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Recent result about Loc (N_{ie}) - hollow fuzzy modules and related concepts

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Abstract

Throughout the work, demonstrate and study (LC (N_{ie})-H-F M) as generalization of concepts (L(N_{ie})-hollow modules). We give several fundamental properties about this concept.

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Introduction

We stated that module M is an LP-hollow module, and that each of its little submodules is contained in a single prime submodule. [6] (in this paper use N_{ie} to mention that N_{ie}-submodules is Nothieron fuzzy submodule, H-F M to mention that hollow fuzzy module & cyc to mention cyclic) Our article contains from four section. Section one reminded us of certain definitions and qualities that we required. Some fundamental LC (N_{ie})-H-F -M features are disputed in section two, Characterization of LC (N_{ie}) – H-F M. section three, include relation between of (L-hollow, and LC (N_{ie}) - hollow) – H-F M.

1." Preliminaries:

Definition 1.1: [1] Let χ_t: M → [0, 1] be a fuzzy set in M, x ∈ M, t ∈(0,1] defined by

$X_t(y)$ = {1 if t = y ∈(0,1]

0 if t ≠ y, when x = 0 and t = 1, χt is referred to as Ƒ- singleton

0_1 (y) = {1 if t = y

0 if t ≠ y, is Ƒ-singleton fuzzy zero singleton "

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Definition 1.2: [8, 9] Assume that X is the simple F-module of R-module M. It has one appropriate F-submodule, which is 0_1.

Definition 1.3: [2, 12] Fuzzy module X of R-module M, & B is F-submodule of X, then if each F-submodule A of X is such that B+ Ă = X, then B is small F-submodule in X. This suggests that Ă = B ".

Lemma (1.4): [7]: If χ is cyc F- Module, then χ_t is cyc module, $\forall$ k $\in$(0,1] .A are only a direct summand of A."

2. Characterization of LC (N_{ie}) – H-F M

The ideas of LC (N) – H-F M are covered in this section, in addition to a review of these modules' fundamental features.

Definition 2.1: F-module χ called is LC (**N**$_{ie}$) - H-F M, where χ has unique Nothieron f- subm contains each small Ƒ- subm oχ. That is to say, in terms of inclusion, it has the Ascending chain condition. That is, if each fuzzy collection of rising F-submodule sequences finally turns into a constant.

Example 2.2: 1- M = Z_9 of Module Z, χ: M → [0, 1] where

X(x) = {1 if x ∈ M, Ă M → [0,1] s.t Ą $_{(x)}$ = {1 if x ∈ N

0,5 otherwise 0.0833 otherwise, (Ă: χ)={1 if x ∈ N

0.25 Otherwise

Thus, Ă $_t$ = N is submodule of χ_t & χ_t = M is f-module. But Ă $_t$ =N =3Z unique prime subm of X_t [6] [5], implies that is unique Nothieron submodule of χ . And hence X is LC (**N**$_{ie}$) - H-F M. Implies that Ă is Nothieron f- subm of X = Z_9 contain all small f- sub. since There exists: $4_{\frac{1}{3}}$. $6_{\frac{1}{2}}$ = $24_{\frac{1}{3}} \subseteq$ Ă for fuzzy singleton $4_{\frac{1}{3}}$of R, then Ă (24) =1 $\ngtr \frac{1}{3}$ & $6_{\frac{1}{3}} \geq \subseteq$ (Ă: χ)=3Z, Ă (6) =1≥ $\frac{1}{3}$, but $4_{\frac{1}{3}}$ $\not\subseteq$ (Ă: χ) =0.25 $\ngtr \frac{1}{3}$. Implies that Ă is Nothieron submodule of χ = Z_9 contain all small fuzzy submodule, hence χ is LC (**N**$_{ie}$) - H-F M.

2- While X=Z_6 as Z-module is not Lc (**N**$_{ie}$) - hollow f- module.

Ă: M→ [0, 1] s, t, (Ă: χ)={ 1 if x ∈ N ,N∈ 2Z

0.25 Otherwise

Proof: M = Z_6, R = Z, χ:M→ [0, 1] s ,t $_{(\chi)}$={1 if x ∈ M

0.25 Otherwise

B: M → [0, 1] s, t B(x) = {1 if x ∈ M

0.25 Otherwise, where K=3Z

(Ăχ) (r) = {1 if R ∈ Z

0.1428 Otherwise ∀ t ∈ (0, 1]

Hence χ is f- module, Ă $_t$ = N, B_t = K are two subm of χ_t, ∀ t ∈(0,1], Since Ă$_t$, B_t are two prime submodule of χ_t as Z-module by [3] .Implies that two Nothieron submodule of X_t as Z-module by definition .Therefore A,B are two

Nothieron fuzzy submodule of χ. Since $4_{\frac{1}{3}} . 5_{\frac{1}{2}} = 19_{\frac{1}{3}} \subseteq$ Ă for fuzzy singleton $4_{\frac{1}{3}}$ of R. Then Ą (10) =1 $\geq \frac{1}{3}$ and $4_{\frac{1}{3}} \subseteq$ (Ą: χ) =2Z, (Ą: χ) (4) =1 $\geq \frac{1}{3}$. Similarly, B_t. In addition, B_t is not small submodule of X_t [6]. Therefore, X is not is LC (N) - H-F M.

Proposition 2.3: If A is Nothieron F- sub of F- module χ. Then A_t is Nie- F- subm of χ_t for all t ∈ (0, 1], immodestly by [11].

Remark 2.4: Every prime fuzzy submodule is Nothieron fuzzy submodule.

Proposition 2.5: F- module χ is Lc (**N**$_{ie}$) - H-F M ⇔ X_t is Lc (**N**$_{ie}$) H-F M ∀ t, of R-module M.

Proof: ⟹ if χ Lc (**N**$_{ie}$) - H-F M, therefore every f-subm in χ is unique **N**$_{ie}$ small F-submodule of consists of all of F-subm. Assume that Ą is F- submodule of X. Thus, for a given f-sub B in Ą of χ, Ă + B = χ , $(Ă + B)_t = \chi_t$ ∀ t ∈(0,1] [7]. Then $Ă_t$ is submodule in χ_t. .That's mean there exists $r_t\ \alpha_k \leq Ą_t$ for f- singleton r_t of R & $\alpha_k \leq \chi$, now either $r_t \leq (Ă: \chi)_t$ or $\alpha_k \leq$ (Ă ; χ), wherever $(Ă: \chi)_t = \{r_t : r_t\ \chi_t \leq Ă_t$, r_t F- singleton of R }. However, because Ă prime F-subm and contains every small F- submodule of X, Ă is Nie F-subm and contains every small F- submodule of χ and so on. by (remark 2.3). Therefore A_t is **N**$_{ie}$ F-submodule of X_t by example (2.2), hence X_t is Lc (**N**$_{ie}$) - H-F M ∀ t ∈(0,1] ⟸ Let χ_t be a LC (N)-H-F M. Suppose that B_t is **N**$_{ie}$ submodule of X_t. then By example (2.2) part one, That's mean there exists $r_t\ \alpha_k \leq A_t$ for f- singleton r_t of R & $\alpha_k \leq \chi$, hence, either $r_t \leq$ (Ă: χ) or $\alpha_k \leq$ (Ă: χ), wherever (Ă: χ) = { r_t : $r_t\ \chi \leq$ Ă, r_t f- singleton of R }. B is small f-**δ** ubm of χ .Therefore χ is Lc (**N**$_{ie}$) - H-F M

Remarks and examples 2.6: The Z-module Q is Lc (**N**$_{ie}$) - H-F M, for instance, although the converse isn't always true. We may show this as follows: Each Lc - H-F M is Lc (**N**$_{ie}$) - H-F M.**:** Suppose that X: Q → [0, 1],

$X_{(x)}$ = {1 if x ∈ M, Ă: Q → [0, 1], A(x) = {1 if x ∈ M
0 Otherwise 0 Otherwise where N= $\langle \overline{0} \rangle$
(Ă: X) (r) = {1 if r ∈ Z
0.1428 Otherwise ∀ t ∈ (0, 1]

Hence χ is f- module, A is f- sub of χ , Since $Ă_t = \langle \overline{0} \rangle$ is small fuzzy sub of X_0= Q, by [4], that is only Nothieron fuzzy submodule in Q, Since$4_{\frac{1}{3}} . 0_{\frac{1}{2}} = 0_{\frac{1}{3}} \subseteq$ Ă for F-singleton$4_{\frac{1}{3}}$ of R, then $0_{\frac{1}{2}}$ (Ă: X) = $\langle \overline{0} \rangle$ also Ă (0) =1 $\geq \frac{1}{3}$, $4_{\frac{1}{3}} \subseteq$ (Ă: X) = $\langle \overline{0} \rangle$ and hence (Ă: X) (4) =1 $\geq \frac{1}{3}$. Whilst Q is not Lc - H-F M, where Q it has not unique maximal fuzzy submodule contain all fuzzy small submodule.

1- Each LC (**N**$_{ie}$) - H-F M is not necessity be simple fuzzy module.

For example, Z-module Z_{25} Lc ($\mathbf{N}_{ie}$) - H-F M, nevertheless not simple F-module **proof:** Let R=Z, M=Z_{25}, X: M → [0, 1], X(x) = {1 if x ∈ M,

0.5 Otherwise

Ă: M ⟶ [0, 1], s.t A(x) = {1 if x ∈ N

0.076923 Otherwise where N= 5Z

(A: X) (r) = {1 if r ∈ N

0.08333 Otherwise ∀ t ∈ (0, 1]

Hence X and A are F- modules (F-submodule) $Ă_t$ = N, X_t = M. There exist $4_{\frac{1}{3}}$. $10_{\frac{1}{2}}$=$40_{\frac{1}{3}}$ ⊆ A for F- singleton$4_{\frac{1}{3}}$ of R, then $40_{\frac{1}{3}}$ ⊆ (**Ă**: X) =5Z since **Ă** (40) =1 ≥ $\frac{1}{3}$ and $10_{\frac{1}{2}}$ ⊆ (**Ă**: X)=5Z, (**Ă**: X) (10) =1 ≥ $\frac{1}{2}$ .Therefore Ă is unique Nothieron f- sub of χ. But Z_{25} is not fuzzy simple fuzzy module. 3-The fuzzy Z-module Z is not is Lc ($\mathbf{N}_{ie}$) - H-F M, because X_t= Z haven't unique Nothieron f-sub.4- A f- subm of Lc ($\mathbf{N}_{ie}$) - - H-F M, not necessity be Lc ($\mathbf{N}_{ie}$) - H-F M .Since a fuzzy Z- module Z is f- subm of the fuzzy module Q which is Lc ($\mathbf{N}_{ie}$) - H-F M by (1), whilst Z is not Lc ($\mathbf{N}_{ie}$) - H-F M.5-"If an R- module X is finitely generated fuzzy module , the each LC ($\mathbf{N}_{ie}$) - H-F M is Loc - H-F M."6-Every H-F M is LC ($\mathbf{N}_{ie}$) - H-F M. B nevertheless the converse is not always correct. By instance, the fuzzy Z-module Q is LC ($\mathbf{N}_{ie}$) - H-F M. Since (0_1) is the only prime f- subm of Q [10],[11] implies that is the only Nothieron f- subm of Q by (remark2,3), where it is small F- submodule of Q but Q is not H-F M by (1).

Proposition 2.7: Epimorphic image of LC (N) - H-F M is LC (N) - H-F M.

Proof: Suppose that χ_1 is LC ($\mathbf{N}_{ie}$) -H-F M, take $\boldsymbol{f}$: $\chi_1 \rightarrow \chi_2$is epimorphism where χ_2 is R-module. Let Ă is unique prime Ƒ- subm of χ_2 ,with Ă + C= X_2 where C is proper Ƒ- subm, with $(Ă_t+ C)_t = (\chi_2)_t$, by [4] and hence is unique Nothieron f- subm of χ_2 [def]. Now $\boldsymbol{f^{-1}(A)}$ is unique nothieron Ƒ- subm of X_1 .So from the other hand, $\boldsymbol{f^{-1}(A)}$=X_1, also $\boldsymbol{f}$ ($\boldsymbol{f^{-1}(A)}$) = $\boldsymbol{f}$ (χ_1) =χ_2 implies that Ă =χ_2. consequently we have $(Ă)_t= (\chi_2)_t$, ∀ t ∈ (0, 1] by [1], which contradiction, since Ă is unique prime f- subm of X_2, hence Ă unique nothieron f- subm of X_2 (remark 2.3). Therefore $\boldsymbol{f^{-1}(A)}$ is unique Nothieron f- subm of χ_1. But χ_1 is LC ($\mathbf{N}_{ie}$) - H-F M. Thus ($\boldsymbol{f^{-1}(A)}$ contain each small f- subm of χ_1Also $\boldsymbol{f}$ ($\boldsymbol{f^{-1}(A)}$)Ą is small f-sub of $f(\chi_1)$ since Ă small f-sub of χ_2.Therefore X2 is LC ($\mathbf{N}_{ie}$) - H-F M (2.3)

3. Relation between (LC and LC (N)) H-F M.

Every LC – H-F M is LC ($\mathbf{N}_{ie}$) - H-F M, where an R- module χ is LC - H-F M. e, if χ has a unique maximal Ƒ- subm that include each small Ƒ- subm of χ. we introduced that in the previous section. Also we demonstration the converse is not true by various examples .the part examine conditions under LC ($\mathbf{N}_{ie}$) - H-F M could be LC – H-F M, χ is LC – H-F M ⇔ χ is LC ($\mathbf{N}_{ie}$) – H and finitely produced f- module .

Proposition 3.1: A F- module χ is LC – H-F M $\Leftrightarrow$ χ is LC (N) - hollow and has unique Nothieron f-subm, of R-module M.

Proof: $\Longrightarrow$ Let χ is LC – H-F M, therefore χ is LC ($\mathbf{N}_{ie}$) – H-F M immediately by remark (2.5) (1). Hence χ has unique maximal fuzzy module, also by (2.1) χ has unique prime f-submodule by [1]. χ Has unique Nothieron fuzzy submodule by (3.2).$\Leftarrow$ Letχ is LC ($\mathbf{N}_{ie}$) - hollow and has unique nothieron f-subm say A, sufficient evidence to demonstrate that X is CYC f- module, let $x_t \subseteq \chi$ and $x \notin A_t$ $\forall$ $\in(0,1]$, then $(rx + A)_t=(\chi)_t$ $\forall$ $t \in(0,1]$. Since, χ is LC ($\mathbf{N}_{ie}$) - H-F M M implies that Ă is unique Nothieron fuzzy submodule of χ contain all a small fuzzy submodule of χ , χ = Rx. hence X is CYC F- module (2.1), hence χ is LC - H-F M. let $x_t \subseteq \chi$ and $x \notin A_t$ $\forall$ $\in(0,1]$, then $(Rx + \breve{A})_t =(\chi)_t$ ¸ $\forall$ $t \in(0,1]$. Since X is LC (N) - H-F M implies that Ă is unique prime fuzzy submodule of χ contain all small fuzzy submodules of, χ and hence X= (x_t). Hence χ is CYC H-F M, and by definition (2.1), i implies that X LC - H-F M.

Proposition 3.2: F-Module χ is Lc – H-F M $\Leftrightarrow$ χ Lc ($\mathbf{N}_{ie}$) - H, F-Rad (χ) $\neq$ χ.

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