area of study within graph labeling theory. Boonklurb, Ruamkaew, and Singhun [5] provided a method for orienting the edges of $C(c \times a)$ by constructing edge-graceful labeling with direction under certain conditions. Behera [4] showed that some coronas are graceful of 2nd kind under certain conditions. Sebastian and Kureethara [15] introduced pendant number of graphs by using path decomposition of a graph. Thejeshwi and Kirupa [18] addressed the concept of graceful labeling to improve the efficiency and structure of communication network addressing. Graham and Sloane [8] established a link between the maximum size of a harmonious subgraph of a complete graph and the largest size $n_\gamma(k)$ of a set of integers modulo n forming an additive basis. Shivarajkumar, Sriraj and Hedge [16] discussed that a class of unicyclic directed wheel graphs is graceful. Annamalai, Meenakshi and Adhimoolam [2] introduced the concept of "power of three acyclic graphs" and demonstrated that these graphs possess graceful labeling. Hrnciar and Haviar [9] demonstrated that all trees with a diameter of five are graceful. Kumar et al. [12] demonstrated that some graceful hairy cycle admits an alpha labeling. Jinsiqintuya [10] discussed the concept of graceful digraphs, which are directed graphs where a specific type of vertex labeling induces a unique labeling of the edges. Sumathi and Ramani [17] introduced the concept of arithmetic sequential graceful labeling. Dhananjalee and Ekanayake [6] explored graceful labeling within graph theory, specifically focusing on pendant graphs. Akerina and Sugeng [1] investigated the graceful labeling of a particular variation of the multiple-fan graph with k pendants. Kumar, Anbarasan and Pushparaj [19] proved edge-odd graceful labeling for the Cartesian product of two paths.

The present work confines graceful labeling of coronas along with 2nd kind under various conditions, which are new and different from any other previous work. It is shown that cycles with two pendant edges joined to each vertex exhibit either graceful labeling or graceful labeling of 2nd kind under certain conditions as given in respective theorems. The pendant edges are also joined either at alternate positions of a cycle to prove the corona graphs are graceful or graceful of 2nd kind with respective conditions.

2. Preliminaries

Definition 2.1: A cycle is a closed path with more than 3 vertices. A cycle with m vertices is represented as C_m.

Definition 2.2: A pendant vertex is defined as a vertex that has a degree of one. Similarly, a pendant edge can be defined as an edge if one end vertex is of degree 1.

Definition 2.3: Corona graph means a cyclic graph by attaching n pendant edges at some or all of its vertices. Usually, it is denoted as $C_m \odot nK_1$.

Definition 2.4: Let $G = (V, E)$ denotes a graph where $V(G)$ represents vertex set and $E(G)$ denotes edge set, with total number of edges being $|E(G)| = q$. A graceful labeling refers to vertex labeling defined by injective function $f : V \rightarrow [0, 1, ...q]$, producing edge labeling $\phi(xy) = |f(x) - f(y)| = \{1, 2, ..., q\}$, where $x, y \in V$. The graph satisfies the graceful labeling is known as a graceful graph.

Definition 2.5: Graceful labeling of a corona is of 2nd kind, if a pendant vertex is assigned with $2m$, where m is the number of vertices.

Figure 1 provides example of graceful labeling of 2nd kind.

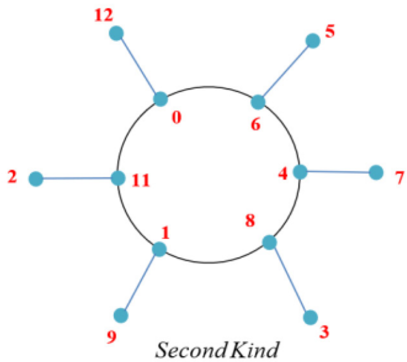

Figure 1

Graceful labeling of 2nd Kind.

3. Main Results

Theorem 3.1: *Every corona* $C_m \odot 2K_1$ *has a graceful labeling of* 2^{nd} *kind when* $m \equiv 0(mod\,4); m > 2.$

Proof: Let us consider $w_i, i=1,2,\ldots,m$ be vertices of cycle C_m and $C_m \odot 2K_1$ has $3m$ edges. Let $(x_i, y_i),\ i=1,2,\ldots,m$ be the pair of pendant vertices for the cycle C_m. Then the following functions produce the vertex labeling of $C_m \odot 2K_1$ when $m \equiv 0(mod\,4);\ m > 2.$

$$f(w_i)=\begin{cases} 0, & i=m \\ \dfrac{3i}{2}, & i=2,4,6,\ldots\ldots\ldots,m-2 \\ 3m+1-3\left(\dfrac{i+1}{2}\right), & i=1,3,5,\ldots\ldots\ldots\ldots,\dfrac{m}{2}-1 \\ 3m-3\left(\dfrac{i+1}{2}\right), & i=\dfrac{m}{2}+1,\dfrac{m}{2}+3,\ldots\ldots\ldots,m-1 \end{cases}$$

$$f(x_i)=\begin{cases} 3m, & i=m \\ 3m-\dfrac{3i}{2}, & i=2,4,\ldots\ldots\ldots,\dfrac{m}{2}-2 \\ 3\left(\dfrac{i-1}{2}\right)+1, & i=1,3,\ldots\ldots\ldots,m-1 \\ 3m-\dfrac{3}{2}i-1, & i=\dfrac{m}{2},\dfrac{m}{2}+2,\ldots\ldots\ldots,m-2 \end{cases}$$

$$f(y_i)=\begin{cases} 3m-1, & i=m \\ 3m-\dfrac{3i}{2}-1, & i=2,4,........,\dfrac{m}{2}-2 \\ 3\left(\dfrac{i-1}{2}\right)+2, & i=1,3,........,m-1 \\ 3m-\dfrac{3}{2}i-2, & i=\dfrac{m}{2},\dfrac{m}{2}+2,.......,m-2 \end{cases}$$

Its corresponding edge labeling is given by,

$$\left|f(w_i)-f(w_1)\right|=3m-2, i=m$$

$$|f(w_i)-f(w_{i+1})|=\begin{cases} \dfrac{3m}{2}, & i=m-1 \\ 3m-3i-2, & i=1,2,......,\dfrac{m}{2}-1 \\ 3m-3i-3, & i=\dfrac{m}{2},\dfrac{m}{2}+1,.......,m-2 \end{cases}$$

$$|f(w_i)-f(x_i)|=\begin{cases} 3m-3i, & i=1,2,3,......,\dfrac{m}{2}-1 \\ 3m-3i-1, & i=\dfrac{m}{2},\dfrac{m}{2}+1,.......,m-1 \\ 3m, & i=m \end{cases}$$

$$|f(w_i)-f(y_i)|=\begin{cases} 3m-3i-1, & i=1,2,3,......,\dfrac{m}{2}-1 \\ 3m-3i-2, & i=\dfrac{m}{2},\dfrac{m}{2}+1,.......,m-1 \\ 3m-1, & i=m \end{cases}$$

Since the edge labeling of the above graph belongs to $\{1,2,3,...,3m\}$ with pendant vertex assigned the label $2m$. Hence the graph satisfies graceful labeling of 2^{nd} kind. ■

Illustration 3.1: If $m\equiv 0(mod\,4)$; $m>2$, let $m=16$. Then $C_{16}\odot 2K_1$ has a graceful labeling of 2^{nd} kind, which is explained in Figure 2 as follows.

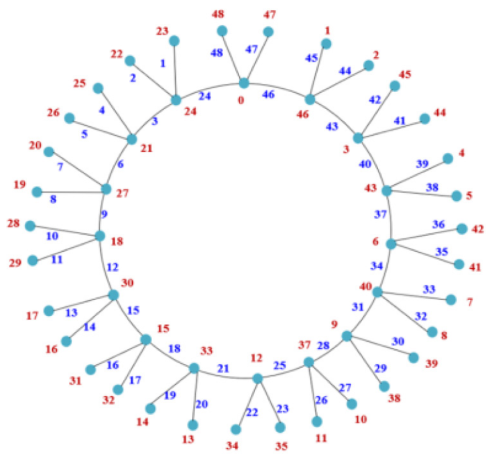

Figure 2

$C_{16} \odot 2K_1$ with graceful labeling of 2^{nd} kind.

From Figure 2, it is clear that the graph $C_{16} \odot 2K_1$ is graceful of 2^{nd} kind since each edge is labeled with dissimilar integers belonging to $\{1,2,3,...,48\}$ and 32 is assigned to the pendant vertex.

Theorem 3.2: *Every corona* $C_m \odot 2K_1$ *has a graceful labeling, when* $m \equiv 3(mod\, 4)$; $m > 2$.

Proof: Let us consider $w_i, i = 1,2,...,m$ be vertices of cycle C_m and $C_m \odot 2K_1$ has $3m$ edges. Let (x_i, y_i), $i = 1,2,...,m$ be the pair of pendant vertices for the cycle C_m. Then the following functions produce the vertex labeling of $C_m \odot 2K_1$ when $m \equiv 3(mod\, 4)$; $m > 2$.

$$f(w_i) = \begin{cases} 0, & i = m \\ \dfrac{3i}{2}, & i = 2,4,........,m-1 \\ 3m+1-3\left(\dfrac{i+1}{2}\right), & i = 1,3,.........,\left\lceil \dfrac{m}{2} \right\rceil - 1 \\ 3m-3\left(\dfrac{i+1}{2}\right), & i = \left\lceil \dfrac{m}{2} \right\rceil + 1, \left\lceil \dfrac{m}{2} \right\rceil + 3,.......,m-2 \end{cases}$$

$$f(x_i)=\begin{cases} 3m, & i=m \\ 3m-\dfrac{3i}{2}, & i=2,4,\ldots\ldots..,\left\lceil\dfrac{m}{2}\right\rceil-2 \\ 3\left(\dfrac{i-1}{2}\right)+1, & i=1,3,5,\ldots\ldots..,m-2 \\ 3m-\dfrac{3}{2}i-1, & i=\left\lceil\dfrac{m}{2}\right\rceil,\left\lceil\dfrac{m}{2}\right\rceil+2,\ldots\ldots.,m-1 \end{cases}$$

$$f(y_i)=\begin{cases} 3m-1, & i=m \\ 3m-\dfrac{3i}{2}-1, & i=2,4,6,\ldots\ldots..,\left\lceil\dfrac{m}{2}\right\rceil-2 \\ 3\left(\dfrac{i-1}{2}\right)+2, & i=1,3,\ldots\ldots..,m-2 \\ 3m-\dfrac{3}{2}i-2, & i=\left\lceil\dfrac{m}{2}\right\rceil,\left\lceil\dfrac{m}{2}\right\rceil+2,\ldots\ldots.,m-1 \end{cases}$$

Its corresponding edge labeling is given by

$$\left|f(w_i)-f(w_1)\right|=3m-2, i=m$$

$$|f(w_i)-f(w_{i+1})|=\begin{cases} \dfrac{3m-3}{2}, & i=m-1 \\ 3m-3i-2, & i=1,2,\ldots,\left\lceil\dfrac{m}{2}\right\rceil-1 \\ 3m-3i-3, & i=\left\lceil\dfrac{m}{2}\right\rceil,\left\lceil\dfrac{m}{2}\right\rceil+1,\ldots,m-2 \end{cases}$$

$$|f(w_i)-f(x_i)|=\begin{cases} 3m-3i, & i=1,2,3,\ldots\ldots,\left\lceil\dfrac{m}{2}\right\rceil-1 \\ 3m-3i-1, & i=\left\lceil\dfrac{m}{2}\right\rceil,\left\lceil\dfrac{m}{2}\right\rceil+1,\ldots\ldots.,m-1 \\ 3m, & i=m \end{cases}$$

$$|f(w_i)-f(y_i)|=\begin{cases} 3m-3i-1, & i=1,2,3,\ldots\ldots,\left\lceil \frac{m}{2}\right\rceil-1 \\ 3m-3i-2, & i=\left\lceil \frac{m}{2}\right\rceil,\left\lceil \frac{m}{2}\right\rceil+1,\ldots\ldots,m-1 \\ 3m-1, & i=m \end{cases}$$

Edge labeling belongs to $\{1,2,\ldots,3m\}$. Hence the graph is graceful.

Illustration 3.2: If $m\equiv 3(mod\,4); m>2$, let $m=15$. Then $C_{15}\odot 2K_1$ has a graceful labeling, which is explained in Figure 3 as follows.

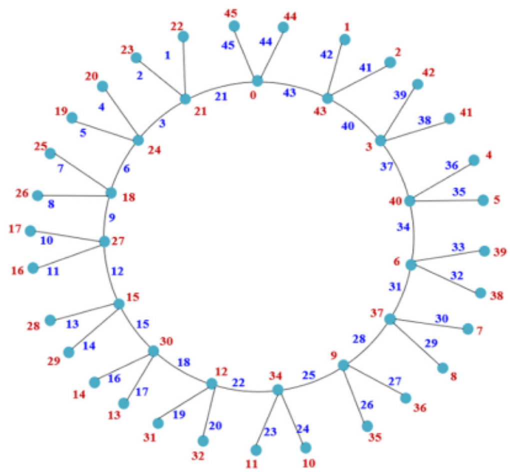

Figure 3

$C_{15}\odot 2K_1$ **with graceful labeling.**

From Figure 3, it is clear that the graph $C_{15}\odot 2K_1$ is graceful, since each edge is labeled with dissimilar integers belonging to $\{1,2,3,\ldots,45\}$.

Theorem 3.3: *Every alternate corona* $C_m\odot 2K_1$ *has a graceful labeling of* 2^{nd} *kind, if* $m\equiv 0(mod\,8)$; $m>3$.

Proof: Let us consider $w_i, i=1,2,\ldots,m$ be vertices of cycle C_m and $C_m\odot 2K_1$ has $2m$ edges. Let (x_i, y_i), $i=1,2,\ldots,m$ be the pair of alternate pendant vertices for the cycle C_m. Then the following functions produce the vertex labeling of $C_m\odot 2K_1$ when $m\equiv 0(mod\,8)$; $m>3$.

$$f(w_i)=\begin{cases} 0, & i=m-1 \\ 2m-2, & i=m \\ \dfrac{m}{2}, & i=m-2 \\ \dfrac{i+1}{2}, & i=1,3,5,.........,m-3 \\ 2m-2-\dfrac{3i}{2}, & i=2,4,6,.........,\dfrac{3m}{4}-2 \\ 2m-3-\dfrac{3i}{2}, & i=\dfrac{3m}{4},\dfrac{3m}{4}+2,\dfrac{3m}{4}+4,.........,m-4 \end{cases}$$

$$f(x_i)=\begin{cases} 2m, & i=m-1 \\ 2m-1, & i=m \\ 2m-3(\dfrac{i+1}{2}), & i=1,3,5,..........,\dfrac{3m}{4}-3 \\ 2m-3(\dfrac{i+1}{2})-1, & i=\dfrac{3m}{4}-1,\dfrac{3m}{4}+1,.......m-3 \\ 2m-\dfrac{3i}{2}-1, & i=2,4,6,........,\dfrac{3m}{4}-2 \\ 2m-\dfrac{3i}{2}-2, & i=\dfrac{3m}{4},\dfrac{3m}{4}+2,\dfrac{3m}{4}+4,.........,m-2 \end{cases}$$

Its corresponding edge labeling is given by

$$\left|f(w_i)-f(w_1)\right|=2m-3, i=m$$

$$|f(w_i)-f(w_{i+1})|=\begin{cases} 2m-2, & i=m-1 \\ \dfrac{m}{2}, & i=m-2 \\ 2m-2i-4, & i=1,3,5,............,\dfrac{3m}{4}-3 \\ 2m-2i-5, & i=\dfrac{3m}{4}-1,\dfrac{3m}{4}+1,.......m-3 \\ 2m-2i-3, & i=2,4,6,........,\dfrac{3m}{4}-2 \\ 2m-2i-4, & i=\dfrac{3m}{4},\dfrac{3m}{4}+2,\dfrac{3m}{4}+4,.........,m-4 \end{cases}$$

$$|f(w_i)-f(x_i)|=\begin{cases} 2m-2i-2, & i=1,3,5,\ldots\ldots\ldots..,\dfrac{3m}{4}-3 \\ 2m-2i-3, & i=\dfrac{3m}{4}-1,\dfrac{3m}{4}+1,\ldots\ldots.m-3 \\ 2m, & i=m-1 \end{cases}$$

$$|f(w_i)-f(x_{i+1})|=\begin{cases} 2m-2i-3, & i=1,3,5,\ldots\ldots\ldots..,\dfrac{3m}{4}-3 \\ 2m-2i-4, & i=\dfrac{3m}{4}-1,\dfrac{3m}{4}+1,\ldots\ldots.m-3 \\ 2m-1, & i=m-1 \end{cases}$$

Since labeled edges belong to $\{1,2,3,\ldots,2m\}$ with pendant vertex assigned the label $2m$. Hence the graph satisfies graceful labeling of 2^{nd} kind.

Illustration 3.3: If $m\equiv 0(mod\,8)$; $m>3$, let $m=16$. Then $C_{16}\odot 2K_1$ has a graceful labeling of 2^{nd} kind, which is explained in Figure 4 as follows.

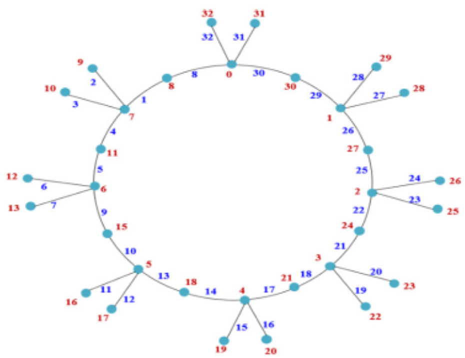

Figure 4

$C_{16}\odot 2K_1$ **with graceful labeling of 2^{nd} kind.**

From Figure 4, it is clear that the above corona $C_{16}\odot 2K_1$ is graceful of 2^{nd} kind, since each edge is labeled with dissimilar integers belonging to $\{1,2,3,\ldots,32\}$ and 32 is assigned to the pendant vertex.

Theorem 3.4: *Every alternate corona* $C_m \odot 2K_1$ *has a graceful labeling of* 2^{nd} *kind, if* $m \equiv 2(mod\,8)$; $m > 3$.

Proof: Let us consider $w_i, i = 1,2,...,m$ be vertices of cycle C_m and $C_m \odot 2K_1$ has $2m$ edges. Let (x_i, y_i), $i = 1,2,...,m$ be the pair of alternate pendant vertices for the corona C_m. Then the following functions produce the vertex labeling of $C_m \odot 2K_1$ when $m \equiv 2(mod\,8)$; $m > 3$.

$$f(w_i) = \begin{cases} 0, & i = m-1 \\ 2m-2, & i = m \\ \dfrac{m+2}{2}, & i = m-2 \\ \dfrac{i+1}{2}, & i = 1,3,.........,m-3 \\ 2m-2-\dfrac{3i}{2}, & i = 2,4,6,.........,\dfrac{3m-2}{4}-3 \\ 2m-3-\dfrac{3i}{2}, & i = \dfrac{3m-2}{4}-1, \dfrac{3m-2}{4}+1, \dfrac{3m}{4}+4,.........,m-4 \end{cases}$$

$$f(x_i) = \begin{cases} 2m, & i = m-1 \\ 2m-1, & i = m \\ 2m-3(\dfrac{i+1}{2}), & i = 1,3,5,..........,\dfrac{3m-2}{4}-2 \\ 2m-3(\dfrac{i+1}{2})-1, & i = \dfrac{3m-2}{4}, \dfrac{3m-2}{4}+2,.......m-3 \\ 2m-\dfrac{3i}{2}-1, & i = 2,4,6,........,\dfrac{3m-2}{4}-1 \\ 2m-\dfrac{3i}{2}-2, & i = \dfrac{3m-2}{4}+1, \dfrac{3m-2}{4}+3,.........,m-4 \\ 2m-\dfrac{3i}{2}-3, & i = m-2 \end{cases}$$

Its corresponding edge labeling is given by

$$\left| f(w_i) - f(w_1) \right| = 2m-3, i = m$$

$$|f(w_i)-f(w_{i+1})|=\begin{cases} 2m-2, & i=m-1 \\ \dfrac{m+2}{2}, & i=m-2 \\ 2m-2i-4, & i=1,3,5,..........,\dfrac{3m-2}{4}-4 \\ 2m-2i-5, & i=\dfrac{3m-2}{4}-2,\dfrac{3m-2}{4},.......m-5 \\ 2m-2i-3, & i=2,4,6,........,\dfrac{3m-2}{4}-3 \\ 2m-2i-4, & i=\dfrac{3m-2}{4}-1,\dfrac{3m-2}{4}+1,\dfrac{3m}{4}+4,........,m-4 \\ 2m-2i-4, & i=m-3 \end{cases}$$

$$|f(w_i)-f(x_i)|=\begin{cases} 2m-2i-2, & i=1,3,5,..........,\dfrac{3m-2}{4}-2 \\ 2m-2i-3, & i=\dfrac{3m-2}{4},\dfrac{3m-2}{4}+2,.......m-3 \\ 2m, & i=m-1 \end{cases}$$

$$|f(w_i)-f(x_{i+1})|=\begin{cases} 2m-2i-3, & i=1,3,5,..........,\dfrac{3m-2}{4}-2 \\ 2m-2i-4, & i=\dfrac{3m-2}{4},\dfrac{3m-2}{4}+2,.......m-5 \\ 2m-2i-5, & i=m-3 \\ 2m-1, & i=m-1 \end{cases}$$

Since labeled edges belong to $\{1,2,3,...,2m\}$ with pendant vertex assigned the label $2m$. Hence the graph satisfies graceful labeling of 2nd kind.

Illustration 3.4: If $m\equiv 2(mod\,8)$; $m>3$, let $m=18$. Then $C_{18}\odot 2K_1$ has a graceful labeling of 2nd kind, which is explained in Figure 5 as follows.

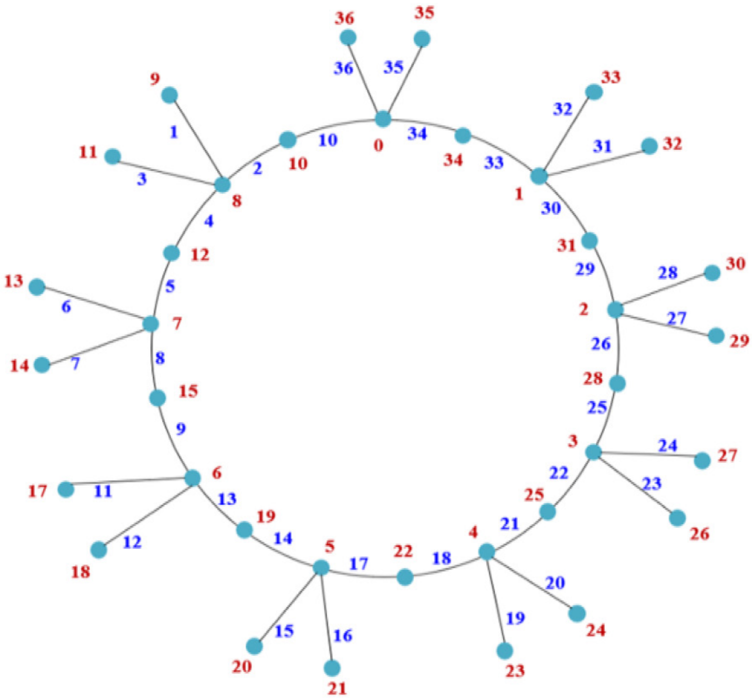

Figure 5

$C_{18} \odot 2K_1$ with graceful labeling of 2^{nd} kind.

From Figure 5, it is clear that the above corona $C_{18} \odot 2K_1$ is graceful of 2^{nd} kind since each edge is labeled with dissimilar integers belonging to $\{1,2,3,\ldots,36\}$ and 36 is assigned to the pendant vertex.

Theorem 3.5: *Every alternate corona $C_m \odot 2K_1$ has a graceful labeling of 2^{nd} kind, if $m \equiv 4(mod\, 8)$; $m > 3$.*

Proof: Let us consider $w_i, i = 1,2,\ldots,m$ be vertices of cycle C_m and $C_m \odot 2K_1$ has $2m$ edges. Let $(x_i, y_i), i = 1,2,\ldots,m$ be the pair of alternate pendant vertices for the cycle C_m. Then the following functions produce the vertex labeling of $C_m \odot 2K_1$ when $m \equiv 4(mod\, 8)$; $m > 3$.

$$f(w_i)=\begin{cases}0, & i=m-1\\ 2m-2, & i=m\\ \frac{m}{2}, & i=m-2\\ \frac{i+1}{2}, & i=1,3,.........,m-3\\ 2m-2-\frac{3i}{2}, & i=2,4,.........,\frac{3m}{4}-3\\ 2m-3-\frac{3i}{2}, & i=\frac{3m}{4}-1,\frac{3m}{4}+1,\frac{3m}{4}+3,.........,m-4\end{cases}$$

$$f(x_i)=\begin{cases}2m, & i=m-1\\ 2m-1, & i=m\\ 2m-3(\frac{i+1}{2}), & i=1,3,5,..........,\frac{3m}{4}-2\\ 2m-3(\frac{i+1}{2})-1, & i=\frac{3m}{4},\frac{3m}{4}+2,.......m-3\\ 2m-\frac{3i}{2}-1, & i=2,4,6,........,\frac{3m}{4}-1\\ 2m-\frac{3i}{2}-2, & i=\frac{3m}{4}+1,\frac{3m}{4}+3,.........,m-2\end{cases}$$

Its corresponding edge labeling is given by

$$|f(w_i)-f(w_1)|=2m-3, i=m$$

$$|f(w_i)-f(w_{i+1})|=\begin{cases}2m-2, & i=m-1\\ \frac{m}{2}, & i=m-2\\ 2m-2i-4, & i=1,3,5,............,\frac{3m}{4}-4\\ 2m-2i-5, & i=\frac{3m}{4}-2,\frac{3m}{4},\frac{3m}{4}+2,.......m-5\\ 2m-2i-3, & i=2,4,6,........,\frac{3m}{4}-3\\ 2m-2i-4, & i=\frac{3m}{4}-1,\frac{3m}{4}+1,.........,m-4\end{cases}$$

$$|f(w_i)-f(x_i)|=\begin{cases}2m-2i-2, & i=1,3,5,\ldots\ldots\ldots\ldots,\frac{3m}{4}-2\\ 2m-2i-3, & i=\frac{3m}{4},\frac{3m}{4}+2,\ldots\ldots.m-3\\ 2m, & i=m-1\end{cases}$$

$$|f(w_i)-f(x_{i+1})|=\begin{cases}2m-2i-3, & i=1,3,5,\ldots\ldots\ldots\ldots,\frac{3m}{4}-2\\ 2m-2i-4, & i=\frac{3m}{4},\frac{3m}{4}+2,\ldots\ldots.m-3\\ 2m-1, & i=m-1\end{cases}$$

Since labeled edges belong to $\{1,2,3,\ldots,2m\}$ with pendant vertex assigned the label $2m$. Hence, the graph satisfies graceful labeling of 2nd kind.

Illustration 3.5: If $m\equiv 4(mod\ 8)$; $m>3$, let $m=12$. Then $C_{12}\odot 2K_1$ has a graceful labeling of 2nd kind, which is explained in Figure 6 as follows.

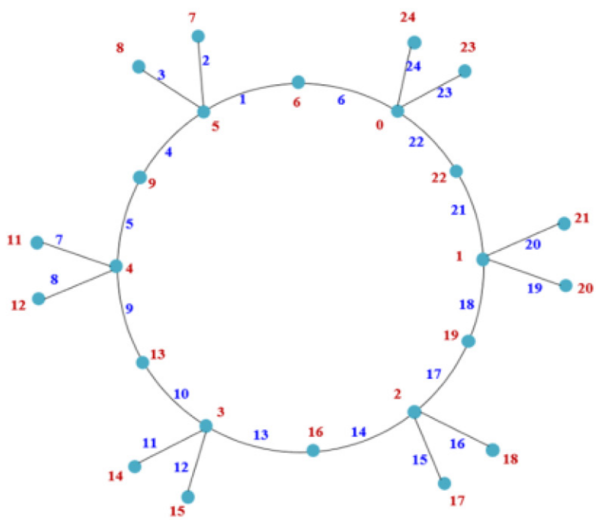

Figure 6

$C_{12}\odot 2K_1$ with graceful labeling of 2nd kind.

From Figure 6, it is clear that the graph $C_{12} \odot 2K_1$ is graceful of 2^{nd} kind, since each edge is labeled with dissimilar integers belonging to $\{1, 2, 3, ..., 24\}$ and 24 is assigned to the pendant vertex.

5. Discussion and Conclusion

This paper awakens the graceful labeling of a class of coronas satisfying certain conditions, along with the graceful labeling of 2^{nd} kind. In some cases, it has been proven that a class of coronas, which are formed by attaching two pendant edges of a cycle, are either graceful or have a graceful labeling of 2^{nd} kind. Furthermore, coronas with alternate pendant edges either have a graceful labeling or a graceful labeling of 2^{nd} kind. This work will help a researcher to extend the coronas either by attaching more than two pendant edges at each vertex or every possible alternate vertex of a cycle to prove it graceful or graceful labeling of the 1^{st}, 2^{nd} and 3^{rd} kind. Also, one can extend the pendant edges of a cycle more than once to verify the graceful labeling of all kinds.

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