

Some generalizations of derivations in posets

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Abstract

Consider a poset T and let F be a function from T to itself. The primary objective of this paper is to present and examine the concept of (F, F) -derivations on T , serving as a generalization of the concept of derivations on T . Additionally, we thoroughly examine the characteristics of the (F, F) -derivations on a poset. Moreover, we provide several examples to show that the different limitations included in the hypothesis of our results are not unnecessary.

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1. Introduction

Although derivation was first applied to rings and algebras many years ago, (viz.; [1], [2], [3], [4], [5], [6], [7], [8], and [9]). The standard derivative has proven to be a valuable tool in many mathematical science fields. But by 1957, Herstein [10] and Posner [11] had simultaneously obtained several significant results, particularly for prime rings, which gave rise to a new interest in the study of algebraic derivations. Many researchers have studied lattices in recent years using generalized types and their derivations to corroborate several important discoveries (see, for example, [12], [13], [14], [15], [16], [17], [18], [19] and their references). Xin [20] studied the notion of Lattice-specific fixed sets of derivations in

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2012 and came up with a few exceptional results. Zhang and Li [21] created the idea of derivation on posets in 2017 and established a few essential attributes of ideals and procedures associated with derivations. Some researchers generalized derivations on posets (for example, see [22] and [23]).

A few notations will be presented for the remaining of the section. Throughout this paper $(T, \leq)$ always denotes a poset. The notation for an element $s \in T$ is $\downarrow s = \{r \in T : r \leq s\}$ as well as $\uparrow s = \{r \in T : s \leq r\}$. For a subset $U \subseteq T$, we denote $l(U) = \{t \in T : t \leq w \text{ for each } w \in U\}$ the "lower cone" of U and $u(U) = \{t \in T : w \leq t \text{ for every } w \in U\}$ the "upper cone" of U . Immediately, it is clear that both have monotone compositions $u(l(-))$ and $l(u(-))$ and are antitones. Additionally, we get $u(l(u(-))) = u(-)$ and $l(u(l(-))) = l(-)$. It is possible to define $l(W) = l(a_1, a_2, \dots, a_n)$ as well as $u(W) = u(a_1, a_2, \dots, a_n)$ if $W = \{a_1, a_2, \dots, a_n\}$ is a finite subset of T . Also, when $U_1 \subseteq T$ and $U_2 \subseteq T$, therefore $l(U_1, U_2)$ and $u(U_1, U_2)$ both will be indicated for $l(U_1 \cup U_2)$ and $u(U_1 \cup U_2)$ respectively. $\downarrow W = \{v \in T : v \leq w \text{ for some } w \in W\}$. is what we write for a subset $W \subseteq T$.

This is how the paper is structured. There are various definitions and results in Section 2 that are crucial for the following sections. The concept of (F, F) -derivations of partially ordered structures is introduced in Section 3 together with define some crucial characteristics involving recently established concepts. Section 4: In Section 4, we further investigate the fixed sets with (F, F) -derivations on posets. Eventually, at last, Section 5 not only demonstrates basic findings connected to By demonstrating the ideals of T , two (F, F) -derivations on posets always commute, as we demonstrate. These results apply to the outcomes of of Zhang and Li's derivations are cited by [21].

2. Basic Concepts

Some of the definitions and conclusions in this part can be applied to the following sections. Let's talk about the next one first.

Definition 2.1: ([24]) Let $(T, \leq)$ be a partially ordered set and $g: T \rightarrow T$ be a map. We say that g is increasing if $s \leq t$, then $g(s) \leq g(t)$ holds for each $s, t \in T$.

Definition 2.2: ([24]) Consider T be a poset and $g: T \rightarrow T$ exist as a map. If g is "increasing", "bijective" together with its inverse function g^{-1} is "increasing", then g is referred to be an automorphism.

Remark 2.3: Recall that $g: T \rightarrow T$ is referred to as an automorphism, meaning that $g(0) = 0$ if The smallest element in the partially ordered set $(T, \leq)$ is 0.

Definition 2.4: ([25]) Given a poset T , $z \in T$ and $S \subseteq T$. If $s \leq z$ for each $s \in S$, so we say that z is an upper bound. A subset S of T is directed if it is not empty and there is an upper bound in S for every finite subset of S . (From the non-empty, it suffices to suppose that there is an upper bound in S for each pair of elements in S). Additionally, if $S = \downarrow S$, we call S "a lower set". Suppose J_1 , a subset of T , is "directed lower set", it is called an ideal.

Definition 2.5: ([26]) Assume that T is a partially ordered set. A map $g: T \rightarrow T$ is said to be a lower homomorphism on T if $u(g(l(s, t))) = u(l(g(s), g(t)))$ holds for every $s, t \in T$.

Keep in mind that Zhang and Li [21] started researching "derivations" on a poset T in 2017 and provided the following definition:

Definition 2.6: [21, definition 2.1] Consider a poset T and a map $G: T \rightarrow T$. If G satisfies the following requirements, we refer to it as "a derivation" on T :

- (i) $G(l(s, t)) = l(u(l(G(s), t), l(s, G(t))))$ for each s and t in T ;
- (ii) $l(G(u(r, s))) = l(u(G(r), G(s)))$ for each s and t in T .

Corollary 2.7: ([21], Theorem 2.2) Let T represent a poset and $G: S \rightarrow S$ represent a function. Given T , G is "a derivation" if and only if

- (i) $G(l(r, s)) = l(Gr, s)) = l(r, Gs)$ for all $r, s \in T$;
- (ii) $l(G(u(r, s))) = l(u(Gr, Gs))$ for all $r, s \in T$.

A derivation of T is G . The fixed set of G in T is represented by $\text{Fix}_G(T)$ and defined by $\{t \in T: G(t) = t\}$.

Corollary 2.8: [21, Proposition 3.3] Let G be a derivation of T , where 0 is the smallest element. Then the subsequent hold:

- (i) $0 \in \text{Fix}_G(T)$;
- (ii) $s \in \text{Fix}_G(T)$, whenever $t \in \text{Fix}_G(T)$, and $s \leq t$;
- (iii) For each $r, s \in \text{Fix}_G(T)$, if T is directed, there is $z \in \text{Fix}_G(T)$ such that $r \leq z, s \leq z$.

Corollary 2.9: [21, Corollary 3.3] Suppose that T is a directed poset with the value 0 as the least element, thus an ideal of T is $\text{Fix}_G(T)$.

Corollary 2.10: [[21], Theorem 3.4] Let G be "a derivation" on T . Next, for any $s \in T$, $\text{Fix}_G(T) \cap l(s) = l(G(s))$.

Corollary 2.11: [21, Theorem 4.4] If there are two derivations of T , G_1 and G_2 . Then they commute.

Corollary 2.12: [21, Theorem 4.5] Let σ_1 and σ_2 be two derivations on T . Then $\sigma_1 \leq \sigma_2$ if and only if $\sigma_2 \sigma_1 = \sigma_1$.

3. On "(F, F)-derivations" of posets

The study of "(F, F)-derivations" on points is covered in this section. We define the term "(F, F)-derivations on posets". We introduce the concept of "(F, F)-derivations" on posets in the following manner, drawing inspiration from the definition of "derivations" in posets [21, Definition 2.1]:

Definition 3.1: Consider a poset T and $g, F: T \rightarrow T$ are maps. If g meets the following criteria, we say that it is "a (F, F)-derivation" on T .

- (i) $g(l(s, z)) = l(u(l(g(F(s))), z), l(s, g(F(t))))$ for every $s, z \in T$;
- (ii) $l(g(u(s, z))) = l(u(g(s), g(z)))$ for every $s, z \in T$.

Remark 3.2: Obviously, every derivation is a (F, F)-derivation on T when $F = id_T$ where id_T is the identity map on T . However, generally, the opposite is not true. The example that follows aids this fact.

Example 3.3: Let a poset $(T, \leq) = (\{0, 1, 2\}, \leq)$. Define the functions

$g, F: \{0, 1, 2\} \rightarrow \{0, 1, 2\}$ by $F(z) = 0 = g(z)$ where $z \in \{0, 1\}$ and $F(2) = g(2) = 1$. After that, verifying that g is a (F, F)-derivation on T is simple. However, g is not a derivation on T .

To validate the main theorem in this section, we require a few auxiliary results, which will be applied in the follow-up. Let's start by the next lemma which an extension [21, Proposition 2.1].

Lemma 3.4: *Let T be a poset and $F:T \rightarrow T$ be a function. If g is "a (F, F) -derivation" on T . Thus, the following claims are valid:*

- (i) $g(z) \leq z$ for every $z \in T$.
- (ii) $g(s) \leq g(z)$ whenever $s \leq z$, for every $s, z \in T$.
- (iii) If T has an ideal J , then $g(J) \subseteq J$.
- (iv) If the element 0 is the least in T , then $g(0) = 0$.

Proof:

- (i) Since $g:T \rightarrow T$ is a (F, F) -derivation. Then, we have

$$\begin{aligned} g(l(w_1, w_1)) &= l(u(l(g(F(w_1))), w_1), l(w_1, g(F(w_1)))) \\ &= l(u(l(g(F(w_1))), w_1)) \\ &= l(g(F(w_1)), w_1) \text{ for every } w_1 \in T. \end{aligned}$$

So, we have $g(l(w_1)) = g(l(w_1, w_1)) = l(g(F(w_1)), w_1)$ for each $w_1 \in T$. But, $g(w_1) \in g(l(w_1)) = g(l(w_1, w_1))$, and hence we get $g(w_1) \in l(g(F(w_1)), w_1)$. Thus $g(w_1) \leq w_1$ for all $w_1 \in T$.

- (ii) Suppose that $s, z \in T$ such that $s \leq z$. Next, we're going to

$$\begin{aligned} l(g(u(s, z))) &= l(g(u(z))) \\ &= l(u(g(s), g(z))). \end{aligned}$$

But $g(s) \in l(u(g(s), g(z)))$, so $g(s) \in l(g(u(z)))$. Hence, for any $s, z \in T$, $g(s) \leq g(z)$.

- (iii) There exists $a \in J$ such that $g(a) = z$, so $z \in g(J)$ and J is an ideal of T . Using (i) to obtain $g(j) \leq j$ so $z \leq j$, but However, J is T 's ideal. Hence $z \in J$. This has shown $g(J) \subseteq J$.

- (iv) If the least element 0 is in T . Then by applying (i), we get $0 \leq g(0) \leq 0$. Hence $g(0) = 0$.

Lemma 3.5: *Let T be a poset, $F:T \rightarrow T$ be a function and g be a (F, F) -derivation on T . The following claims are therefore accurate:*

- (i) If $g(l(s)) = l(t)$, then $g(s) = t$ for each $s, t \in T$;
- (ii) If $g(u(s)) = u(t)$, then $g(s) = z$ for all $s, t \in T$.

Proof:

- (i) Be certain that $g(l(s)) = l(t)$. Consequently, we get $t \in g(l(s))$ since $t \in l(t)$. Consequently, $z \in l(s)$ exists such that $g(z) = t$. By

$z \in l(s)$ then $g(z) \leq g(s)$, the above relation yields that $t \leq g(s)$. Conversely $g(s) \in g(l(s)) = l(t)$, so $g(s) \leq t$. Hence $g(s) = t$ for every $s, t \in T$.

- (ii) By using a comparable method with the required modification, we may demonstrate (ii).

The following theorem, which is the primary result of this section, is extended in [21, Theorem 2.2].

Theorem 3.6: Suppose T is a poset as well as $g, F: T \rightarrow T$ are functions such that for any $r \in T$, $r \leq F(r)$, and F is an automorphism. g is therefore a " (F, F) -derivation" of T if and only if

- (i) $g(l(s, z)) = l(g(F(s)), z) = l(s, g(F(z)))$ for every $s, z \in T$;
(ii) $l(g(u(s, z))) = l(u(g(s), g(z)))$ for every $s, z \in T$.

Proof: Demonstrating the identity between the condition (i) in Definition 3.1 and the condition (i) in this theorem is the just remaining step. Assuming first that this theorem's condition is met, then for all $s, w \in T$.

$$\begin{aligned} g(l(s, w)) &= l(g(F(s)), w) = l(s, g(F(w))), \\ &= l(u(l(g(F(s)), w))) \\ &= l(u(l(g(F(s)), w), l(s, g(F(w)))). \end{aligned}$$

Secondly, Assuming that g is "a (F, F) -derivation" on T , we obtain

$$\begin{aligned} l(g(F(s)), w) &= l(u(l(g(F(s)), w))) \subseteq l(u(l(g(F(s)), w), l(s, g(F(w)))) \\ &= g(l(s, w)) \text{ for all } s, w \in T. \end{aligned}$$

This implies that

$$l(g(F(s)), w) \subseteq g(l(s, w)) \text{ for each } s, w \in T. \quad (1)$$

Conversely, let $z \in g(l(s, w))$, then there is $v \in l(s, w)$ satisfying $g(v) = z$. By Lemma 3.4 (i), (ii), and $id_T \leq F$ on T we have $g(v) \leq g(s) \leq g(F(s))$, $g(v) \leq g(w) \leq w$. So $z = g(v) \in l(g(F(s)), w)$. Thus, we get

$$g(l(s, w)) \subseteq l(g(F(s)), w) \text{ for each } s, w \in T. \quad (2)$$

Using Eqs. (1) and (2), we get $g(l(s, w)) = l(g(F(s)), w)$, for any $s, w \in T$. Similarly we can prove $g(l(s, w)) = l(s, g(F(w)))$, for all $s, w \in T$.

When we enter $F = id_T$, From Theorem 3.6, Corollary 2.7 follows immediately.

Theorem 3.7: Let g be a (F, F) -derivation associated with a function F on T and let $id_T \leq F$ of T . Then, for each $s, z \in T$, $g(l(s, z)) = l(g(s), g(z))$.

Proof: First assume that $w \in l(g(s), g(z))$. Then, $w \leq g(s)$, $w \leq g(z)$. Using $id_T \leq F$ on T and Lemma 3.4 (i), we obtain $w \leq g(F(s))$ and $w \leq z$. Then we have $w \in l(g(F(s)), z)$. By Theorem 3.6(i), $l(g(s), g(z)) \subseteq g(l(s, z))$. Conversely, we suppose that $w \in g(l(s, z))$, so there is $v \in l(s, z)$ such that $g(v) = w$. Thus by Lemma 3.4(ii), we obtain $w = g(v) \leq g(s)$, $w = g(v) \leq g(z)$. Thus $w \in l(g(s), g(z))$. Hence $g(l(s, z)) \subseteq l(g(s), g(z))$ for all $s, z \in T$. This implies that $g(l(s, z)) = l(g(s), g(z))$ for every $s, z \in T$.

4. The fixed points of (F, F) -derivations on posets

Consider that T is a poset in this section, $F: T \rightarrow T$ is a map such that $id_T \leq F$ on T , and a (F, F) -derivation of T is g . An element $s \in T$ is known as a (F, F) -fixed point of g in T if $g(s) = F(s)$ or $g(s) = s$. We will denote $Fix_{g,F}(T)$ for the set $\{s \in T: g(s) = F(s) \text{ or } g(s) = s\}$ of all (F, F) -fixed points of g in T , and $g(T) = \{g(s): s \in T\}$. Obviously, $Fix_g(T) = \{s \in T: g(s) = s\} = Fix_{g, id_T}(T)$.

This section's first result generalizes Theorem 3.1 in [21], is now proven.

Theorem 4.1: Given a poset T , let $F: T \rightarrow T$ be a bijective map such that $t \leq F(t)$ for all $t \in T$. Also, let g be a " (F, F) -derivation" on T such that F commutes with g on T . Then the next claims are true:

- (i) $g(z) \in Fix_{g,F}(T)$ for all $z \in T$;
- (ii) $Fix_{g,F}(T) = g(T)$.

Proof: (i) To prove that for each $w_1 \in T$, $g(w_1) \in Fix_{g,F}(T)$. Demonstrating that $g(g(w_1)) = F(g(w_1)) = g(F(w_1))$ or $g(g(w_1)) = g(w_1)$, for all $w_1 \in T$, is sufficient. Based on Lemma 3.4 and Theorem 3.6 (i), we have

$$\begin{aligned} g(g(l(w_1))) &= g(g(l(w_1, w_1))) \\ &= g(l(w_1, g(F(w_1)))) \\ &= l(g(F(w_1)), g(F(w_1))) \\ &= l(g(F(w_1))) \text{ for all } w_1 \in T \end{aligned}$$

The use of Lemma 3.7 and Lemma 3.5 implies that $g(g(w_1)) = g(F(w_1))$ for all $w_1 \in T$. Since F commutes with g on T , then

$g(g(w_1)) = F(g(w_1))$ for every $w_1 \in T$. This means that for every $w_1 \in T$, $g(w_1) \in \text{Fix}_{g,F}(T)$.

To demonstrate (ii), for every $w_1 \in T$, $g(w_1) \in \text{Fix}_{g,F}(T)$ is obtained by part (i). This leads to $\text{Fix}_{g,F}(T) \subseteq g(T)$. Conversely, we assume that $w_1 \in \text{Fix}_{g,F}(T)$, so get $F(w_1) = g(w_1)$ or $w_1 = g(w_1)$, for all $w_1 \in T$. Firstly if $F(w_1) = g(w_1)$ then applications of F is bijective and commutes with g on T , yields that $w_1 = F^{-1}(g(w_1)) = g(F^{-1}(w_1))$. So $w_1 \in g(T)$. Secondly if $w_1 = g(w_1)$, then also $w_1 \in g(T)$. Therefore, we find that $\text{Fix}_{g,F}(T) = g(T)$. This fully demonstrates the theorem.

The following example demonstrates that Theorem 4.1 requires commutes with g on T and that conditions F is a bijective map.

Example 4.2: Consider a poset " $(T, \leq) = (\{1,2,3\}, \leq)$ ". Describe the maps

" $\sigma, F: T \rightarrow T$ " as follow: " $\sigma(1) = 1 = \sigma(2)$ ", " $\sigma(3) = 2$ ", " $F(1) = 1$ " and " $F(2) = 2 = F(3)$ ". After that, verifying that σ is a (F, F) -derivation on T is simple. Furthermore, F is not a bijective map, do not commutes with σ and $\text{id}_T \leq F$ on T . But we find that $\sigma(\sigma(3)) = 1 \neq 2 = F(\sigma(3))$ and $\sigma(\sigma(3)) = 1 \neq 2 = g(3)$ i.e., $\sigma(3) \notin \text{Fix}_{\sigma,F}(T)$. Therefore, it is impossible to leave out the presumed restriction in the hypothesis of Theorem 4.1.

Proposition 4.3: For an automorphism F of T , take f, g be two " (F, F) -derivations" such that F commutes with f, g , and $\text{id}_T \leq F$ on T . Then " $\text{Fix}_{f,F}(T) = \text{Fix}_{g,F}(T)$ " if and only if " $f = g$ ".

Proof: There is just one clear way if $f = g$. On the other hand, we assume that $w_1 \in T$ and $\text{Fix}_{f,F}(T) = \text{Fix}_{g,F}(T)$. Next, we want to demonstrate that $f = g$. Theorem 4.1(i) states that $f(w_1) \in \text{Fix}_{f,F}(T) = \text{Fix}_{g,F}(T)$. Consequently, $g(f(w_1)) = F(f(w_1))$ or $g(f(w_1)) = f(w_1)$ must be true. In similar way, we get $f(g(w_1)) = F(g(w_1))$ or $f(g(w_1)) = g(w_1)$. Now we have four cases

Case (1): if $F(f(w_1)) = g(f(w_1))$ and $f(g(w_1)) = F(g(w_1))$, then application of Lemma 3.4 (i),(ii) yields that $F(f(w_1)) = F(g(w_1))$ for every $w_1 \in T$. Using F is bijective map we have $g(w_1) = f(w_1)$ for every $w_1 \in T$ i.e. $g = f$.

Case (2): if $g(f(w_1)) = f(w_1)$ and $f(g(w_1)) = g(w_1)$, then using Lemma 3.4 (i),(ii) we have $g(w_1) = f(w_1)$ for all $w_1 \in T$. Thus $g = f$.

Case (3): given that $g(f(w_1)) = F(f(w_1)) = f(F(w_1))$ and $f(g(w_1)) = g(w_1)$, then by Lemma 3.4 (i),(ii) and $id_T \leq F$ on T we get $g(f(w_1)) = f(w_1)$ for all $w_1 \in T$. Therefore similarly as case 2 we have $f = g$.

Case (4): if $g(f(w_1)) = f(w_1)$ and $f(g(w_1)) = F(g(w_1))$, then similarly as case 3 we have $g = f$.

Proposition 4.4: Consider g be "a (F, F) -derivation" on T , and let T be a poset for which 0 is the least element. If F is a bijective map, commutes with g on T and $id_T \leq F$ of T , Therefore, the following claims are true:

- (i) $Fix_{g,F}(T) \neq \emptyset$;
- (ii) $s \in Fix_{g,F}(T)$; if $z \in Fix_{g,F}(T)$, and $s \leq z$
- (iii) For any $s, t \in Fix_{g,F}(T)$, there exists $z \in Fix_{g,F}(T)$ such that $s \leq z, t \leq z$. if T is directed.

Proof:

- (i) We obtain $0 = g(0)$, by Lemma 3.4 (iv), which implies that $0 \in Fix_{g,F}(T) \neq \emptyset$.
- (ii) With $z \in Fix_{g,F}(T)$, and $s \leq z$, we can assume that $g(z) = F(z)$ or $g(z) = z$. Firstly if $g(z) = z$, consequently, using Theorem 3.6(i), we obtain $g(l(s)) = g(l(z, s)) = (l(g(F(z)), s)) = l(F(F(g(z))), s) = l(F(z), s) = l(s)$. Application of Lemma 3.5(i) gives $g(s) = s$ and henceforward we conclude that $s \in Fix_F(T)$. Secondly if $g(z) = F(z)$, then $g(l(s)) = g(l(z, s)) = (l(g(F(z)), s)) = l(F(F(g(z))), s) = l(F(F(z)), s) = l(s)$. Also by Lemma 3.5(i) we have $g(s) = s$ and so $s \in Fix_F(T)$.
- (iii) T is assumed to be directed. There exists $w_1 \in P$ for every $r, s \in T$, so that $s \leq w_1$ and $r \leq w_1$. Since $r, s \in Fix_{g,F}(T)$, then we write $(g(s) = F(s) \text{ or } g(s) = s)$ and $(g(r) = F(r) \text{ or } g(r) = r)$. So we have four cases:

Case (1): if $g(s) = F(s)$ and $g(r) = F(r)$, so $F(s) = g(s) \leq g(w_1)$ and $F(r) = g(r) \leq g(w_1)$. Now taking $z = F^{-1}(g(w_1)) = g(F^{-1}(w_1))$. Hence, we obtain $z \in Fix_{g,F}(T)$ using Theorem 4.1(ii).

Case (2): if $g(s) = s$ and $g(r) = r$, then $s = g(s) \leq g(w_1)$ and $r = g(r) \leq g(w_1)$. Applying Theorem 4.1(ii) once more, we arrive at $z \in \text{Fix}_{g,F}(T)$ after taking $z = g(w_1)$.

Case (3): if $g(s) = F(s)$ and $g(r) = r$, then $F(s) = g(s) \leq g(w_1)$ and $r = g(r) \leq g(w_1)$. So $s = F^{-1}(g(s)) = g(F^{-1}(s) \leq F^{-1}(g(w_1))) = g(F^{-1}(w_1)) \leq g(w_1)$. By Theorem 4.1(ii), we get $z \in \text{Fix}_{g,F}(T)$ after taking $z = g(w_1)$.

Case (4): if $g(s) = s$ and $g(r) = F(r)$, then similarly as case(3) we have $z = g(w_1)$. We also obtain $z \in \text{Fix}_{g,F}(T)$ by Theorem **Error! Reference source not found.**(ii).

Corollary 4.5: If T is a directed poset, then $\text{Fix}_{g,F}(T)$ is an ideal of T , where its least element is 0.

Theorem 4.6: Consider g be "a (F, F) -derivation" on T where $F: T \rightarrow T$ be a function such that $gF = Fg$, F is an automorphism and $\text{id}_T \leq F$ on T . Then, $\text{Fix}_{g,F}(T) \cap l(F(z)) = l(F(g(z)))$ for all $z \in T$.

Proof: Assuming that $s, v \in T$, we first determine that $s \in \text{Fix}_{g,F}(T) \cap l(F(v))$. Then, we get $(g(s) = F(s) \text{ or } g(s) = s)$ and $s \leq F(v)$. Then we have two cases:

Case (1): if $g(s) = F(s)$ and $s \leq F(v)$ then $g(s) \leq g(F(v)) = F(g(v))$. Henceforward we conclude $F(s) \leq F(g(v))$. But F is an automorphism and $\text{id}_T \leq F$ on T then we have $s \leq g(v) \leq F(g(v))$. Thus $s \in l(F(g(v)))$, and therefore we obtain $\text{Fix}_{g,F}(T) \cap l(F(v)) \subseteq l(F(g(v)))$.

Case (2): if $g(s) = s$ and $s \leq F(v)$, then $g(s) \leq g(F(v)) = F(g(v))$. Henceforward we conclude $s \leq F(g(v))$. Thus $s \in l(F(g(v)))$. Therefore we get $\text{Fix}_{g,F}(T) \cap l(F(v)) \subseteq l(F(g(v)))$.

Hence from cases (1) and (2) $\text{Fix}_{g,F}(T) \cap l(F(v)) \subseteq l(F(g(v)))$, we have for all $v \in T$. Conversely we have $F(g(v)) = g(F(v)) \leq v \leq F(v)$ and $g(F(v)) \in \text{Fix}_{g,F}(T)$. This yields $g(F(v)) \in \text{Fix}_{g,F}(T) \cap l(F(v))$. Therefore, for every $v \in T$, we obtain $l(F(g(v))) \subseteq \text{Fix}_{g,F}(T) \cap l(F(v))$ by Proposition 4.4(ii). Therefore for every $v \in T$, $\text{Fix}_{g,F}(T) \cap l(F(v)) = l(F(g(v)))$.

The above theorem immediately leads to the corollary 2.10 when we insert $F = id_T$.

5. Structural properties of posets involving (F, F) -derivations

The least element in the poset T in this section is 0 , and $F: T \rightarrow T$ is a bijective map such that $id_T \leq F$ on T .

We will begin with the following result which an extension [21, Proposition 4.3].

Theorem 5.1: *Let $F: T \rightarrow T$ be an automorphism such that $id_T \leq F$ on T and let σ be "a (F, F) -derivation" on T . Next, let T have two ideals, U_1 and U_2 . Then, the assertions listed below are accurate:*

- (i) $\sigma(U_1) \subseteq \sigma(U_2)$, whenever $U_1 \subseteq U_2$,
- (ii) $\sigma(U_1 \cap U_2) = \sigma(U_1) \cap \sigma(U_2)$.

Proof:

- (i) Its clear to prove part (i) by using U_1, U_2 are ideals of T .
- (ii) Is evident that, $\sigma(U_1 \cap U_2) \subseteq \sigma(U_1) \cap \sigma(U_2)$. Conversely let, say $t \in \sigma(U_1) \cap \sigma(U_2)$. Consequently, $u_1 \in U_1, u_2 \in U_2$ exist so that $\sigma(u_1) = t$ and $\sigma(u_2) = t$. Accordingly, theorem 3.6 (i) gives us $\sigma(l(u_1, \sigma(u_2))) = l(\sigma(F(u_1)), \sigma(u_2)) = l(F(\sigma(u_1)), \sigma(u_2)) = l(F(t), t) = l(F(t), t)$. Therefore, $F(t) \in \sigma(l(u_1, \sigma(u_2)))$ and $F(t) \in l(F(t), t)$. Consequently, $z \in l(u_1, \sigma(u_2))$ exists such that $\sigma(z) = F(t)$. Then, $t = F^{-1}(\sigma(z)) = \sigma(F^{-1}(z))$. We obtain $F^{-1}(z) \leq z \leq u_1$ and $F^{-1}(z) \leq \sigma(u_1) \leq u_2$ by applying $id_T \leq F$ on T and Lemma 3.4 (i). Thus $z \in U_1 \cap U_2$, and, consequently, $t \in \sigma(U_1 \cap U_2)$. As a result, $\sigma(U_1) \cap \sigma(U_2) \subseteq \sigma(U_1 \cap U_2)$. This means that $\sigma(U_1 \cap U_2) = \sigma(U_1) \cap \sigma(U_2)$.

The subsequent Theorem is an extension of Theorem 4.4 in [21].

Theorem 5.2: *Assume f and g are two " (F, F) -derivations" of T so that $F: T \rightarrow T$ is a bijective map, $id_T \leq F$, and F commutes with f, g on T . It follows that f and g must commute.*

Proof: Two " (F, F) -derivations" on T are assumed to be f and g . Thus, we have

$$\begin{aligned}
fg(l(w_1)) &= f(g(l(w_1, w_1))) \\
&= f(l(w_1, g(F(w_1)))), \text{Theorem 3.6,} \\
&= l(f(F(w_1)), g(F(w_1))) \text{ foreach } w_1 \in T,
\end{aligned}$$

and

$$\begin{aligned}
gf(l(w_1)) &= g(f(l(w_1, w_1))) \\
&= g(l(f(F(w_1)), w_1)), \text{Theorem 3.6,} \\
&= l(f(F(w_1)), g(F(w_1))) \text{ for all } w_1 \in T.
\end{aligned}$$

This leads to $fg(l(w)) = gf(l(w))$. But $fg(w) = f(g(w)) \in fg(l(w))$, thus $fg(w) \in gf(l(w))$. Next, there is $s \in l(w)$ such that $fg(w) = g(s)$. By Lemma 3.4 (ii), $g(s) \leq g(f(w))$, and consequently $fg(w) \leq gf(w)$, for every $w \in T$. Similarly, we can prove that $gf(w) \leq fg(w)$, for each $w \in T$. Henceforth, we infer that $fg(w) = gf(w)$ for every $w \in T$.

Put $F = id_T$ in Theorem 5.2 we obtain Corollary 2.11.

The following example demonstrates how the circumstances of $id_T \leq F$ and F commutes with f, g on T are essential in Theorem 5.2 and can not be omitted.

Example 5.3: Let a poset $(T, \leq) = (\mathbb{R}, \leq)$. Define the following functions $f, g, F: T \rightarrow T$ as follow: $f(w) = w - 1, g(w) = \frac{1}{2}w$ and $F(w) = w - 1$ for all $w \in T$. Accordingly, f, g are " (F, F) -derivations" on T , which can be easily verified. However, we find that $fg \neq gf$ on T , meaning that f, g do not commute. As a result, Theorem 5.2 requires F to commute with f, g and $id_T \leq F$ of T .

The key conclusion of this study is the theorem that follows which generalizes Theorem 4.5, proved in [21].

Theorem 5.4: Two " (F, F) -derivations" of T , f and g , F commutes with f, g and $id_T \leq F$ on T , are given. Then, " $f \leq g$ " if and only if " $f = gf$ ".

Proof: Initially, allow us assume that there are two " (F, F) -derivations" on T , f and g , so that $f \leq g$. After that, $f(w_1) \in Fix_{f,F}(T)$ that is, $F(f(w_1)) = f(f(w_1))$ or $f(w_1) = f(f(w_1))$ for every $w_1 \in T$. Given $w_1 \in T$, there are two cases:

Case (1): If $f(f(w_1)) = f(w_1)$, then by Lemma 3.4 (i) yields $gf(w_1) \leq f(w_1)$. But $f(f(w_1)) = f(w_1) \leq g(f(w_1))$. Thus $gf(w_1) = f(w_1)$.

Case (2): If $F(f(w_1)) = f(f(w_1))$, then using $id_T \leq F$ and $f \leq g$ on T we get $f(w_1) \leq f(f(w_1)) \leq g(f(w_1))$. Thus $f(w_1) \leq g(f(w_1))$. However, $g(f(w_1)) \leq f(w_1)$. Thus, $g(f(w_1)) = f(w_1)$. Then from cases (1),(2) we have $gf = f$. Conversely, we suppose that $f(w_1) = gf(w_1) = fg(w_1) \leq g(w_1)$ for all $w_1 \in T$. It follows from this that $f(w_1) \leq g(w_1)$ for every $w_1 \in T$. Therefore, we conclude that $f \leq g$. We have finished the theorem's proof.

We have Corollary 2.12 as a direct result of Theorem 3.6 if we put $F = id_T$.

Our final example demonstrates that Theorem 5.4 requires that F commutes with f and g on T .

Example 5.5: Let a poset $(T, \leq) = (\mathbb{R}, \leq)$. As follows, define $F, f, g: T \rightarrow T$: For any $w \in T$, $F(w) = w + 1$, $f(w) = \frac{1}{4}w$ and $g(t) = \frac{1}{2}$. Demonstrating that f, g are " (F, F) -derivations" on T is then simple. Furthermore, we observe that for every $w \in T - \{0\}$, $gf(w) = \frac{1}{8}w \neq \frac{1}{4}w = f(w)$, even though $f \leq g$, on T . It is therefore impossible to leave out the presumed restriction.

Data availability statement

No data associated with this investigation.

Conflict of Interest Statement

According to the authors, they have no conflicting interests.

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