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Relationship between the inverse sum indeg index of graph operations and their factors

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Abstract

The inverse sum indeg (ISI) index of a simple graph $\mathcal{K}$ is expressed by

$$ISI(\mathcal{K}) = \sum_{\varsigma_{\mathcal{K}}\varsigma'_{\mathcal{K}} \in E_{\mathcal{K}}} \frac{d_{\mathcal{K}}(\varsigma_{\mathcal{K}})d_{\mathcal{K}}(\varsigma'_{\mathcal{K}})}{d_{\mathcal{K}}(\varsigma_{\mathcal{K}}) + d_{\mathcal{K}}(\varsigma'_{\mathcal{K}})},$$

in which $E_{\mathcal{K}}$ represents the edge set of the graph $\mathcal{K}$ and $d_{\mathcal{K}}(\varsigma_{\mathcal{K}})$ indicates the degree of the vertex $\varsigma_{\mathcal{K}}$ in $\mathcal{K}$. The ISI index is one of the 20 selected indices from a group of 148 discrete Adriatic indices that have been proposed as remarkable predictors of physico-chemical

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properties of molecular structures. In addition, this index has been detected as an excellent measure for predicting the total surface area of isomers of octane. In this study, we establish a comparison between the ISI index for various standard graph operations, including the sum, corona product, disjunctive product, Cartesian product, lexicographic product, and strong product, alongside the ISI index of their respective factors. This comparison is achieved by presenting new and sharp upper and lower bounds on the ISI index for the aforementioned graph operations, expressed based on the ISI indices of their respective factors. The necessary and sufficient conditions for the inequalities to hold as equalities are also analyzed. In the specific case where all factors are regular graphs, we provide exact formulas for the ISI index. Utilizing these results, we present precise values for the ISI index of various graphs of mathematical or chemical significance, including cone graphs, windmill graphs, thorn graphs, C_4-nanotori, Rook's graphs, prism graphs, and closed fence graphs.

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1. Introduction

Let $\mathcal{K}$ represent a finite simple graph, with $V_{\mathcal{K}}$ and $E_{\mathcal{K}}$ indicating its sets of vertices and edges, respectively. For $\varsigma_{\mathcal{K}} \in V_{\mathcal{K}}$, the degree $d_{\mathcal{K}}(\varsigma_{\mathcal{K}})$ is the number of vertices $\varsigma'_{\mathcal{K}} \in V_{\mathcal{K}}$ for which $\varsigma_{\mathcal{K}}\varsigma'_{\mathcal{K}} \in E_{\mathcal{K}}$. The notations $\delta_{\mathcal{K}}$ and $\Delta_{\mathcal{K}}$ represent the minimum degree and maximum degree of $\mathcal{K}$, respectively. A graph $\mathcal{K}$ is referred to as κ-regular graph, whenever $d_{\mathcal{K}}(\varsigma_{\mathcal{K}}) = \kappa$ for every $\varsigma_{\mathcal{K}} \in V_{\mathcal{K}}$.

Molecular graphs or *chemical graphs* of chemical compounds are labeled graphs whose vertices indicate the atoms of the compounds and edges indicate the chemical bonds. *Topological indices* are two-dimensional descriptors that can be computed from the chemical graphs, do not relate to the method the graph is depicted or labeled, and have no requirement of energy minimization of the molecular structure. These invariants are utilized in advancing quantitative structure-activity (QSAR) and quantitative structure-property (QSPR) relationships. Topological indices are categorized into various types, with vertex-degree-based (VDB) indices being particularly noteworthy. They serve as valuable tools for examining a range of properties of chemical structures, such as boiling point, enthalpy of vaporization, entropy, rotational radius, and viscosity. Dozens of VDB indices with different levels of applications have been established so far in the mathematical-chemistry literature [1]. Among the diversity of VDB topological indices, Zagreb indices [2, 3], Randić connectivity index [4, 5], atom-bond connectivity index [6], harmonic index [7], sum connectivity index [8], geometric-arithmetic index [9], inverse sum indeg index [10], symmetric division deg index [10, 11], variable sum exdeg index [12, 13], atom-bond sum-connectivity index [14, 15], arithmetic-geometric index [16, 17], and VL index [18] can be mentioned.

The *Zagreb indices* are recognized as two of the leading VDB invariants. They were proposed by Gutman and Trinajstić [2] in 1972 and further developed by Gutman *et al*. [3] in 1975. They are formulated by

$$M_1(\mathcal{K}) = \sum_{\varsigma_{\mathcal{K}} \in V_{\mathcal{K}}} d_{\mathcal{K}}(\varsigma_{\mathcal{K}})^2 = \sum_{\varsigma_{\mathcal{K}}\varsigma'_{\mathcal{K}} \in E_{\mathcal{K}}} \left(d_{\mathcal{K}}(\varsigma_{\mathcal{K}}) + d_{\mathcal{K}}(\varsigma'_{\mathcal{K}})\right),$$

$$M_2(\mathcal{K}) = \sum_{\varsigma_{\mathcal{K}}\varsigma'_{\mathcal{K}} \in E_{\mathcal{K}}} d_{\mathcal{K}}(\varsigma_{\mathcal{K}}) d_{\mathcal{K}}(\varsigma'_{\mathcal{K}}).$$

These invariants are employed in the exploration of molecular complexity, ZE-isomerism, chirality, and hetero-systems, indicating their potential use in the creation of multi-linear regression models. For further insights into extremal results and bounds, see [19, 20], and for their modifications, refer to [21, 22].

Another member of the populous family of VDB topological indices which is worthy of attention is the *inverse sum indeg* (ISI) index. It was put forward by Vukičević and Gašperov [10] in 2010 and further developed by Sedlar *et al*. [23] in 2015. This invariant is expressed by

$$ISI(\mathcal{K}) = \sum_{\varsigma_{\mathcal{K}}\varsigma'_{\mathcal{K}} \in E_{\mathcal{K}}} \frac{1}{\frac{1}{d_{\mathcal{K}}(\varsigma_{\mathcal{K}})} + \frac{1}{d_{\mathcal{K}}(\varsigma'_{\mathcal{K}})}} = \sum_{\varsigma_{\mathcal{K}}\varsigma'_{\mathcal{K}} \in E_{\mathcal{K}}} \frac{d_{\mathcal{K}}(\varsigma_{\mathcal{K}}) d_{\mathcal{K}}(\varsigma'_{\mathcal{K}})}{d_{\mathcal{K}}(\varsigma_{\mathcal{K}}) + d_{\mathcal{K}}(\varsigma'_{\mathcal{K}})}.$$

The ISI index is one of the 20 discrete Adriatic indices identified by Vukičević and Gašperov [10] as notable predictors of the physico-chemical properties of chemical structures. It is also proposed as an important measure for predicting the total surface area of isomers of octane. In recent years, the ISI index has garnered considerable attention from both mathematicians and chemists due to its predictive capabilities. For example, Sedlar *et al*. [23] and An and Xiong [24] investigated the extremal problems concerning the ISI index across various special classes of graphs. Falahati-Nezhad *et al*. [25], Ali *et al*. [26] and Bharali *et al*. [27] proposed a number of sharp bounds on the ISI index based on certain graph parameters and connected this invariant to some best-known topological indices. Matejić *et al*. [28] and Gutman *et al*. [29] analyzed some previously-established bounds on the ISI index and presented some new bounds. In addition to extremal problems, the problem of calculating the ISI index for chemical graphs and nano-structures were also considered in various researches. For example, Das and Mondal [30] studied a variant of the ISI index for some molecular graphs with chemical significance. Falahati-Nezhad and Azari [31] investigated the ISI index for several families of nanotubes. Lokesha *et al*. [32] calculated it for polycyclic aromatic hydrocarbons and polyhex nanotubes.

There are various methods for calculating topological indices of chemical graphs related to chemical structures. One effective approach involves constructing larger chemical graphs from smaller ones by applying one or more appropriate graph operations. The topological properties of the resulting graphs, known as *composite graphs*, are highly dependent on their constituent components. Consequently, exploring the relationship between the topological indices of graph operations and their factors is a significant research area within chemical graph theory. The calculation of the ISI index for graph operations and transformations has also been the focus of several recent studies. Lokesha *et al.* [33] provided upper and lower bounds on the ISI index specific splice graphs, taking into account the size, minimum degree, and maximum degree of their factors. Pattabiraman [34] obtained upper and lower bounds on the ISI index for different graph operations, including composition, Cartesian product, join, corona product, and Mycielski graph, in terms of the Zagreb indices, inverse degree, Randić index, harmonic index, and symmetric division degree index of their factors. Additionally, Havare [35] provided upper bounds on the ISI index for the sequential sum, Cartesian product, lexicographic product, and corona product based on the first Zagreb index of their factors. Rani et al. [36] established sharp upper and lower bounds on the ISI index for several graph operations, specifically expressed based on the second Zagreb index of the respective operation itself. By analyzing previous results, it becomes evident that most established inequalities concerning the ISI index of graph operations do not depend on the ISI index of their factors. Furthermore, many of these inequalities are not sharp, meaning that the necessary and sufficient conditions for equality have not been thoroughly examined. In this paper, we concentrate on the connection between the ISI index of composite graphs and their factors. Specifically, we derive sharp bounds (both upper and lower) on the ISI index for certain composite graphs, including the sum, corona, disjunction, Cartesian, lexicographic, and strong products, in terms of the ISI index of their factors and identify the conditions under which equality occurs. Additionally, we provide precise formulas for the ISI index of these graph operations in cases where all factors are regular graphs. These findings allow us to determine the exact amounts of the ISI index for several graphs of chemical or general interest.

2. Main Results

In this section, we assume the simple connected graphs $\mathcal{K}$ and $\mathcal{R}$, which have disjoint vertex sets. Additionally, we assume that, $|V_{\mathcal{K}}| = n_{\mathcal{K}}$, $|V_{\mathcal{R}}| = n_{\mathcal{R}}$, $|E_{\mathcal{K}}| = m_{\mathcal{K}}$ and $|E_{\mathcal{R}}| = m_{\mathcal{R}}$. We begin our discussion with the sum of graphs.

2.1 *Sum*

The *sum* $\mathcal{K}+\mathcal{R}$ is the graph with $V_{\mathcal{K}+\mathcal{R}} = V_{\mathcal{K}} \cup V_{\mathcal{R}}$ and $E_{\mathcal{K}+\mathcal{R}} = E_{\mathcal{K}} \cup E_{\mathcal{R}} \cup \{\varsigma\varsigma' | \varsigma \in V_{\mathcal{K}}, \varsigma' \in V_{\mathcal{R}}\}$ (see [37]). Obviously, $n_{\mathcal{K}+\mathcal{R}} = n_{\mathcal{K}} + n_{\mathcal{R}}$ and $m_{\mathcal{K}+\mathcal{R}} = m_{\mathcal{K}} + m_{\mathcal{R}} + n_{\mathcal{K}} n_{\mathcal{R}}$.

Theorem 2.1: The ISI index of $\mathcal{K}+\mathcal{R}$ is bounded from above as

$$\begin{aligned} ISI(\mathcal{K}+\mathcal{R}) \leq\ & \frac{\Delta_{\mathcal{K}}}{\Delta_{\mathcal{K}}+n_{\mathcal{R}}} ISI(\mathcal{K}) + \frac{\Delta_{\mathcal{R}}}{\Delta_{\mathcal{R}}+n_{\mathcal{K}}} ISI(\mathcal{R}) + \frac{n_{\mathcal{R}}(M_1(\mathcal{K})+n_{\mathcal{R}} m_{\mathcal{K}})}{2(\delta_{\mathcal{K}}+n_{\mathcal{R}})} \\ & + \frac{n_{\mathcal{K}}(M_1(\mathcal{R})+n_{\mathcal{K}} m_{\mathcal{R}})}{2(\delta_{\mathcal{R}}+n_{\mathcal{K}})} + \frac{n_{\mathcal{K}} n_{\mathcal{R}} (\Delta_{\mathcal{K}}+n_{\mathcal{R}})(\Delta_{\mathcal{R}}+n_{\mathcal{K}})}{\Delta_{\mathcal{K}}+\Delta_{\mathcal{R}}+n_{\mathcal{K}}+n_{\mathcal{R}}}. \end{aligned} \tag{2.1}$$

Equality holds iff $\mathcal{K}$ and $\mathcal{R}$ are regular.

Proof: From the definition,

$$ISI(\mathcal{K}+\mathcal{R}) = \sum_{\varsigma\varsigma' \in E(\mathcal{K}+\mathcal{R})} \frac{d_{\mathcal{K}+\mathcal{R}}(\varsigma)\, d_{\mathcal{K}+\mathcal{R}}(\varsigma')}{d_{\mathcal{K}+\mathcal{R}}(\varsigma) + d_{\mathcal{K}+\mathcal{R}}(\varsigma')}.$$

The edges of $\mathcal{K}+\mathcal{R}$ can be partitioned into the subsets E_1, E_2 and E_3, where $E_1 = \{\varsigma\varsigma' | \varsigma, \varsigma' \in V_{\mathcal{K}}\}$, $E_2 = \{\varsigma\varsigma' | \varsigma, \varsigma' \in V_{\mathcal{R}}\}$ and $E_3 = \{\varsigma\varsigma' | \varsigma \in V_{\mathcal{K}}, \varsigma' \in V_{\mathcal{R}}\}$.

If $\varsigma\varsigma' \in E_1$, then $d_{\mathcal{K}+\mathcal{R}}(\varsigma) = d_{\mathcal{K}}(\varsigma) + n_{\mathcal{R}}$ and $d_{\mathcal{K}+\mathcal{R}}(\varsigma') = d_{\mathcal{K}}(\varsigma') + n_{\mathcal{R}}$. Therefore

$$\begin{aligned}\frac{d_{\mathcal{K}+\mathcal{R}}(\varsigma)\, d_{\mathcal{K}+\mathcal{R}}(\varsigma')}{d_{\mathcal{K}+\mathcal{R}}(\varsigma)+d_{\mathcal{K}+\mathcal{R}}(\varsigma')} &= \frac{(d_{\mathcal{K}}(\varsigma)+n_{\mathcal{R}})(d_{\mathcal{K}}(\varsigma')+n_{\mathcal{R}})}{d_{\mathcal{K}}(\varsigma)+d_{\mathcal{K}}(\varsigma')+2n_{\mathcal{R}}} \\ &= \frac{d_{\mathcal{K}}(\varsigma)\, d_{\mathcal{K}}(\varsigma')}{d_{\mathcal{K}}(\varsigma)+d_{\mathcal{K}}(\varsigma')+2n_{\mathcal{R}}} + \frac{n_{\mathcal{R}}(d_{\mathcal{K}}(\varsigma)+d_{\mathcal{K}}(\varsigma'))+n_{\mathcal{R}}^2}{d_{\mathcal{K}}(\varsigma)+d_{\mathcal{K}}(\varsigma')+2n_{\mathcal{R}}} \\ &= \left(\frac{d_{\mathcal{K}}(\varsigma)\, d_{\mathcal{K}}(\varsigma')}{d_{\mathcal{K}}(\varsigma)+d_{\mathcal{K}}(\varsigma')} \times \left(1 - \frac{2n_{\mathcal{R}}}{d_{\mathcal{K}}(\varsigma)+d_{\mathcal{K}}(\varsigma')+2n_{\mathcal{R}}}\right)\right) \\ &\quad + \frac{n_{\mathcal{R}}(d_{\mathcal{K}}(\varsigma)+d_{\mathcal{K}}(\varsigma'))+n_{\mathcal{R}}^2}{d_{\mathcal{K}}(\varsigma)+d_{\mathcal{K}}(\varsigma')+2n_{\mathcal{R}}} \\ &\le \frac{\Delta_{\mathcal{K}}}{\Delta_{\mathcal{K}}+n_{\mathcal{R}}} \times \frac{d_{\mathcal{K}}(\varsigma)\, d_{\mathcal{K}}(\varsigma')}{d_{\mathcal{K}}(\varsigma)+d_{\mathcal{K}}(\varsigma')} + \frac{n_{\mathcal{R}}(d_{\mathcal{K}}(\varsigma)+d_{\mathcal{K}}(\varsigma'))+n_{\mathcal{R}}^2}{2(\delta_{\mathcal{K}}+n_{\mathcal{R}})}.\end{aligned}$$

Then

$$\sum_{\varsigma\varsigma'\in E_1} \frac{d_{\mathcal{K}+\mathcal{R}}(\varsigma)\, d_{\mathcal{K}+\mathcal{R}}(\varsigma')}{d_{\mathcal{K}+\mathcal{R}}(\varsigma)+d_{\mathcal{K}+\mathcal{R}}(\varsigma')} \le \frac{\Delta_{\mathcal{K}}}{\Delta_{\mathcal{K}}+n_{\mathcal{R}}}\, ISI(\mathcal{K}) + \frac{n_{\mathcal{R}}(M_1(\mathcal{K})+n_{\mathcal{R}}m_{\mathcal{K}})}{2(\delta_{\mathcal{K}}+n_{\mathcal{R}})}. \tag{2.2}$$

Evidently, equality holds iff $\Delta_{\mathcal{K}} = \delta_{\mathcal{K}}$. By symmetry,

$$\sum_{\varsigma\varsigma'\in E_2} \frac{d_{\mathcal{K}+\mathcal{R}}(\varsigma)\, d_{\mathcal{K}+\mathcal{R}}(\varsigma')}{d_{\mathcal{K}+\mathcal{R}}(\varsigma)+d_{\mathcal{K}+\mathcal{R}}(\varsigma')} \le \frac{\Delta_{\mathcal{R}}}{\Delta_{\mathcal{R}}+n_{\mathcal{K}}}\, ISI(\mathcal{R}) + \frac{n_{\mathcal{K}}(M_1(\mathcal{R})+n_{\mathcal{K}}m_{\mathcal{R}})}{2(\delta_{\mathcal{R}}+n_{\mathcal{K}})}. \tag{2.3}$$

Equality occurs iff $\Delta_{\mathcal{R}} = \delta_{\mathcal{R}}$.

If $\varsigma\varsigma' \in E_3$, where $\varsigma \in V_{\mathcal{K}}$ and $\varsigma' \in V_{\mathcal{R}}$, then $d_{\mathcal{K}+\mathcal{R}}(\varsigma) = d_{\mathcal{K}}(\varsigma) + n_{\mathcal{R}}$ and $d_{\mathcal{K}+\mathcal{R}}(\varsigma') = d_{\mathcal{R}}(\varsigma') + n_{\mathcal{K}}$. Thus

$$\frac{1}{d_{\mathcal{K}+\mathcal{R}}(\varsigma)} + \frac{1}{d_{\mathcal{K}+\mathcal{R}}(\varsigma')} \ge \frac{1}{\Delta_{\mathcal{K}}+n_{\mathcal{R}}} + \frac{1}{\Delta_{\mathcal{R}}+n_{\mathcal{K}}}.$$

Equality occurs iff $d_{\mathcal{K}}(\varsigma) = \Delta_{\mathcal{K}}$ and $d_{\mathcal{R}}(\varsigma') = \Delta_{\mathcal{R}}$. Thus

$$\sum_{\varsigma\varsigma'\in E_3} \frac{d_{\mathcal{K}+\mathcal{R}}(\varsigma)\, d_{\mathcal{K}+\mathcal{R}}(\varsigma')}{d_{\mathcal{K}+\mathcal{R}}(\varsigma)+d_{\mathcal{K}+\mathcal{R}}(\varsigma')} \le \frac{n_{\mathcal{K}}n_{\mathcal{R}}(\Delta_{\mathcal{K}}+n_{\mathcal{R}})(\Delta_{\mathcal{R}}+n_{\mathcal{K}})}{\Delta_{\mathcal{K}}+\Delta_{\mathcal{R}}+n_{\mathcal{K}}+n_{\mathcal{R}}}, \tag{2.4}$$

and equality occurs iff for all $\varsigma \in V_{\mathcal{K}}$ and $\varsigma' \in V_{\mathcal{R}}$, $d_{\mathcal{K}}(\varsigma) = \Delta_{\mathcal{K}}$ and $d_{\mathcal{R}}(\varsigma') = \Delta_{\mathcal{R}}$. From (2.2), (2.3) and (2.4), we get (2.1) and the equality holds in (2.1) iff $\mathcal{K}$ and $\mathcal{R}$ are regular.

Following a similar approach to the proof of Theorem 2.1, we can establish the following results.

Theorem 2.2: The ISI index of $\mathcal{K} + \mathcal{R}$ is bounded from below as

$$\begin{aligned}ISI(\mathcal{K}+\mathcal{R}) \ge\ & \frac{\delta_{\mathcal{K}}}{\delta_{\mathcal{K}}+n_{\mathcal{R}}} ISI(\mathcal{K}) + \frac{\delta_{\mathcal{R}}}{\delta_{\mathcal{R}}+n_{\mathcal{K}}} ISI(\mathcal{R}) + \frac{n_{\mathcal{R}}(M_1(\mathcal{K})+n_{\mathcal{R}}m_{\mathcal{K}})}{2(\Delta_{\mathcal{K}}+n_{\mathcal{R}})} \\ & + \frac{n_{\mathcal{K}}(M_1(\mathcal{R})+n_{\mathcal{K}}m_{\mathcal{R}})}{2(\Delta_{\mathcal{R}}+n_{\mathcal{K}})} + \frac{n_{\mathcal{K}}n_{\mathcal{R}}(\delta_{\mathcal{K}}+n_{\mathcal{R}})(\delta_{\mathcal{R}}+n_{\mathcal{K}})}{\delta_{\mathcal{K}}+\delta_{\mathcal{R}}+n_{\mathcal{K}}+n_{\mathcal{R}}}.\end{aligned}$$

Equality occurs iff $\mathcal{K}$ and $\mathcal{R}$ are regular.

Corollary 2.3: For a κ-regular graph $\mathcal{K}$ and a ρ-regular graph $\mathcal{R}$,

$$ISI(\mathcal{K}+\mathcal{R}) = \frac{\kappa n_{\mathcal{K}}(\kappa+n_{\mathcal{R}})+\rho n_{\mathcal{R}}(\rho+n_{\mathcal{K}})}{4} + \frac{n_{\mathcal{K}}n_{\mathcal{R}}(\kappa+n_{\mathcal{R}})(\rho+n_{\mathcal{K}})}{\kappa+\rho+n_{\mathcal{K}}+n_{\mathcal{R}}}.$$

Example 2.4: For the ν-gonal μ-cone graph $C_\nu + \overline{K}_\mu$ and windmill graph $K_1 + \mu K_{\nu-1}$ (see Figure 1), we have

1. $ISI(C_\nu + \bar{K}_\mu) = \frac{\nu(\mu+2)}{2} + \frac{\nu^2\mu(\mu+2)}{\nu+\mu+2}$,
2. $ISI(K_1 + \mu K_{\nu-1}) = \frac{\mu(\nu-1)^2(\nu-2)}{4} + \frac{\mu^2(\nu-1)^2}{\mu+1}$.

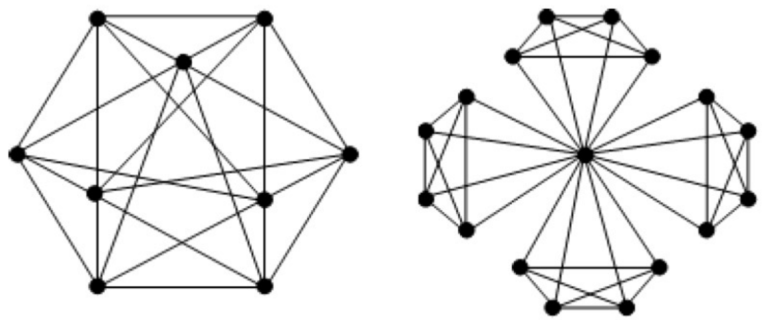

Figure 1

The 6-gonal 3-cone graph $C_6 + \bar{K}_3$ and the windmill graph $K_1 + 4K_4$.

2.2 *Corona product*

The *corona product* $\mathcal{K} \circ \mathcal{R}$ is formed by including one copy of the graph $\mathcal{K}$ and $n_\mathcal{K}$ copies of the graph $\mathcal{R}$. In this construction, the ith vertex of $\mathcal{K}$ is connected to all vertices in the ith copy of $\mathcal{R}$, for each $1 \le i \le n_\mathcal{K}$ (as noted in [37]). Obviously, $n_{\mathcal{K}\circ\mathcal{R}} = n_\mathcal{K} + n_\mathcal{K} n_\mathcal{R}$ and $m_{\mathcal{K}\circ\mathcal{R}} = m_\mathcal{K} + n_\mathcal{K}(n_\mathcal{R} + m_\mathcal{R})$.

Theorem 2.5: *The ISI index of $\mathcal{K} \circ \mathcal{R}$ is bounded from above as*

$$\begin{aligned} ISI(\mathcal{K}\circ\mathcal{R}) \le\ & \frac{\Delta_\mathcal{K}}{\Delta_\mathcal{K}+n_\mathcal{R}} ISI(\mathcal{K}) + \frac{n_\mathcal{K}\Delta_\mathcal{R}}{\Delta_\mathcal{R}+1} ISI(\mathcal{R}) + \frac{n_\mathcal{R}(M_1(\mathcal{K})+n_\mathcal{R} m_\mathcal{K})}{2(\delta_\mathcal{K}+n_\mathcal{R})} \\ & + \frac{n_\mathcal{K}(M_1(\mathcal{R})+m_\mathcal{R})}{2(\delta_\mathcal{R}+1)} + \frac{n_\mathcal{K} n_\mathcal{R}(\Delta_\mathcal{K}+n_\mathcal{R})(\Delta_\mathcal{R}+1)}{\Delta_\mathcal{K}+\Delta_\mathcal{R}+n_\mathcal{R}+1}, \end{aligned} \tag{2.5}$$

and equality occurs iff $\mathcal{K}$ and $\mathcal{R}$ are regular.

Proof: Consider three subsets of $E_{\mathcal{K}\circ\mathcal{R}}$ as $E_1 = \{\varsigma\varsigma' | \varsigma, \varsigma' \in V_\mathcal{K}\}$, $E_2 = \{\varsigma\varsigma' | \varsigma, \varsigma' \in V_\mathcal{R}\}$ and $E_3 = \{\varsigma\varsigma' | \varsigma \in V_\mathcal{K}, \varsigma' \in V_\mathcal{R}\}$. Obviously,

$$\begin{aligned} ISI(\mathcal{K}\circ\mathcal{R}) =\ & \sum_{\varsigma\varsigma'\in E_1} \frac{d_{\mathcal{K}\circ\mathcal{R}}(\varsigma) d_{\mathcal{K}\circ\mathcal{R}}(\varsigma')}{d_{\mathcal{K}\circ\mathcal{R}}(\varsigma)+d_{\mathcal{K}\circ\mathcal{R}}(\varsigma')} + n_\mathcal{K} \sum_{\varsigma\varsigma'\in E_2} \frac{d_{\mathcal{K}\circ\mathcal{R}}(\varsigma) d_{\mathcal{K}\circ\mathcal{R}}(\varsigma')}{d_{\mathcal{K}\circ\mathcal{R}}(\varsigma)+d_{\mathcal{K}\circ\mathcal{R}}(\varsigma')} \\ & + \sum_{\varsigma\varsigma'\in E_3} \frac{d_{\mathcal{K}\circ\mathcal{R}}(\varsigma) d_{\mathcal{K}\circ\mathcal{R}}(\varsigma')}{d_{\mathcal{K}\circ\mathcal{R}}(\varsigma)+d_{\mathcal{K}\circ\mathcal{R}}(\varsigma')}. \end{aligned} \tag{2.6}$$

For $\varsigma\varsigma' \in E_1$, $d_{\mathcal{K}\circ\mathcal{R}}(\varsigma) = d_\mathcal{K}(\varsigma) + n_\mathcal{R}$ and $d_{\mathcal{K}\circ\mathcal{R}}(\varsigma') = d_\mathcal{K}(\varsigma') + n_\mathcal{R}$ and for $\varsigma\varsigma' \in E_2$, $d_{\mathcal{K}\circ\mathcal{R}}(\varsigma) = d_\mathcal{R}(\varsigma) + 1$ and $d_{\mathcal{K}\circ\mathcal{R}}(\varsigma') = d_\mathcal{R}(\varsigma') + 1$. Similar to the argument of Theorem 2.1,

$$\sum_{\varsigma\varsigma'\in E_1} \frac{d_{\mathcal{K}\circ\mathcal{R}}(\varsigma) d_{\mathcal{K}\circ\mathcal{R}}(\varsigma')}{d_{\mathcal{K}\circ\mathcal{R}}(\varsigma)+d_{\mathcal{K}\circ\mathcal{R}}(\varsigma')} \le \frac{\Delta_\mathcal{K}}{\Delta_\mathcal{K}+n_\mathcal{R}} ISI(\mathcal{K}) + \frac{n_\mathcal{R}(M_1(\mathcal{K})+n_\mathcal{R} m_\mathcal{K})}{2(\delta_\mathcal{K}+n_\mathcal{R})}, \tag{2.7}$$

$$\sum_{\varsigma\varsigma'\in E_2} \frac{d_{\mathcal{K}\circ\mathcal{R}}(\varsigma)d_{\mathcal{K}\circ\mathcal{R}}(\varsigma')}{d_{\mathcal{K}\circ\mathcal{R}}(\varsigma)+d_{\mathcal{K}\circ\mathcal{R}}(\varsigma')} \leq \frac{\Delta_{\mathcal{R}}}{\Delta_{\mathcal{R}}+1}\, ISI(\mathcal{R}) + \frac{M_1(\mathcal{R})+m_{\mathcal{R}}}{2(\delta_{\mathcal{R}}+1)}, \tag{2.8}$$

and equality holds in both cases iff $\Delta_{\mathcal{K}} = \delta_{\mathcal{K}}$ and $\Delta_{\mathcal{R}} = \delta_{\mathcal{R}}$.

For $\varsigma\varsigma' \in E_3$ with $\varsigma \in V_{\mathcal{K}}$ and $\varsigma' \in V_{\mathcal{R}}$, $d_{\mathcal{K}\circ\mathcal{R}}(\varsigma) = d_{\mathcal{K}}(\varsigma) + n_{\mathcal{R}}$ and $d_{\mathcal{K}\circ\mathcal{R}}(\varsigma') = d_{\mathcal{R}}(\varsigma') + 1$. Then

$$\frac{d_{\mathcal{K}\circ\mathcal{R}}(\varsigma)d_{\mathcal{K}\circ\mathcal{R}}(\varsigma')}{d_{\mathcal{K}\circ\mathcal{R}}(\varsigma)+d_{\mathcal{K}\circ\mathcal{R}}(\varsigma')} = \frac{(d_{\mathcal{K}}(\varsigma)+n_{\mathcal{R}})(d_{\mathcal{R}}(\varsigma')+1)}{d_{\mathcal{K}}(\varsigma)+d_{\mathcal{R}}(\varsigma')+n_{\mathcal{R}}+1} \leq \frac{(\Delta_{\mathcal{K}}+n_{\mathcal{R}})(\Delta_{\mathcal{R}}+1)}{\Delta_{\mathcal{K}}+\Delta_{\mathcal{R}}+n_{\mathcal{R}}+1}.$$

Therefore

$$\sum_{\varsigma\varsigma'\in E_3} \frac{d_{\mathcal{K}\circ\mathcal{R}}(\varsigma)d_{\mathcal{K}\circ\mathcal{R}}(\varsigma')}{d_{\mathcal{K}\circ\mathcal{R}}(\varsigma)+d_{\mathcal{K}\circ\mathcal{R}}(\varsigma')} \leq \frac{n_{\mathcal{K}}n_{\mathcal{R}}(\Delta_{\mathcal{K}}+n_{\mathcal{R}})(\Delta_{\mathcal{R}}+1)}{\Delta_{\mathcal{K}}+\Delta_{\mathcal{R}}+n_{\mathcal{R}}+1}, \tag{2.9}$$

and equality occurs iff $\Delta_{\mathcal{K}} = \delta_{\mathcal{K}}$ and $\Delta_{\mathcal{R}} = \delta_{\mathcal{R}}$. From (2.6), (2.7), (2.8) and (2.9), we get (2.5) and equality holds in (2.5) iff $\mathcal{K}$ and $\mathcal{R}$ are regular.

By employing reasoning similar to that used in Theorem 2.5, we can derive the subsequent result.

Theorem 2.6: *The ISI index of $\mathcal{K}\circ\mathcal{R}$ is bounded from below as*

$$ISI(\mathcal{K}\circ\mathcal{R}) \geq \frac{\delta_{\mathcal{K}}}{\delta_{\mathcal{K}}+n_{\mathcal{R}}}\, ISI(\mathcal{K}) + \frac{n_{\mathcal{K}}\delta_{\mathcal{R}}}{\delta_{\mathcal{R}}+1}\, ISI(\mathcal{R}) + \frac{n_{\mathcal{R}}(M_1(\mathcal{K})+n_{\mathcal{R}}m_{\mathcal{K}})}{2(\Delta_{\mathcal{K}}+n_{\mathcal{R}})} + \frac{n_{\mathcal{K}}(M_1(\mathcal{R})+m_{\mathcal{R}})}{2(\Delta_{\mathcal{R}}+1)} + \frac{n_{\mathcal{K}}n_{\mathcal{R}}(\delta_{\mathcal{K}}+n_{\mathcal{R}})(\delta_{\mathcal{R}}+1)}{\delta_{\mathcal{K}}+\delta_{\mathcal{R}}+n_{\mathcal{R}}+1}.$$

Equality occurs iff $\mathcal{K}$ and $\mathcal{R}$ are regular graphs.

Corollary 2.7: *For a κ-regular graph $\mathcal{K}$ and a ρ-regular graph $\mathcal{R}$,*

$$ISI(\mathcal{K}\circ\mathcal{R}) = \frac{n_{\mathcal{K}}\kappa(\kappa+n_{\mathcal{R}})+n_{\mathcal{K}}n_{\mathcal{R}}\rho(\rho+1)}{4} + \frac{n_{\mathcal{K}}n_{\mathcal{R}}(\kappa+n_{\mathcal{R}})(\rho+1)}{\kappa+\rho+n_{\mathcal{R}}+1}.$$

Example 2.8: *For the thorn cycle $C_\nu \circ \overline{K}_\mu$ (see Figure 2),*

$$ISI(C_\nu \circ \overline{K}_\mu) = \frac{\nu(\mu+2)}{2} + \frac{\nu\mu(\mu+2)}{\mu+3}.$$

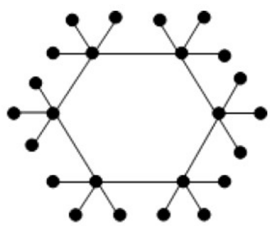

Figure 2
The thorn cycle $C_6 \circ \overline{K}_3$.

2.3 *Disjunctive product*

The *disjunctive product* $\mathcal{K}\vee\mathcal{R}$ is the graph with $V_{\mathcal{K}\vee\mathcal{R}} = V_{\mathcal{K}} \times V_{\mathcal{R}}$ and $E_{\mathcal{K}\vee\mathcal{R}} = \{(\varsigma_{\mathcal{K}},\varsigma_{\mathcal{R}})(\varsigma'_{\mathcal{K}},\varsigma'_{\mathcal{R}}) | \varsigma_{\mathcal{R}},\varsigma'_{\mathcal{R}} \in V_{\mathcal{R}}, \varsigma_{\mathcal{K}}\varsigma'_{\mathcal{K}} \in E_{\mathcal{K}}\} \cup \{(\varsigma_{\mathcal{K}},\varsigma_{\mathcal{R}})(\varsigma'_{\mathcal{K}},\varsigma'_{\mathcal{R}}) | \varsigma_{\mathcal{K}},$

$\varsigma_{\mathcal{K}}, \varsigma'_{\mathcal{K}} \in V_{\mathcal{K}}, \varsigma_{\mathcal{R}}\varsigma'_{\mathcal{R}} \in E_{\mathcal{R}}\}$ (see [37]). Obviously, $n_{\mathcal{K}\vee\mathcal{R}} = n_{\mathcal{K}}n_{\mathcal{R}}$ and $m_{\mathcal{K}\vee\mathcal{R}} = m_{\mathcal{K}}n_{\mathcal{R}}^2 + m_{\mathcal{R}}n_{\mathcal{K}}^2 - 2m_{\mathcal{K}}m_{\mathcal{R}}$.

Theorem 2.9: *Let $\mathcal{K}$ or $\mathcal{R}$ have no isolated vertices. Then the ISI index of $\mathcal{K}\vee\mathcal{R}$ is bounded from above as*

$$\begin{aligned} ISI(\mathcal{K}\vee\mathcal{R}) \leq\ & \frac{2\Delta_{\mathcal{K}}(n_{\mathcal{R}}^2-2m_{\mathcal{R}})^2\, ISI(\mathcal{K})+2\Delta_{\mathcal{R}}(n_{\mathcal{K}}^2-2m_{\mathcal{K}})^2\, ISI(\mathcal{R})}{2n_{\mathcal{R}}\delta_{\mathcal{K}}+2n_{\mathcal{K}}\delta_{\mathcal{R}}-\delta_{\mathcal{K}}\Delta_{\mathcal{R}}-\Delta_{\mathcal{K}}\delta_{\mathcal{R}}} \\ & + \frac{2n_{\mathcal{K}}m_{\mathcal{R}}(n_{\mathcal{R}}^2-2m_{\mathcal{R}})M_1(\mathcal{K})+2n_{\mathcal{R}}m_{\mathcal{K}}(n_{\mathcal{K}}^2-2m_{\mathcal{K}})M_1(\mathcal{R})+4m_{\mathcal{K}}m_{\mathcal{R}}(n_{\mathcal{K}}^2m_{\mathcal{R}}+n_{\mathcal{R}}^2m_{\mathcal{K}})}{2n_{\mathcal{R}}\delta_{\mathcal{K}}+2n_{\mathcal{K}}\delta_{\mathcal{R}}-\delta_{\mathcal{K}}\Delta_{\mathcal{R}}-\Delta_{\mathcal{K}}\delta_{\mathcal{R}}} \\ & - \frac{2n_{\mathcal{R}}^2m_{\mathcal{R}}M_2(\mathcal{K})+2n_{\mathcal{K}}^2m_{\mathcal{K}}M_2(\mathcal{R})-2n_{\mathcal{R}}M_1(\mathcal{R})M_2(\mathcal{K})-2n_{\mathcal{K}}M_1(\mathcal{K})M_2(\mathcal{R})}{2n_{\mathcal{R}}\Delta_{\mathcal{K}}+2n_{\mathcal{K}}\Delta_{\mathcal{R}}-\Delta_{\mathcal{K}}\delta_{\mathcal{R}}-\delta_{\mathcal{K}}\Delta_{\mathcal{R}}} \\ & - \frac{n_{\mathcal{K}}n_{\mathcal{R}}M_1(\mathcal{K})M_1(\mathcal{R})+2M_2(\mathcal{K})M_2(\mathcal{R})}{2n_{\mathcal{R}}\Delta_{\mathcal{K}}+2n_{\mathcal{K}}\Delta_{\mathcal{R}}-\Delta_{\mathcal{K}}\delta_{\mathcal{R}}-\delta_{\mathcal{K}}\Delta_{\mathcal{R}}}. \end{aligned} \tag{2.10}$$

Equality occurs iff $\mathcal{K}$ and $\mathcal{R}$ are regular.

Proof: Consider the subsets of $E_{\mathcal{K}\vee\mathcal{R}}$ as $E_1 = \{(\varsigma_{\mathcal{K}}, \varsigma_{\mathcal{R}})(\varsigma'_{\mathcal{K}}, \varsigma'_{\mathcal{R}}) | \varsigma_{\mathcal{R}}, \varsigma'_{\mathcal{R}} \in V_{\mathcal{R}}, \varsigma_{\mathcal{K}}\varsigma'_{\mathcal{K}} \in E_{\mathcal{K}}\}$, $E_2 = \{(\varsigma_{\mathcal{K}}, \varsigma_{\mathcal{R}})(\varsigma'_{\mathcal{K}}, \varsigma'_{\mathcal{R}}) | \varsigma_{\mathcal{K}}, \varsigma'_{\mathcal{K}} \in V_{\mathcal{K}}, \varsigma_{\mathcal{R}}\varsigma'_{\mathcal{R}} \in E_{\mathcal{R}}\}$ and $E_3 = \{(\varsigma_{\mathcal{K}}, \varsigma_{\mathcal{R}})(\varsigma'_{\mathcal{K}}, \varsigma'_{\mathcal{R}}) | \varsigma_{\mathcal{K}}\varsigma'_{\mathcal{K}} \in E_{\mathcal{K}}, \varsigma_{\mathcal{R}}\varsigma'_{\mathcal{R}} \in E_{\mathcal{R}}\}$. Evidently,

$$\begin{aligned} ISI(\mathcal{K}\vee\mathcal{R}) =\ & \sum_{(\varsigma_{\mathcal{K}},\varsigma_{\mathcal{R}})(\varsigma'_{\mathcal{K}},\varsigma'_{\mathcal{R}})\in E_1} \frac{d_{\mathcal{K}\vee\mathcal{R}}(\varsigma_{\mathcal{K}},\varsigma_{\mathcal{R}})d_{\mathcal{K}\vee\mathcal{R}}(\varsigma'_{\mathcal{K}},\varsigma'_{\mathcal{R}})}{d_{\mathcal{K}\vee\mathcal{R}}(\varsigma_{\mathcal{K}},\varsigma_{\mathcal{R}})+d_{\mathcal{K}\vee\mathcal{R}}(\varsigma'_{\mathcal{K}},\varsigma'_{\mathcal{R}})} \\ & + \sum_{(\varsigma_{\mathcal{K}},\varsigma_{\mathcal{R}})(\varsigma'_{\mathcal{K}},\varsigma'_{\mathcal{R}})\in E_2} \frac{d_{\mathcal{K}\vee\mathcal{R}}(\varsigma_{\mathcal{K}},\varsigma_{\mathcal{R}})d_{\mathcal{K}\vee\mathcal{R}}(\varsigma'_{\mathcal{K}},\varsigma'_{\mathcal{R}})}{d_{\mathcal{K}\vee\mathcal{R}}(\varsigma_{\mathcal{K}},\varsigma_{\mathcal{R}})+d_{\mathcal{K}\vee\mathcal{R}}(\varsigma'_{\mathcal{K}},\varsigma'_{\mathcal{R}})} \\ & - \sum_{(\varsigma_{\mathcal{K}},\varsigma_{\mathcal{R}})(\varsigma'_{\mathcal{K}},\varsigma'_{\mathcal{R}})\in E_3} \frac{d_{\mathcal{K}\vee\mathcal{R}}(\varsigma_{\mathcal{K}},\varsigma_{\mathcal{R}})d_{\mathcal{K}\vee\mathcal{R}}(\varsigma'_{\mathcal{K}},\varsigma'_{\mathcal{R}})}{d_{\mathcal{K}\vee\mathcal{R}}(\varsigma_{\mathcal{K}},\varsigma_{\mathcal{R}})+d_{\mathcal{K}\vee\mathcal{R}}(\varsigma'_{\mathcal{K}},\varsigma'_{\mathcal{R}})}. \end{aligned} \tag{2.11}$$

Suppose that, $(\varsigma_{\mathcal{K}}, \varsigma_{\mathcal{R}})(\varsigma'_{\mathcal{K}}, \varsigma'_{\mathcal{R}}) \in E_1$. Then $d_{\mathcal{K}\vee\mathcal{R}}(\varsigma_{\mathcal{K}}, \varsigma_{\mathcal{R}}) = n_{\mathcal{R}}d_{\mathcal{K}}(\varsigma_{\mathcal{K}}) + n_{\mathcal{K}}d_{\mathcal{R}}(\varsigma_{\mathcal{R}}) - d_{\mathcal{K}}(\varsigma_{\mathcal{K}})d_{\mathcal{R}}(\varsigma_{\mathcal{R}})$ and $d_{\mathcal{K}\vee\mathcal{R}}(\varsigma'_{\mathcal{K}}, \varsigma'_{\mathcal{R}}) = n_{\mathcal{R}}d_{\mathcal{K}}(\varsigma'_{\mathcal{K}}) + n_{\mathcal{K}}d_{\mathcal{R}}(\varsigma'_{\mathcal{R}}) - d_{\mathcal{K}}(\varsigma'_{\mathcal{K}})d_{\mathcal{R}}(\varsigma'_{\mathcal{R}})$. Therefore

$$\begin{aligned} & n_{\mathcal{R}}(d_{\mathcal{K}}(\varsigma_{\mathcal{K}}) + d_{\mathcal{K}}(\varsigma'_{\mathcal{K}})) + n_{\mathcal{K}}(d_{\mathcal{R}}(\varsigma_{\mathcal{R}}) + d_{\mathcal{R}}(\varsigma'_{\mathcal{R}})) - d_{\mathcal{K}}(\varsigma_{\mathcal{K}})d_{\mathcal{R}}(\varsigma_{\mathcal{R}}) \\ & -d_{\mathcal{K}}(\varsigma'_{\mathcal{K}})d_{\mathcal{R}}(\varsigma'_{\mathcal{R}}) \\ & = \left(n_{\mathcal{R}} - \frac{1}{2}\, d_{\mathcal{R}}(\varsigma'_{\mathcal{R}})\right) d_{\mathcal{K}}(\varsigma'_{\mathcal{K}}) + \left(n_{\mathcal{K}} - \frac{1}{2}\, d_{\mathcal{K}}(\varsigma'_{\mathcal{K}})\right) d_{\mathcal{R}}(\varsigma'_{\mathcal{R}}) \\ & + \left(n_{\mathcal{R}} - \frac{1}{2}\, d_{\mathcal{R}}(\varsigma_{\mathcal{R}})\right) d_{\mathcal{K}}(\varsigma_{\mathcal{K}}) + \left(n_{\mathcal{K}} - \frac{1}{2}\, d_{\mathcal{K}}(\varsigma_{\mathcal{K}})\right) d_{\mathcal{R}}(\varsigma_{\mathcal{R}}) \\ & \geq \left(n_{\mathcal{R}} - \frac{1}{2}\, d_{\mathcal{R}}(\varsigma'_{\mathcal{R}})\right) \delta_{\mathcal{K}} + \left(n_{\mathcal{K}} - \frac{1}{2}\, d_{\mathcal{K}}(\varsigma'_{\mathcal{K}})\right) \delta_{\mathcal{R}} + \left(n_{\mathcal{R}} - \frac{1}{2}\, d_{\mathcal{R}}(\varsigma_{\mathcal{R}})\right) \delta_{\mathcal{K}} \\ & + \left(n_{\mathcal{K}} - \frac{1}{2}\, d_{\mathcal{K}}(\varsigma_{\mathcal{K}})\right) \delta_{\mathcal{R}} \\ & \geq 2n_{\mathcal{R}}\delta_{\mathcal{K}} + 2n_{\mathcal{K}}\delta_{\mathcal{R}} - \delta_{\mathcal{K}}\Delta_{\mathcal{R}} - \Delta_{\mathcal{K}}\delta_{\mathcal{R}}. \end{aligned} \tag{2.12}$$

Hence for every edge $(\varsigma_{\mathcal{K}}, \varsigma_{\mathcal{R}})(\varsigma'_{\mathcal{K}}, \varsigma'_{\mathcal{R}}) \in E_1$,

$$\begin{aligned} & \frac{d_{\mathcal{K}\vee\mathcal{R}}(\varsigma_{\mathcal{K}},\varsigma_{\mathcal{R}})d_{\mathcal{K}\vee\mathcal{R}}(\varsigma'_{\mathcal{K}},\varsigma'_{\mathcal{R}})}{d_{\mathcal{K}\vee\mathcal{R}}(\varsigma_{\mathcal{K}},\varsigma_{\mathcal{R}})+d_{\mathcal{K}\vee\mathcal{R}}(\varsigma'_{\mathcal{K}},\varsigma'_{\mathcal{R}})} \\ & = \frac{(n_{\mathcal{R}}d_{\mathcal{K}}(\varsigma_{\mathcal{K}})+n_{\mathcal{K}}d_{\mathcal{R}}(\varsigma_{\mathcal{R}})-d_{\mathcal{K}}(\varsigma_{\mathcal{K}})d_{\mathcal{R}}(\varsigma_{\mathcal{R}}))\,(n_{\mathcal{R}}d_{\mathcal{K}}(\varsigma'_{\mathcal{K}})+n_{\mathcal{K}}d_{\mathcal{R}}(\varsigma'_{\mathcal{R}})-d_{\mathcal{K}}(\varsigma'_{\mathcal{K}})d_{\mathcal{R}}(\varsigma'_{\mathcal{R}}))}{n_{\mathcal{R}}(d_{\mathcal{K}}(\varsigma_{\mathcal{K}})+d_{\mathcal{K}}(\varsigma'_{\mathcal{K}}))+n_{\mathcal{K}}(d_{\mathcal{R}}(\varsigma_{\mathcal{R}})+d_{\mathcal{R}}(\varsigma'_{\mathcal{R}}))-d_{\mathcal{K}}(\varsigma_{\mathcal{K}})d_{\mathcal{R}}(\varsigma_{\mathcal{R}})-d_{\mathcal{K}}(\varsigma'_{\mathcal{K}})d_{\mathcal{R}}(\varsigma'_{\mathcal{R}})} \end{aligned}$$

$$= \frac{(n_{\mathcal{R}}^2 - n_{\mathcal{R}} d_{\mathcal{R}}(\varsigma_{\mathcal{R}}) - n_{\mathcal{R}} d_{\mathcal{R}}(\varsigma'_{\mathcal{R}}) + d_{\mathcal{R}}(\varsigma_{\mathcal{R}}) d_{\mathcal{R}}(\varsigma'_{\mathcal{R}}))\, d_{\mathcal{K}}(\varsigma_{\mathcal{K}})\, d_{\mathcal{K}}(\varsigma'_{\mathcal{K}})}{n_{\mathcal{R}}(d_{\mathcal{K}}(\varsigma_{\mathcal{K}}) + d_{\mathcal{K}}(\varsigma'_{\mathcal{K}})) + n_{\mathcal{K}}(d_{\mathcal{R}}(\varsigma_{\mathcal{R}}) + d_{\mathcal{R}}(\varsigma'_{\mathcal{R}})) - d_{\mathcal{K}}(\varsigma_{\mathcal{K}}) d_{\mathcal{R}}(\varsigma_{\mathcal{R}}) - d_{\mathcal{K}}(\varsigma'_{\mathcal{K}}) d_{\mathcal{R}}(\varsigma'_{\mathcal{R}})}$$
$$+ \frac{n_{\mathcal{K}} n_{\mathcal{R}}\, (d_{\mathcal{R}}(\varsigma'_{\mathcal{R}}) d_{\mathcal{K}}(\varsigma_{\mathcal{K}}) + d_{\mathcal{R}}(\varsigma_{\mathcal{R}}) d_{\mathcal{K}}(\varsigma'_{\mathcal{K}})) - n_{\mathcal{K}} d_{\mathcal{R}}(\varsigma_{\mathcal{R}}) d_{\mathcal{R}}(\varsigma'_{\mathcal{R}})\, (d_{\mathcal{K}}(\varsigma_{\mathcal{K}}) + d_{\mathcal{K}}(\varsigma'_{\mathcal{K}}))}{n_{\mathcal{R}}(d_{\mathcal{K}}(\varsigma_{\mathcal{K}}) + d_{\mathcal{K}}(\varsigma'_{\mathcal{K}})) + n_{\mathcal{K}}(d_{\mathcal{R}}(\varsigma_{\mathcal{R}}) + d_{\mathcal{R}}(\varsigma'_{\mathcal{R}})) - d_{\mathcal{K}}(\varsigma_{\mathcal{K}}) d_{\mathcal{R}}(\varsigma_{\mathcal{R}}) - d_{\mathcal{K}}(\varsigma'_{\mathcal{K}}) d_{\mathcal{R}}(\varsigma'_{\mathcal{R}})}$$
$$+ \frac{n_{\mathcal{K}}^2 d_{\mathcal{R}}(\varsigma_{\mathcal{R}}) d_{\mathcal{R}}(\varsigma'_{\mathcal{R}})}{n_{\mathcal{R}}(d_{\mathcal{K}}(\varsigma_{\mathcal{K}}) + d_{\mathcal{K}}(\varsigma'_{\mathcal{K}})) + n_{\mathcal{K}}(d_{\mathcal{R}}(\varsigma_{\mathcal{R}}) + d_{\mathcal{R}}(\varsigma'_{\mathcal{R}})) - d_{\mathcal{K}}(\varsigma_{\mathcal{K}}) d_{\mathcal{R}}(\varsigma_{\mathcal{R}}) - d_{\mathcal{K}}(\varsigma'_{\mathcal{K}}) d_{\mathcal{R}}(\varsigma'_{\mathcal{R}})}$$
$$\le \frac{2\Delta_{\mathcal{K}}(n_{\mathcal{R}}^2 - n_{\mathcal{R}} d_{\mathcal{R}}(\varsigma_{\mathcal{R}}) - n_{\mathcal{R}} d_{\mathcal{R}}(\varsigma'_{\mathcal{R}}) + d_{\mathcal{R}}(\varsigma_{\mathcal{R}}) d_{\mathcal{R}}(\varsigma'_{\mathcal{R}}))}{2n_{\mathcal{R}}\delta_{\mathcal{K}} + 2n_{\mathcal{K}}\delta_{\mathcal{R}} - \delta_{\mathcal{K}}\Delta_{\mathcal{R}} - \Delta_{\mathcal{K}}\delta_{\mathcal{R}}} \left(\frac{d_{\mathcal{K}}(\varsigma_{\mathcal{K}}) d_{\mathcal{K}}(\varsigma'_{\mathcal{K}})}{d_{\mathcal{K}}(\varsigma_{\mathcal{K}}) + d_{\mathcal{K}}(\varsigma'_{\mathcal{K}})} \right)$$
$$+ \frac{n_{\mathcal{K}} n_{\mathcal{R}}\, (d_{\mathcal{R}}(\varsigma'_{\mathcal{R}}) d_{\mathcal{K}}(\varsigma_{\mathcal{K}}) + d_{\mathcal{R}}(\varsigma_{\mathcal{R}}) d_{\mathcal{K}}(\varsigma'_{\mathcal{K}})) - n_{\mathcal{K}} d_{\mathcal{R}}(\varsigma_{\mathcal{R}}) d_{\mathcal{R}}(\varsigma'_{\mathcal{R}})\, (d_{\mathcal{K}}(\varsigma_{\mathcal{K}}) + d_{\mathcal{K}}(\varsigma'_{\mathcal{K}}))}{2n_{\mathcal{R}}\delta_{\mathcal{K}} + 2n_{\mathcal{K}}\delta_{\mathcal{R}} - \delta_{\mathcal{K}}\Delta_{\mathcal{R}} - \Delta_{\mathcal{K}}\delta_{\mathcal{R}}}$$
$$+ \frac{n_{\mathcal{K}}^2 d_{\mathcal{R}}(\varsigma_{\mathcal{R}}) d_{\mathcal{R}}(\varsigma'_{\mathcal{R}})}{2n_{\mathcal{R}}\delta_{\mathcal{K}} + 2n_{\mathcal{K}}\delta_{\mathcal{R}} - \delta_{\mathcal{K}}\Delta_{\mathcal{R}} - \Delta_{\mathcal{K}}\delta_{\mathcal{R}}}.$$

Therefore

$$\begin{aligned} &\sum_{(\varsigma_{\mathcal{K}},\varsigma_{\mathcal{R}})(\varsigma'_{\mathcal{K}},\varsigma'_{\mathcal{R}}) \in E_1} \frac{d_{\mathcal{K}\vee\mathcal{R}}(\varsigma_{\mathcal{K}},\varsigma_{\mathcal{R}}) d_{\mathcal{K}\vee\mathcal{R}}(\varsigma'_{\mathcal{K}},\varsigma'_{\mathcal{R}})}{d_{\mathcal{K}\vee\mathcal{R}}(\varsigma_{\mathcal{K}},\varsigma_{\mathcal{R}}) + d_{\mathcal{K}\vee\mathcal{R}}(\varsigma'_{\mathcal{K}},\varsigma'_{\mathcal{R}})} \\ &= \sum_{\varsigma_{\mathcal{R}},\varsigma'_{\mathcal{R}} \in V_{\mathcal{R}}} \sum_{\varsigma_{\mathcal{K}}\varsigma'_{\mathcal{K}} \in E_{\mathcal{K}}} \frac{d_{\mathcal{K}\vee\mathcal{R}}(\varsigma_{\mathcal{K}},\varsigma_{\mathcal{R}}) d_{\mathcal{K}\vee\mathcal{R}}(\varsigma'_{\mathcal{K}},\varsigma'_{\mathcal{R}})}{d_{\mathcal{K}\vee\mathcal{R}}(\varsigma_{\mathcal{K}},\varsigma_{\mathcal{R}}) + d_{\mathcal{K}\vee\mathcal{R}}(\varsigma'_{\mathcal{K}},\varsigma'_{\mathcal{R}})} \\ &\le \frac{2\Delta_{\mathcal{K}}(n_{\mathcal{R}}^2 - 2m_{\mathcal{R}})^2\, ISI(\mathcal{K}) + 2n_{\mathcal{K}} m_{\mathcal{R}}(n_{\mathcal{R}}^2 - 2m_{\mathcal{R}}) M_1(\mathcal{K}) + 4m_{\mathcal{K}} n_{\mathcal{K}}^2 m_{\mathcal{R}}^2}{2n_{\mathcal{R}}\delta_{\mathcal{K}} + 2n_{\mathcal{K}}\delta_{\mathcal{R}} - \delta_{\mathcal{K}}\Delta_{\mathcal{R}} - \Delta_{\mathcal{K}}\delta_{\mathcal{R}}}. \end{aligned} \tag{2.13}$$

Equality holds iff $\Delta_{\mathcal{K}} = \delta_{\mathcal{K}}$ and $\Delta_{\mathcal{R}} = \delta_{\mathcal{R}}$. By symmetry,

$$\begin{aligned} &\sum_{(\varsigma_{\mathcal{K}},\varsigma_{\mathcal{R}})(\varsigma'_{\mathcal{K}},\varsigma'_{\mathcal{R}}) \in E_2} \frac{d_{\mathcal{K}\vee\mathcal{R}}(\varsigma_{\mathcal{K}},\varsigma_{\mathcal{R}}) d_{\mathcal{K}\vee\mathcal{R}}(\varsigma'_{\mathcal{K}},\varsigma'_{\mathcal{R}})}{d_{\mathcal{K}\vee\mathcal{R}}(\varsigma_{\mathcal{K}},\varsigma_{\mathcal{R}}) + d_{\mathcal{K}\vee\mathcal{R}}(\varsigma'_{\mathcal{K}},\varsigma'_{\mathcal{R}})} \\ &\le \frac{2\Delta_{\mathcal{R}}(n_{\mathcal{K}}^2 - 2m_{\mathcal{K}})^2\, ISI(\mathcal{R}) + 2n_{\mathcal{R}} m_{\mathcal{K}}(n_{\mathcal{K}}^2 - 2m_{\mathcal{K}}) M_1(\mathcal{R}) + 4m_{\mathcal{R}} n_{\mathcal{R}}^2 m_{\mathcal{K}}^2}{2n_{\mathcal{R}}\delta_{\mathcal{K}} + 2n_{\mathcal{K}}\delta_{\mathcal{R}} - \delta_{\mathcal{K}}\Delta_{\mathcal{R}} - \Delta_{\mathcal{K}}\delta_{\mathcal{R}}}. \end{aligned} \tag{2.14}$$

Equality holds iff $\Delta_{\mathcal{K}} = \delta_{\mathcal{K}}$ and $\Delta_{\mathcal{R}} = \delta_{\mathcal{R}}$.

Assume that, $(\varsigma_{\mathcal{K}},\varsigma_{\mathcal{R}})(\varsigma'_{\mathcal{K}},\varsigma'_{\mathcal{R}}) \in E_3$. Then $d_{\mathcal{K}\vee\mathcal{R}}(\varsigma_{\mathcal{K}},\varsigma_{\mathcal{R}}) = n_{\mathcal{R}} d_{\mathcal{K}}(\varsigma_{\mathcal{K}}) + n_{\mathcal{K}} d_{\mathcal{R}}(\varsigma_{\mathcal{R}}) - d_{\mathcal{K}}(\varsigma_{\mathcal{K}}) d_{\mathcal{R}}(\varsigma_{\mathcal{R}})$ and $d_{\mathcal{K}\vee\mathcal{R}}(\varsigma'_{\mathcal{K}},\varsigma'_{\mathcal{R}}) = n_{\mathcal{R}} d_{\mathcal{K}}(\varsigma'_{\mathcal{K}}) + n_{\mathcal{K}} d_{\mathcal{R}}(\varsigma'_{\mathcal{R}}) - d_{\mathcal{K}}(\varsigma'_{\mathcal{K}}) d_{\mathcal{R}}(\varsigma'_{\mathcal{R}})$. Similar to Eq. (2.12), we obtain:

$$\begin{aligned} &n_{\mathcal{R}}(d_{\mathcal{K}}(\varsigma_{\mathcal{K}}) + d_{\mathcal{K}}(\varsigma'_{\mathcal{K}})) + n_{\mathcal{K}}(d_{\mathcal{R}}(\varsigma_{\mathcal{R}}) + d_{\mathcal{R}}(\varsigma'_{\mathcal{R}})) - d_{\mathcal{K}}(\varsigma_{\mathcal{K}}) d_{\mathcal{R}}(\varsigma_{\mathcal{R}}) \\ &- d_{\mathcal{K}}(\varsigma'_{\mathcal{K}}) d_{\mathcal{R}}(\varsigma'_{\mathcal{R}}) \\ &\le 2n_{\mathcal{R}}\Delta_{\mathcal{K}} + 2n_{\mathcal{K}}\Delta_{\mathcal{R}} - \Delta_{\mathcal{K}}\delta_{\mathcal{R}} - \delta_{\mathcal{K}}\Delta_{\mathcal{R}}. \end{aligned}$$

Hence for every edge $e = (\varsigma_{\mathcal{K}},\varsigma_{\mathcal{R}})(\varsigma'_{\mathcal{K}},\varsigma'_{\mathcal{R}}) \in E_3$,

$$\frac{d_{\mathcal{K}\vee\mathcal{R}}(\varsigma_{\mathcal{K}},\varsigma_{\mathcal{R}}) d_{\mathcal{K}\vee\mathcal{R}}(\varsigma'_{\mathcal{K}},\varsigma'_{\mathcal{R}})}{d_{\mathcal{K}\vee\mathcal{R}}(\varsigma_{\mathcal{K}},\varsigma_{\mathcal{R}}) + d_{\mathcal{K}\vee\mathcal{R}}(\varsigma'_{\mathcal{K}},\varsigma'_{\mathcal{R}})}$$
$$= \frac{(n_{\mathcal{R}} d_{\mathcal{K}}(\varsigma_{\mathcal{K}}) + n_{\mathcal{K}} d_{\mathcal{R}}(\varsigma_{\mathcal{R}}) - d_{\mathcal{K}}(\varsigma_{\mathcal{K}}) d_{\mathcal{R}}(\varsigma_{\mathcal{R}}))\, (n_{\mathcal{R}} d_{\mathcal{K}}(\varsigma'_{\mathcal{K}}) + n_{\mathcal{K}} d_{\mathcal{R}}(\varsigma'_{\mathcal{R}}) - d_{\mathcal{K}}(\varsigma'_{\mathcal{K}}) d_{\mathcal{R}}(\varsigma'_{\mathcal{R}}))}{n_{\mathcal{R}}(d_{\mathcal{K}}(\varsigma_{\mathcal{K}}) + d_{\mathcal{K}}(\varsigma'_{\mathcal{K}})) + n_{\mathcal{K}}(d_{\mathcal{R}}(\varsigma_{\mathcal{R}}) + d_{\mathcal{R}}(\varsigma'_{\mathcal{R}})) - d_{\mathcal{K}}(\varsigma_{\mathcal{K}}) d_{\mathcal{R}}(\varsigma_{\mathcal{R}}) - d_{\mathcal{K}}(\varsigma'_{\mathcal{K}}) d_{\mathcal{R}}(\varsigma'_{\mathcal{R}})}$$
$$\ge \frac{(n_{\mathcal{R}}^2 - n_{\mathcal{R}} d_{\mathcal{R}}(\varsigma_{\mathcal{R}}) - n_{\mathcal{R}} d_{\mathcal{R}}(\varsigma'_{\mathcal{R}}) + d_{\mathcal{R}}(\varsigma_{\mathcal{R}}) d_{\mathcal{R}}(\varsigma'_{\mathcal{R}}))\, d_{\mathcal{K}}(\varsigma_{\mathcal{K}})\, d_{\mathcal{K}}(\varsigma'_{\mathcal{K}})}{2n_{\mathcal{R}}\Delta_{\mathcal{K}} + 2n_{\mathcal{K}}\Delta_{\mathcal{R}} - \Delta_{\mathcal{K}}\delta_{\mathcal{R}} - \delta_{\mathcal{K}}\Delta_{\mathcal{R}}}$$
$$+ \frac{n_{\mathcal{K}} n_{\mathcal{R}}(d_{\mathcal{R}}(\varsigma'_{\mathcal{R}}) d_{\mathcal{K}}(\varsigma_{\mathcal{K}}) + d_{\mathcal{R}}(\varsigma_{\mathcal{R}}) d_{\mathcal{K}}(\varsigma'_{\mathcal{K}})) - n_{\mathcal{K}} d_{\mathcal{R}}(\varsigma_{\mathcal{R}}) d_{\mathcal{R}}(\varsigma'_{\mathcal{R}})(d_{\mathcal{K}}(\varsigma_{\mathcal{K}}) + d_{\mathcal{K}}(\varsigma'_{\mathcal{K}}))}{2n_{\mathcal{R}}\Delta_{\mathcal{K}} + 2n_{\mathcal{K}}\Delta_{\mathcal{R}} - \Delta_{\mathcal{K}}\delta_{\mathcal{R}} - \delta_{\mathcal{K}}\Delta_{\mathcal{R}}}$$
$$+ \frac{n_{\mathcal{K}}^2 d_{\mathcal{R}}(\varsigma_{\mathcal{R}}) d_{\mathcal{R}}(\varsigma'_{\mathcal{R}})}{2n_{\mathcal{R}}\Delta_{\mathcal{K}} + 2n_{\mathcal{K}}\Delta_{\mathcal{R}} - \Delta_{\mathcal{K}}\delta_{\mathcal{R}} - \delta_{\mathcal{K}}\Delta_{\mathcal{R}}}.$$

Therefore

$$\sum_{(\varsigma_{\mathcal{K}},\varsigma_{\mathcal{R}})(\varsigma'_{\mathcal{K}},\varsigma'_{\mathcal{R}}) \in E_3} \frac{d_{\mathcal{K}\vee\mathcal{R}}(\varsigma_{\mathcal{K}},\varsigma_{\mathcal{R}}) d_{\mathcal{K}\vee\mathcal{R}}(\varsigma'_{\mathcal{K}},\varsigma'_{\mathcal{R}})}{d_{\mathcal{K}\vee\mathcal{R}}(\varsigma_{\mathcal{K}},\varsigma_{\mathcal{R}}) + d_{\mathcal{K}\vee\mathcal{R}}(\varsigma'_{\mathcal{K}},\varsigma'_{\mathcal{R}})}$$

$$= \sum_{\varsigma_{\mathcal{R}}\varsigma'_{\mathcal{R}}\in E_{\mathcal{R}}} \sum_{\varsigma_{\mathcal{K}}\varsigma'_{\mathcal{K}}\in E_{\mathcal{K}}} \left(\frac{d_{\mathcal{K}\vee\mathcal{R}}(\varsigma_{\mathcal{K}},\varsigma_{\mathcal{R}})d_{\mathcal{K}\vee\mathcal{R}}(\varsigma'_{\mathcal{K}},\varsigma'_{\mathcal{R}})}{d_{\mathcal{K}\vee\mathcal{R}}(\varsigma_{\mathcal{K}},\varsigma_{\mathcal{R}})+d_{\mathcal{K}\vee\mathcal{R}}(\varsigma'_{\mathcal{K}},\varsigma'_{\mathcal{R}})} + \frac{d_{\mathcal{K}\vee\mathcal{R}}(\varsigma_{\mathcal{K}},\varsigma'_{\mathcal{R}})d_{\mathcal{K}\vee\mathcal{R}}(\varsigma'_{\mathcal{K}},\varsigma_{\mathcal{R}})}{d_{\mathcal{K}\vee\mathcal{R}}(\varsigma_{\mathcal{K}},\varsigma'_{\mathcal{R}})+d_{\mathcal{K}\vee\mathcal{R}}(\varsigma'_{\mathcal{K}},\varsigma_{\mathcal{R}})}\right)$$

$$\geq \frac{2(n_{\mathcal{R}}^2 m_{\mathcal{R}} - n_{\mathcal{R}}M_1(\mathcal{R}) + M_2(\mathcal{R}))M_2(\mathcal{K})}{2n_{\mathcal{R}}\Delta_{\mathcal{K}} + 2n_{\mathcal{K}}\Delta_{\mathcal{R}} - \Delta_{\mathcal{K}}\delta_{\mathcal{R}} - \delta_{\mathcal{K}}\Delta_{\mathcal{R}}}$$
$$+ \frac{n_{\mathcal{K}}n_{\mathcal{R}}M_1(\mathcal{K})M_1(\mathcal{R}) - 2n_{\mathcal{K}}M_1(\mathcal{K})M_2(\mathcal{R}) + 2n_{\mathcal{K}}^2 m_{\mathcal{K}}M_2(\mathcal{R})}{2n_{\mathcal{R}}\Delta_{\mathcal{K}} + 2n_{\mathcal{K}}\Delta_{\mathcal{R}} - \Delta_{\mathcal{K}}\delta_{\mathcal{R}} - \delta_{\mathcal{K}}\Delta_{\mathcal{R}}}. \tag{2.15}$$

Equality holds iff $\Delta_{\mathcal{K}} = \delta_{\mathcal{K}}$ and $\Delta_{\mathcal{R}} = \delta_{\mathcal{R}}$.

From (2.11), (2.13), (2.14) and (2.15), we get (2.10). In addition, equality holds in (2.10) iff $\mathcal{K}$ and $\mathcal{R}$ are regular.

Similar to the reasoning presented in the proof of Theorem 2.9, we can derive the subsequent results.

Theorem 2.10: *Let $\mathcal{K}$ or $\mathcal{R}$ have no isolated vertices. Then the ISI index of $\mathcal{K} \vee \mathcal{R}$ is bounded from below as*

$$ISI(\mathcal{K}\vee\mathcal{R}) \geq \frac{2\delta_{\mathcal{K}}(n_{\mathcal{R}}^2 - 2m_{\mathcal{R}})^2\, ISI(\mathcal{K}) + 2\delta_{\mathcal{R}}(n_{\mathcal{K}}^2 - 2m_{\mathcal{K}})^2\, ISI(\mathcal{R})}{2n_{\mathcal{R}}\Delta_{\mathcal{K}} + 2n_{\mathcal{K}}\Delta_{\mathcal{R}} - \Delta_{\mathcal{K}}\delta_{\mathcal{R}} - \delta_{\mathcal{K}}\Delta_{\mathcal{R}}}$$
$$+ \frac{2n_{\mathcal{K}}m_{\mathcal{R}}(n_{\mathcal{R}}^2 - 2m_{\mathcal{R}})M_1(\mathcal{K}) + 2n_{\mathcal{R}}m_{\mathcal{K}}(n_{\mathcal{K}}^2 - 2m_{\mathcal{K}})M_1(\mathcal{R}) + 4m_{\mathcal{K}}m_{\mathcal{R}}(n_{\mathcal{K}}^2 m_{\mathcal{R}} + n_{\mathcal{R}}^2 m_{\mathcal{K}})}{2n_{\mathcal{R}}\Delta_{\mathcal{K}} + 2n_{\mathcal{K}}\Delta_{\mathcal{R}} - \Delta_{\mathcal{K}}\delta_{\mathcal{R}} - \delta_{\mathcal{K}}\Delta_{\mathcal{R}}}$$
$$- \frac{2n_{\mathcal{R}}^2 m_{\mathcal{R}}M_2(\mathcal{K}) + 2n_{\mathcal{K}}^2 m_{\mathcal{K}}M_2(\mathcal{R}) - 2n_{\mathcal{R}}M_1(\mathcal{R})M_2(\mathcal{K}) - 2n_{\mathcal{K}}M_1(\mathcal{K})M_2(\mathcal{R})}{2n_{\mathcal{R}}\delta_{\mathcal{K}} + 2n_{\mathcal{K}}\delta_{\mathcal{R}} - \delta_{\mathcal{K}}\Delta_{\mathcal{R}} - \Delta_{\mathcal{K}}\delta_{\mathcal{R}}}$$
$$- \frac{n_{\mathcal{K}}n_{\mathcal{R}}M_1(\mathcal{K})M_1(\mathcal{R}) + 2M_2(\mathcal{K})M_2(\mathcal{R})}{2n_{\mathcal{R}}\delta_{\mathcal{K}} + 2n_{\mathcal{K}}\delta_{\mathcal{R}} - \delta_{\mathcal{K}}\Delta_{\mathcal{R}} - \Delta_{\mathcal{K}}\delta_{\mathcal{R}}}.$$

Equality occurs iff $\mathcal{K}$ and $\mathcal{R}$ are regular.

Corollary 2.11: *For a κ-regular graph $\mathcal{K}$ and a ρ-regular graph $\mathcal{R}$,*

$$ISI(\mathcal{K}\vee\mathcal{R}) = \frac{n_{\mathcal{K}}n_{\mathcal{R}}(\kappa n_{\mathcal{R}} + \rho n_{\mathcal{K}} - \kappa\rho)^2}{4}.$$

Example 2.12: For the graphs $C_\nu \vee C_\mu$, $K_\nu \vee K_\mu$ and $C_\nu \vee K_\mu$ (see Figure 3),

1. $ISI(C_\nu \vee C_\mu) = \nu\mu(\nu + \mu - 2)^2$,
2. $ISI(K_\nu \vee K_\mu) = \frac{\nu\mu(\nu\mu - 1)}{4}$,
3. $ISI(C_\nu \vee K_\mu) = \frac{\nu\mu(\nu\mu - \nu + 2)}{4}$.

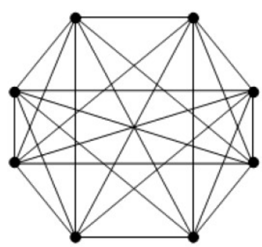

Figure 3
The graph $C_4 \vee K_2$.

2.4 *Cartesian product*

The *Cartesian product* $\mathcal{K} \square \mathcal{R}$ is a graph whose vertex set is given by $V(\mathcal{K} \square \mathcal{R}) = V_{\mathcal{K}} \times V_{\mathcal{R}}$, and whose edge set is defined as $E(\mathcal{K} \square \mathcal{R}) = \{(\varsigma_{\mathcal{K}}, \varsigma_{\mathcal{R}})(\varsigma'_{\mathcal{K}}, \varsigma_{\mathcal{R}}) | \varsigma_{\mathcal{R}} \in V_{\mathcal{R}}, \varsigma_{\mathcal{K}}\varsigma'_{\mathcal{K}} \in E_{\mathcal{K}}\} \cup \{(\varsigma_{\mathcal{K}}, \varsigma_{\mathcal{R}})(\varsigma_{\mathcal{K}}, \varsigma'_{\mathcal{R}}) | \varsigma_{\mathcal{K}} \in V_{\mathcal{K}}, \varsigma_{\mathcal{R}}\varsigma'_{\mathcal{R}} \in E_{\mathcal{R}}\}$ (see [37]). Obviously, $n_{\mathcal{K} \square \mathcal{R}} = n_{\mathcal{K}} n_{\mathcal{R}}$ and $m_{\mathcal{K} \mathrm{W} \mathcal{R}} = m_{\mathcal{K}} n_{\mathcal{R}} + n_{\mathcal{K}} m_{\mathcal{R}}$.

Theorem 2.13: *Let $\mathcal{K}$ or $\mathcal{R}$ have no isolated vertices. Then the ISI index of $\mathcal{K} \square \mathcal{R}$ is bounded from above as*

$$ISI(\mathcal{K} \square \mathcal{R}) \leq \frac{n_{\mathcal{R}}\Delta_{\mathcal{K}}}{\Delta_{\mathcal{K}}+\delta_{\mathcal{R}}} ISI(\mathcal{K}) + \frac{n_{\mathcal{K}}\Delta_{\mathcal{R}}}{\Delta_{\mathcal{R}}+\delta_{\mathcal{K}}} ISI(\mathcal{R}) + \frac{3(m_{\mathcal{R}}M_1(\mathcal{K})+m_{\mathcal{K}}M_1(\mathcal{R}))}{2(\delta_{\mathcal{K}}+\delta_{\mathcal{R}})}, \tag{2.16}$$

and equality occurs iff $\mathcal{K}$ and $\mathcal{R}$ are regular.

Proof: The set of $E_{\mathcal{K} \square \mathcal{R}}$ is divided into the subsets E_1 and E_2 as follows:

$$E_1 = \{(\varsigma_{\mathcal{K}}, \varsigma_{\mathcal{R}})(\varsigma'_{\mathcal{K}}, \varsigma_{\mathcal{R}}) | \varsigma_{\mathcal{R}} \in V_{\mathcal{R}}, \varsigma_{\mathcal{K}}\varsigma'_{\mathcal{K}} \in E_{\mathcal{K}}\},$$
$$E_2 = \{(\varsigma_{\mathcal{K}}, \varsigma_{\mathcal{R}})(\varsigma_{\mathcal{K}}, \varsigma'_{\mathcal{R}}) | \varsigma_{\mathcal{K}} \in V_{\mathcal{K}}, \varsigma_{\mathcal{R}}\varsigma'_{\mathcal{R}} \in E_{\mathcal{R}}\}.$$

For $(\varsigma_{\mathcal{K}}, \varsigma_{\mathcal{R}})(\varsigma'_{\mathcal{K}}, \varsigma_{\mathcal{R}}) \in E_1$, $d_{\mathcal{K} \square \mathcal{R}}(\varsigma_{\mathcal{K}}, \varsigma_{\mathcal{R}}) = d_{\mathcal{K}}(\varsigma_{\mathcal{K}}) + d_{\mathcal{R}}(\varsigma_{\mathcal{R}})$ and $d_{\mathcal{K} \square \mathcal{R}}(\varsigma'_{\mathcal{K}}, \varsigma_{\mathcal{R}}) = d_{\mathcal{K}}(\varsigma'_{\mathcal{K}}) + d_{\mathcal{R}}(\varsigma_{\mathcal{R}})$. Moreover,

$$\frac{2d_{\mathcal{R}}(\varsigma_{\mathcal{R}})}{d_{\mathcal{K}}(\varsigma_{\mathcal{K}})+d_{\mathcal{K}}(\varsigma'_{\mathcal{K}})+2d_{\mathcal{R}}(\varsigma_{\mathcal{R}})} \geq \frac{d_{\mathcal{R}}(\varsigma_{\mathcal{R}})}{\Delta_{\mathcal{K}}+d_{\mathcal{R}}(\varsigma_{\mathcal{R}})} \geq \frac{\delta_{\mathcal{R}}}{\Delta_{\mathcal{K}}+\delta_{\mathcal{R}}}.$$

Hence for every $(\varsigma_{\mathcal{K}}, \varsigma_{\mathcal{R}})(\varsigma'_{\mathcal{K}}, \varsigma_{\mathcal{R}}) \in E_1$,

$$\begin{aligned}
\frac{d_{\mathcal{K} \square \mathcal{R}}(\varsigma_{\mathcal{K}},\varsigma_{\mathcal{R}})d_{\mathcal{K} \square \mathcal{R}}(\varsigma'_{\mathcal{K}},\varsigma_{\mathcal{R}})}{d_{\mathcal{K} \square \mathcal{R}}(\varsigma_{\mathcal{K}},\varsigma_{\mathcal{R}})+d_{\mathcal{K} \square \mathcal{R}}(\varsigma'_{\mathcal{K}},\varsigma_{\mathcal{R}})} &= \frac{(d_{\mathcal{K}}(\varsigma_{\mathcal{K}})+d_{\mathcal{R}}(\varsigma_{\mathcal{R}}))(d_{\mathcal{K}}(\varsigma'_{\mathcal{K}})+d_{\mathcal{R}}(\varsigma_{\mathcal{R}}))}{d_{\mathcal{K}}(\varsigma_{\mathcal{K}})+d_{\mathcal{K}}(\varsigma'_{\mathcal{K}})+2d_{\mathcal{R}}(\varsigma_{\mathcal{R}})} \\
&= \frac{d_{\mathcal{K}}(\varsigma_{\mathcal{K}})d_{\mathcal{K}}(\varsigma'_{\mathcal{K}})}{d_{\mathcal{K}}(\varsigma_{\mathcal{K}})+d_{\mathcal{K}}(\varsigma'_{\mathcal{K}})+2d_{\mathcal{R}}(\varsigma_{\mathcal{R}})} + \frac{d_{\mathcal{R}}(\varsigma_{\mathcal{R}})(d_{\mathcal{K}}(\varsigma_{\mathcal{K}})+d_{\mathcal{K}}(\varsigma'_{\mathcal{K}}))+d_{\mathcal{R}}(\varsigma_{\mathcal{R}})^2}{d_{\mathcal{K}}(\varsigma_{\mathcal{K}})+d_{\mathcal{K}}(\varsigma'_{\mathcal{K}})+2d_{\mathcal{R}}(\varsigma_{\mathcal{R}})} \\
&= \frac{d_{\mathcal{K}}(\varsigma_{\mathcal{K}})d_{\mathcal{K}}(\varsigma'_{\mathcal{K}})}{d_{\mathcal{K}}(\varsigma_{\mathcal{K}})+d_{\mathcal{K}}(\varsigma'_{\mathcal{K}})} \times \left(1 - \frac{2d_{\mathcal{R}}(\varsigma_{\mathcal{R}})}{d_{\mathcal{K}}(\varsigma_{\mathcal{K}})+d_{\mathcal{K}}(\varsigma'_{\mathcal{K}})+2d_{\mathcal{R}}(\varsigma_{\mathcal{R}})}\right) \\
&\quad + \frac{d_{\mathcal{R}}(\varsigma_{\mathcal{R}})(d_{\mathcal{K}}(\varsigma_{\mathcal{K}})+d_{\mathcal{K}}(\varsigma'_{\mathcal{K}}))+d_{\mathcal{R}}(\varsigma_{\mathcal{R}})^2}{d_{\mathcal{K}}(\varsigma_{\mathcal{K}})+d_{\mathcal{K}}(\varsigma'_{\mathcal{K}})+2d_{\mathcal{R}}(\varsigma_{\mathcal{R}})} \\
&\leq \frac{\Delta_{\mathcal{K}}}{\Delta_{\mathcal{K}}+\delta_{\mathcal{R}}}\left(\frac{d_{\mathcal{K}}(\varsigma_{\mathcal{K}})d_{\mathcal{K}}(\varsigma'_{\mathcal{K}})}{d_{\mathcal{K}}(\varsigma_{\mathcal{K}})+d_{\mathcal{K}}(\varsigma'_{\mathcal{K}})}\right) + \frac{d_{\mathcal{R}}(\varsigma_{\mathcal{R}})(d_{\mathcal{K}}(\varsigma_{\mathcal{K}})+d_{\mathcal{K}}(\varsigma'_{\mathcal{K}}))+d_{\mathcal{R}}(\varsigma_{\mathcal{R}})^2}{2(\delta_{\mathcal{K}}+\delta_{\mathcal{R}})}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\sum_{(\varsigma_{\mathcal{K}},\varsigma_{\mathcal{R}})(\varsigma'_{\mathcal{K}},\varsigma_{\mathcal{R}}) \in E_1} \frac{d_{\mathcal{K} \square \mathcal{R}}(\varsigma_{\mathcal{K}},\varsigma_{\mathcal{R}})d_{\mathcal{K} \square \mathcal{R}}(\varsigma'_{\mathcal{K}},\varsigma_{\mathcal{R}})}{d_{\mathcal{K} \square \mathcal{R}}(\varsigma_{\mathcal{K}},\varsigma_{\mathcal{R}})+d_{\mathcal{K} \square \mathcal{R}}(\varsigma'_{\mathcal{K}},\varsigma_{\mathcal{R}})} \\
&= \sum_{\varsigma_{\mathcal{R}} \in V_{\mathcal{R}}} \sum_{\varsigma_{\mathcal{K}}\varsigma'_{\mathcal{K}} \in E_{\mathcal{K}}} \frac{d_{\mathcal{K} \square \mathcal{R}}(\varsigma_{\mathcal{K}},\varsigma_{\mathcal{R}})d_{\mathcal{K} \square \mathcal{R}}(\varsigma'_{\mathcal{K}},\varsigma_{\mathcal{R}})}{d_{\mathcal{K} \square \mathcal{R}}(\varsigma_{\mathcal{K}},\varsigma_{\mathcal{R}})+d_{\mathcal{K} \square \mathcal{R}}(\varsigma'_{\mathcal{K}},\varsigma_{\mathcal{R}})} \\
&\leq \frac{n_{\mathcal{R}}\Delta_{\mathcal{K}}}{\Delta_{\mathcal{K}}+\delta_{\mathcal{R}}} ISI(\mathcal{K}) + \frac{2m_{\mathcal{R}}M_1(\mathcal{K})+m_{\mathcal{K}}M_1(\mathcal{R})}{2(\delta_{\mathcal{K}}+\delta_{\mathcal{R}})}.
\end{aligned} \tag{2.17}$$

Equality holds in (2.17) iff $\Delta_{\mathcal{K}} = \delta_{\mathcal{K}}$ and $\Delta_{\mathcal{R}} = \delta_{\mathcal{R}}$. Similarly,

$$\begin{aligned}
&\sum_{(\varsigma_{\mathcal{K}},\varsigma_{\mathcal{R}})(\varsigma_{\mathcal{K}},\varsigma'_{\mathcal{R}}) \in E_2} \frac{d_{\mathcal{K} \square \mathcal{R}}(\varsigma_{\mathcal{K}},\varsigma_{\mathcal{R}})d_{\mathcal{K} \square \mathcal{R}}(\varsigma_{\mathcal{K}},\varsigma'_{\mathcal{R}})}{d_{\mathcal{K} \square \mathcal{R}}(\varsigma_{\mathcal{K}},\varsigma_{\mathcal{R}})+d_{\mathcal{K} \square \mathcal{R}}(\varsigma_{\mathcal{K}},\varsigma'_{\mathcal{R}})} \\
&\leq \frac{n_{\mathcal{K}}\Delta_{\mathcal{R}}}{\Delta_{\mathcal{R}}+\delta_{\mathcal{K}}} ISI(\mathcal{R}) + \frac{2m_{\mathcal{K}}M_1(\mathcal{R})+m_{\mathcal{R}}M_1(\mathcal{K})}{2(\delta_{\mathcal{K}}+\delta_{\mathcal{R}})},
\end{aligned} \tag{2.18}$$

with quality iff $\Delta_{\mathcal{K}} = \delta_{\mathcal{K}}$ and $\Delta_{\mathcal{R}} = \delta_{\mathcal{R}}$. Using (2.17) and (2.18), we get (2.16). In addition, equality holds in (2.16) iff $\mathcal{K}$ and $\mathcal{R}$ are regular.

Theorem 2.14: *Let $\mathcal{K}$ or $\mathcal{R}$ have no isolated vertex. Then the ISI index of $\mathcal{K} \,\Box\, \mathcal{R}$ is bounded from below as*

$$ISI(\mathcal{K} \,\Box\, \mathcal{R}) \geq \frac{n_{\mathcal{R}}\delta_{\mathcal{K}}}{\delta_{\mathcal{K}}+\Delta_{\mathcal{R}}} ISI(\mathcal{K}) + \frac{n_{\mathcal{K}}\delta_{\mathcal{R}}}{\delta_{\mathcal{R}}+\Delta_{\mathcal{K}}} ISI(\mathcal{R}) + \frac{3(m_{\mathcal{R}}M_1(\mathcal{K})+m_{\mathcal{K}}M_1(\mathcal{R}))}{2(\Delta_{\mathcal{K}}+\Delta_{\mathcal{R}})}.$$

Equality occurs iff $\mathcal{K}$ and $\mathcal{R}$ are regular.

The proof follows a similar approach to that of Theorem 2.13, and therefore, we will omit it.

Corollary 2.15: *For a κ-regular graph $\mathcal{K}$ and a ρ-regular graph $\mathcal{R}$,*

$$ISI(\mathcal{K} \,\Box\, \mathcal{R}) = \frac{n_{\mathcal{K}}n_{\mathcal{R}}(\kappa+\rho)^2}{4}.$$

Example 2.16: For the C_4-nanotorus $C_\nu \,\Box\, C_\mu$ (see Figure 4), Rook's graph $K_\nu \,\Box\, K_\mu$ (see Figure 5) and prism graph $K_2 \,\Box\, C_\mu$ (see Figure 5),

1. $ISI(C_\nu \,\Box\, C_\mu) = 4\nu\mu$,
2. $ISI(K_\nu \,\Box\, K_\mu) = \frac{\nu\mu(\nu+\mu-2)^2}{4}$,
3. $ISI(K_2 \,\Box\, C_\mu) = \frac{9\mu}{2}$.

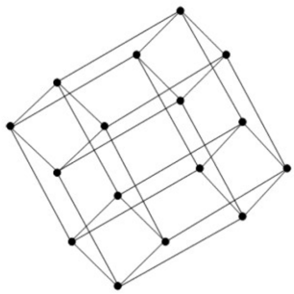

Figure 4
The C_4-nanotorus $C_4 \,\Box\, C_4$.

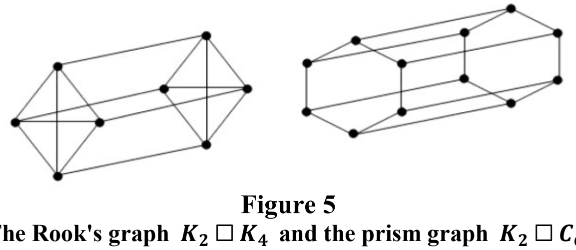

Figure 5
The Rook's graph $K_2 \,\Box\, K_4$ and the prism graph $K_2 \,\Box\, C_6$.

2.5 *Lexicographic product*

The *lexicographic product* $\mathcal{K}[\mathcal{R}]$ is the graph with $V_{\mathcal{K}[\mathcal{R}]} = V_{\mathcal{K}} \times V_{\mathcal{R}}$ and $E_{\mathcal{K}[\mathcal{R}]} = \{(\varsigma_{\mathcal{K}}, \varsigma_{\mathcal{R}})(\varsigma'_{\mathcal{K}}, \varsigma'_{\mathcal{R}}) | \varsigma_{\mathcal{R}}, \varsigma'_{\mathcal{R}} \in V_{\mathcal{R}}, \varsigma_{\mathcal{K}}\varsigma'_{\mathcal{K}} \in E_{\mathcal{K}}\} \cup \{(\varsigma_{\mathcal{K}}, \varsigma_{\mathcal{R}})(\varsigma_{\mathcal{K}}, \varsigma'_{\mathcal{R}}) | \varsigma_{\mathcal{K}} \in V_{\mathcal{K}}, \varsigma_{\mathcal{R}}\varsigma'_{\mathcal{R}} \in E_{\mathcal{R}}\}$ (see [37]). Obviously, $n_{\mathcal{K}[\mathcal{R}]} = n_{\mathcal{K}} n_{\mathcal{R}}$ and $m_{\mathcal{K}[\mathcal{R}]} = m_{\mathcal{K}} n_{\mathcal{R}}^2 + m_{\mathcal{R}} n_{\mathcal{K}}$.

Theorem 2.17: *Let $\mathcal{K}$ or $\mathcal{R}$ have no isolated vertices. Then the ISI index of $\mathcal{K}[\mathcal{R}]$ is bounded from above as*

$$\begin{aligned} ISI(\mathcal{K}[\mathcal{R}]) \leq\ & \frac{n_{\mathcal{R}}^4\, \Delta_{\mathcal{K}}\, ISI(\mathcal{K})}{n_{\mathcal{R}}\Delta_{\mathcal{K}} + \delta_{\mathcal{R}}} + \frac{n_{\mathcal{K}}\, \Delta_{\mathcal{R}}\, ISI(\mathcal{R})}{n_{\mathcal{R}}\delta_{\mathcal{K}} + \Delta_{\mathcal{R}}} \\ & + \frac{3m_{\mathcal{R}} n_{\mathcal{R}}^2 M_1(\mathcal{K}) + 2m_{\mathcal{K}} n_{\mathcal{R}} M_1(\mathcal{R}) + 4m_{\mathcal{K}} m_{\mathcal{R}}^2}{2(n_{\mathcal{R}}\delta_{\mathcal{K}} + \delta_{\mathcal{R}})}, \end{aligned} \tag{2.19}$$

and equality occurs iff $\mathcal{K}$ and $\mathcal{R}$ are regular.

Proof: The edges of the graph $\mathcal{K}[\mathcal{R}]$ can be divided into two subsets, labeled as E_1 and E_2, where

$$E_1 = \{(\varsigma_{\mathcal{K}}, \varsigma_{\mathcal{R}})(\varsigma'_{\mathcal{K}}, \varsigma'_{\mathcal{R}}) | \varsigma_{\mathcal{R}}, \varsigma'_{\mathcal{R}} \in V_{\mathcal{R}}, \varsigma_{\mathcal{K}}\varsigma'_{\mathcal{K}} \in E_{\mathcal{K}}\},$$
$$E_2 = \{(\varsigma_{\mathcal{K}}, \varsigma_{\mathcal{R}})(\varsigma_{\mathcal{K}}, \varsigma'_{\mathcal{R}}) | \varsigma_{\mathcal{R}}\varsigma'_{\mathcal{R}} \in E_{\mathcal{R}}, \varsigma_{\mathcal{K}} \in V_{\mathcal{K}}\}.$$

For $(\varsigma_{\mathcal{K}}, \varsigma_{\mathcal{R}})(\varsigma'_{\mathcal{K}}, \varsigma'_{\mathcal{R}}) \in E_1$, $d_{\mathcal{K}[\mathcal{R}]}(\varsigma_{\mathcal{K}}, \varsigma_{\mathcal{R}}) = n_{\mathcal{R}} d_{\mathcal{K}}(\varsigma_{\mathcal{K}}) + d_{\mathcal{R}}(\varsigma_{\mathcal{R}})$ and $d_{\mathcal{K}[\mathcal{R}]}(\varsigma'_{\mathcal{K}}, \varsigma'_{\mathcal{R}}) = n_{\mathcal{R}} d_{\mathcal{K}}(\varsigma'_{\mathcal{K}}) + d_{\mathcal{R}}(\varsigma'_{\mathcal{R}})$. Moreover

$$\begin{aligned} \frac{n_{\mathcal{R}}^2 \left(d_{\mathcal{K}}(\varsigma_{\mathcal{K}}) + d_{\mathcal{K}}(\varsigma'_{\mathcal{K}})\right)}{n_{\mathcal{R}}(d_{\mathcal{K}}(\varsigma_{\mathcal{K}}) + d_{\mathcal{K}}(\varsigma'_{\mathcal{K}})) + d_{\mathcal{R}}(\varsigma_{\mathcal{R}}) + d_{\mathcal{R}}(\varsigma'_{\mathcal{R}})} &\leq \frac{n_{\mathcal{R}}^2 \left(d_{\mathcal{K}}(\varsigma_{\mathcal{K}}) + d_{\mathcal{K}}(\varsigma'_{\mathcal{K}})\right)}{n_{\mathcal{R}}(d_{\mathcal{K}}(\varsigma_{\mathcal{K}}) + d_{\mathcal{K}}(\varsigma'_{\mathcal{K}})) + 2\delta_{\mathcal{R}}} \\ = n_{\mathcal{R}} - \frac{2\, n_{\mathcal{R}}\, \delta_{\mathcal{R}}}{n_{\mathcal{R}}(d_{\mathcal{K}}(\varsigma_{\mathcal{K}}) + d_{\mathcal{K}}(\varsigma'_{\mathcal{K}})) + 2\delta_{\mathcal{R}}} &\leq \frac{n_2^2 \Delta_{\mathcal{K}}}{n_{\mathcal{R}}\Delta_{\mathcal{K}} + \delta_{\mathcal{R}}}. \end{aligned}$$

Hence for every $(\varsigma_{\mathcal{K}}, \varsigma_{\mathcal{R}})(\varsigma'_{\mathcal{K}}, \varsigma'_{\mathcal{R}}) \in E_1$,

$$\begin{aligned} \frac{d_{\mathcal{K}[\mathcal{R}]}(\varsigma_{\mathcal{K}}, \varsigma_{\mathcal{R}}) d_{\mathcal{K}[\mathcal{R}]}(\varsigma'_{\mathcal{K}}, \varsigma'_{\mathcal{R}})}{d_{\mathcal{K}[\mathcal{R}]}(\varsigma_{\mathcal{K}}, \varsigma_{\mathcal{R}}) + d_{\mathcal{K}[\mathcal{R}]}(\varsigma'_{\mathcal{K}}, \varsigma'_{\mathcal{R}})} &= \frac{(n_{\mathcal{R}} d_{\mathcal{K}}(\varsigma_{\mathcal{K}}) + d_{\mathcal{R}}(\varsigma_{\mathcal{R}}))(n_{\mathcal{R}} d_{\mathcal{K}}(\varsigma'_{\mathcal{K}}) + d_{\mathcal{R}}(\varsigma'_{\mathcal{R}}))}{n_{\mathcal{R}}(d_{\mathcal{K}}(\varsigma_{\mathcal{K}}) + d_{\mathcal{K}}(\varsigma'_{\mathcal{K}})) + d_{\mathcal{R}}(\varsigma_{\mathcal{R}}) + d_{\mathcal{R}}(\varsigma'_{\mathcal{R}})} \\ &= \frac{n_{\mathcal{R}}^2 d_{\mathcal{K}}(\varsigma_{\mathcal{K}}) d_{\mathcal{K}}(\varsigma'_{\mathcal{K}})}{n_{\mathcal{R}}(d_{\mathcal{K}}(\varsigma_{\mathcal{K}}) + d_{\mathcal{K}}(\varsigma'_{\mathcal{K}})) + d_{\mathcal{R}}(\varsigma_{\mathcal{R}}) + d_{\mathcal{R}}(\varsigma'_{\mathcal{R}})} \\ &\quad + \frac{n_{\mathcal{R}}(d_{\mathcal{R}}(\varsigma'_{\mathcal{R}}) d_{\mathcal{K}}(\varsigma_{\mathcal{K}}) + d_{\mathcal{R}}(\varsigma_{\mathcal{R}}) d_{\mathcal{K}}(\varsigma'_{\mathcal{K}})) + d_{\mathcal{R}}(\varsigma_{\mathcal{R}}) d_{\mathcal{R}}(\varsigma'_{\mathcal{R}})}{n_{\mathcal{R}}(d_{\mathcal{K}}(\varsigma_{\mathcal{K}}) + d_{\mathcal{K}}(\varsigma'_{\mathcal{K}})) + d_{\mathcal{R}}(\varsigma_{\mathcal{R}}) + d_{\mathcal{R}}(\varsigma'_{\mathcal{R}})} \\ &\leq \frac{n_2^2 \Delta_{\mathcal{K}}}{n_{\mathcal{R}}\Delta_{\mathcal{K}} + \delta_{\mathcal{R}}} \left(\frac{d_{\mathcal{K}}(\varsigma_{\mathcal{K}}) d_{\mathcal{K}}(\varsigma'_{\mathcal{K}})}{d_{\mathcal{K}}(\varsigma_{\mathcal{K}}) + d_{\mathcal{K}}(\varsigma'_{\mathcal{K}})} \right) \\ &\quad + \frac{n_{\mathcal{R}}(d_{\mathcal{R}}(\varsigma'_{\mathcal{R}}) d_{\mathcal{K}}(\varsigma_{\mathcal{K}}) + d_{\mathcal{R}}(\varsigma_{\mathcal{R}}) d_{\mathcal{K}}(\varsigma'_{\mathcal{K}})) + d_{\mathcal{R}}(\varsigma_{\mathcal{R}}) d_{\mathcal{R}}(\varsigma'_{\mathcal{R}})}{2(n_{\mathcal{R}}\delta_{\mathcal{K}} + \delta_{\mathcal{R}})}. \end{aligned}$$

Therefore

$$\begin{aligned} & \sum_{(\varsigma_{\mathcal{K}}, \varsigma_{\mathcal{R}})(\varsigma'_{\mathcal{K}}, \varsigma'_{\mathcal{R}}) \in E_1} \frac{d_{\mathcal{K}[\mathcal{R}]}(\varsigma_{\mathcal{K}}, \varsigma_{\mathcal{R}}) d_{\mathcal{K}[\mathcal{R}]}(\varsigma'_{\mathcal{K}}, \varsigma'_{\mathcal{R}})}{d_{\mathcal{K}[\mathcal{R}]}(\varsigma_{\mathcal{K}}, \varsigma_{\mathcal{R}}) + d_{\mathcal{K}[\mathcal{R}]}(\varsigma'_{\mathcal{K}}, \varsigma'_{\mathcal{R}})} \\ &= \sum_{\varsigma_{\mathcal{R}}, \varsigma'_{\mathcal{R}} \in V_{\mathcal{R}}} \sum_{\varsigma_{\mathcal{K}}\varsigma'_{\mathcal{K}} \in E_{\mathcal{K}}} \frac{d_{\mathcal{K}[\mathcal{R}]}(\varsigma_{\mathcal{K}}, \varsigma_{\mathcal{R}}) d_{\mathcal{K}[\mathcal{R}]}(\varsigma'_{\mathcal{K}}, \varsigma'_{\mathcal{R}})}{d_{\mathcal{K}[\mathcal{R}]}(\varsigma_{\mathcal{K}}, \varsigma_{\mathcal{R}}) + d_{\mathcal{K}[\mathcal{R}]}(\varsigma'_{\mathcal{K}}, \varsigma'_{\mathcal{R}})} \\ &\leq \frac{n_{\mathcal{R}}^4\, \Delta_{\mathcal{K}}\, ISI(\mathcal{K})}{n_{\mathcal{R}}\Delta_{\mathcal{K}} + \delta_{\mathcal{R}}} + \frac{2m_{\mathcal{R}} n_{\mathcal{R}}^2 M_1(\mathcal{K}) + 4m_{\mathcal{K}} m_{\mathcal{R}}^2}{2(n_{\mathcal{R}}\delta_{\mathcal{K}} + \delta_{\mathcal{R}})}. \end{aligned} \tag{2.20}$$

Equality holds iff $\Delta_{\mathcal{K}} = \delta_{\mathcal{K}}$ and $\Delta_{\mathcal{R}} = \delta_{\mathcal{R}}$. Using the proof of Theorem 2.13 and inequality (2.20), we have

$$\sum_{(\varsigma_{\mathcal{K}},\varsigma_{\mathcal{R}})(\varsigma_{\mathcal{K}},\varsigma'_{\mathcal{R}})\in E_2} \frac{d_{\mathcal{K}[\mathcal{R}]}(\varsigma_{\mathcal{K}},\varsigma_{\mathcal{R}})d_{\mathcal{K}[\mathcal{R}]}(\varsigma'_{\mathcal{K}},\varsigma'_{\mathcal{R}})}{d_{\mathcal{K}[\mathcal{R}]}(\varsigma_{\mathcal{K}},\varsigma_{\mathcal{R}})+d_{\mathcal{K}[\mathcal{R}]}(\varsigma'_{\mathcal{K}},\varsigma'_{\mathcal{R}})} \leq \frac{n_{\mathcal{K}}\Delta_{\mathcal{R}} ISI(\mathcal{R})}{n_{\mathcal{R}}\delta_{\mathcal{K}}+\Delta_{\mathcal{R}}} + \frac{2m_{\mathcal{K}}n_{\mathcal{R}}M_1(\mathcal{R})+n_{\mathcal{R}}^2 m_{\mathcal{R}}M_1(\mathcal{K})}{2(n_{\mathcal{R}}\delta_{\mathcal{K}}+\delta_{\mathcal{R}})}, \tag{2.21}$$

and equality occurs iff $\Delta_{\mathcal{K}} = \delta_{\mathcal{K}}$ and $\Delta_{\mathcal{R}} = \delta_{\mathcal{R}}$. From (2.20) and (2.21), we get (2.19). Moreover, equality holds in (2.19) iff $\mathcal{K}$ and $\mathcal{R}$ are regular.

By a reasoning similar to that of Theorem 2.17, we obtain:

Theorem 2.18: *Let $\mathcal{K}$ or $\mathcal{R}$ have no isolated vertices. Then the ISI index of $\mathcal{K}[\mathcal{R}]$ is bounded from below as*

$$ISI(\mathcal{K}[\mathcal{R}]) \geq \frac{n_{\mathcal{R}}^4\, \delta_{\mathcal{K}}\, ISI(\mathcal{K})}{n_{\mathcal{R}}\delta_{\mathcal{K}}+\Delta_{\mathcal{R}}} + \frac{n_{\mathcal{K}}\, \delta_{\mathcal{R}}\, ISI(\mathcal{R})}{n_{\mathcal{R}}\Delta_{\mathcal{K}}+\delta_{\mathcal{R}}} + \frac{3m_{\mathcal{R}}n_{\mathcal{R}}^2 M_1(\mathcal{K})+2m_{\mathcal{K}}n_{\mathcal{R}}M_1(\mathcal{R})+4m_{\mathcal{K}}m_{\mathcal{R}}^2}{2(n_{\mathcal{R}}\Delta_{\mathcal{K}}+\Delta_{\mathcal{R}})},$$

and equality occurs iff $\mathcal{K}$ and $\mathcal{R}$ are regular.

Corollary 2.19: *For a κ-regular graph $\mathcal{K}$ and a ρ-regular graph $\mathcal{R}$,*

$$ISI(\mathcal{K}[\mathcal{R}]) = \frac{n_{\mathcal{K}}n_{\mathcal{R}}(n_{\mathcal{R}}\kappa+\rho)^2}{4}.$$

Example 2.20: For the closed fence graph $C_\nu[K_2]$ (see Figure 6),

$$ISI(C_\nu[K_2]) = \frac{25\nu}{2}.$$

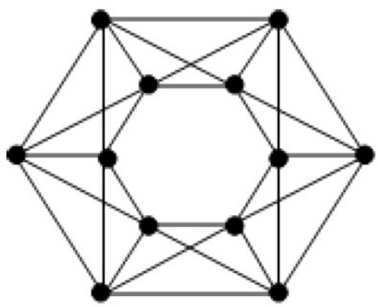

Figure 6
The closed fence graph $C_6[K_2]$.

2.6 *Strong product*

The *strong product* $\mathcal{K} \boxtimes \mathcal{R}$ is the graph with $V(\mathcal{K} \boxtimes \mathcal{R}) = V_{\mathcal{K}} \times V_{\mathcal{R}}$ and $E(\mathcal{K} \boxtimes \mathcal{R}) = \{(\varsigma_{\mathcal{K}},\varsigma_{\mathcal{R}})(\varsigma'_{\mathcal{K}},\varsigma_{\mathcal{R}}) | \varsigma_{\mathcal{R}} \in V_{\mathcal{R}}, \varsigma_{\mathcal{K}}\varsigma'_{\mathcal{K}} \in E_{\mathcal{K}}\} \cup \{(\varsigma_{\mathcal{K}},\varsigma_{\mathcal{R}})(\varsigma_{\mathcal{K}},\varsigma'_{\mathcal{R}}) | \varsigma_{\mathcal{K}} \in V_{\mathcal{K}}, \varsigma_{\mathcal{R}}\varsigma'_{\mathcal{R}} \in E_{\mathcal{R}}\} \cup \{(\varsigma_{\mathcal{K}},\varsigma_{\mathcal{R}})(\varsigma'_{\mathcal{K}},\varsigma'_{\mathcal{R}}) | \varsigma_{\mathcal{K}}\varsigma'_{\mathcal{K}} \in E_{\mathcal{K}}, \varsigma_{\mathcal{R}}\varsigma'_{\mathcal{R}} \in E_{\mathcal{R}}\}$ (see [37]). Obviously, $n_{\mathcal{K}\boxtimes\mathcal{R}} = n_{\mathcal{K}}n_{\mathcal{R}}$ and $m_{\mathcal{K}\boxtimes\mathcal{R}} = m_{\mathcal{K}}n_{\mathcal{R}} + m_{\mathcal{R}}n_{\mathcal{K}} + 2m_{\mathcal{K}}m_{\mathcal{R}}$.

The next theorems can be proved similar to Theorems 2.9-2.19 and hence we don't present their proofs.

Theorem 2.21: *Let $\mathcal{K}$ or $\mathcal{R}$ have no isolated vertices. Then the ISI index of $\mathcal{K} \boxtimes \mathcal{R}$ is bounded from above as*

$$\begin{aligned} ISI(\mathcal{K} \boxtimes \mathcal{R}) \leq\ & \frac{(n_{\mathcal{R}}+m_{\mathcal{R}})\Delta_{\mathcal{K}}(\Delta_{\mathcal{R}}+1)^2 ISI(\mathcal{K})+(n_{\mathcal{K}}\Delta_{\mathcal{R}}(\Delta_{\mathcal{K}}+1)^2+m_{\mathcal{K}}\Delta_{\mathcal{R}})ISI(\mathcal{R})}{\delta_{\mathcal{K}}+\delta_{\mathcal{R}}+\delta_{\mathcal{K}}\delta_{\mathcal{R}}} \\ &+ \frac{m_{\mathcal{R}}M_2(\mathcal{K})+m_{\mathcal{K}}M_2(\mathcal{R})+M_2(\mathcal{K})M_1(\mathcal{R})+M_1(\mathcal{K})M_2(\mathcal{R})+M_2(\mathcal{K})M_2(\mathcal{R})}{\delta_{\mathcal{K}}+\delta_{\mathcal{R}}+\delta_{\mathcal{K}}\delta_{\mathcal{R}}} \\ &+ \frac{(n_{\mathcal{R}}+m_{\mathcal{R}})\Delta_{\mathcal{R}}(\Delta_{\mathcal{R}}+1)M_1(\mathcal{K})+n_{\mathcal{K}}\Delta_{\mathcal{K}}(\Delta_{\mathcal{K}}+1)M_1(\mathcal{R})+M_1(\mathcal{K})M_1(\mathcal{R})}{2(\delta_{\mathcal{K}}+\delta_{\mathcal{R}}+\delta_{\mathcal{K}}\delta_{\mathcal{R}})} \\ &+ \frac{n_{\mathcal{R}}m_{\mathcal{K}}\Delta_{\mathcal{R}}^2+n_{\mathcal{K}}m_{\mathcal{R}}\Delta_{\mathcal{K}}^2}{2(\delta_{\mathcal{K}}+\delta_{\mathcal{R}}+\delta_{\mathcal{K}}\delta_{\mathcal{R}})}. \end{aligned}$$

Equality occurs iff $\mathcal{K}$ and $\mathcal{R}$ are regular.

Theorem 2.22: *Let $\mathcal{K}$ or $\mathcal{R}$ be nonempty graphs. Then the ISI index of $\mathcal{K})\mathcal{R}$ is bounded from below as*

$$\begin{aligned} ISI(\mathcal{K} \boxtimes \mathcal{R}) \geq\ & \frac{(n_{\mathcal{R}}+m_{\mathcal{R}})\delta_{\mathcal{K}}(\delta_{\mathcal{R}}+1)^2 ISI(\mathcal{K})+(n_{\mathcal{K}}\delta_{\mathcal{R}}(\delta_{\mathcal{K}}+1)^2+m_{\mathcal{K}}\delta_{\mathcal{R}})ISI(\mathcal{R})}{\Delta_{\mathcal{K}}+\Delta_{\mathcal{R}}+\Delta_{\mathcal{K}}\Delta_{\mathcal{R}}} \\ &+ \frac{m_{\mathcal{R}}M_2(\mathcal{K})+m_{\mathcal{K}}M_2(\mathcal{R})+M_2(\mathcal{K})M_1(\mathcal{R})+M_1(\mathcal{K})M_2(\mathcal{R})+M_2(\mathcal{K})M_2(\mathcal{R})}{\Delta_{\mathcal{K}}+\Delta_{\mathcal{R}}+\Delta_{\mathcal{K}}\Delta_{\mathcal{R}}} \\ &+ \frac{(n_{\mathcal{R}}+m_{\mathcal{R}})\delta_{\mathcal{R}}(\delta_{\mathcal{R}}+1)M_1(\mathcal{K})+n_{\mathcal{K}}\delta_{\mathcal{K}}(\delta_{\mathcal{K}}+1)M_1(\mathcal{R})+M_1(\mathcal{K})M_1(\mathcal{R})}{2(\Delta_{\mathcal{K}}+\Delta_{\mathcal{R}}+\Delta_{\mathcal{K}}\Delta_{\mathcal{R}})} \\ &+ \frac{n_{\mathcal{R}}m_{\mathcal{K}}\delta_{\mathcal{R}}^2+n_{\mathcal{K}}m_{\mathcal{R}}\delta_{\mathcal{K}}^2}{2(\Delta_{\mathcal{K}}+\Delta_{\mathcal{R}}+\Delta_{\mathcal{K}}\Delta_{\mathcal{R}})}. \end{aligned}$$

Equality occurs iff $\mathcal{K}$ and $\mathcal{R}$ are regular.

Corollary 2.23: *For a κ-regular graph $\mathcal{K}$ and a ρ-regular graph $\mathcal{R}$,*

$$ISI(\mathcal{K} \boxtimes \mathcal{R}) = \frac{n_{\mathcal{K}}n_{\mathcal{R}}(\kappa+\rho+\kappa\rho)^2}{4}.$$

Example 2.24: For the graphs $C_\nu \boxtimes C_\mu$, $K_\nu \boxtimes K_\mu$ and $C_\nu \boxtimes K_\mu$,

(i) $ISI(C_\nu \boxtimes C_\mu) = 16\nu\mu$,

(ii) $ISI(K_\nu \boxtimes K_\mu) = \frac{\nu\mu(\nu\mu-1)}{4}$,

(iii) $ISI(C_\nu \boxtimes K_\mu) = \frac{\nu\mu(3\mu-1)}{4}$.

3. Conclusion

In this paper, we explored the relations between the ISI index of certain graph operations, including the sum, corona product, disjunctive product, Cartesian product, lexicographic product, and strong product, and the ISI index of their respective factors. However, there are still numerous other graph operations and topological indices that are not addressed in this study. Investigating these relationships for additional graph operations, such as the rooted product (refer to [38]), hierarchical product (see [39]), edge-corona and neighborhood edge-corona (see [40]), and edge F-join (see [41]), would be intriguing. Furthermore, examining topological indices from different categoriesâ€”such as entropy-based, eigenvalue-based, and spectrum-based indices under various graph operations could prove beneficial. Future research in these areas could yield significant insights into the physico-chemical features of molecular structures.

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Conflicts of interest

The authors have no competing interest related to this paper.

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